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Flag Algebras in Extremal Graph Theory

PhD Thesis
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Contents

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Introduction

Extremal graph theory, and more generally extremal combinatorics, is a large field with lots of intriguing connections and important applications in many distinct areas of mathematics, physics, and computer science (cf. [**3**]). Typical and deceptively innocent sounding problems include: What is the minimum number of edges in a graph on n vertices guaranteeing existence of a triangle? How many triangles are there in a graph whose number of edges is above that threshold? One can easily imagine more of that sort based on other similar graph properties. Actually, there are plenty of challenges and open problems stimulating extensive research in the area for the several decades.

In many areas in computer science one studies properties of very large graphs – graph models of technological networks. For example the graphs of computers and physical links between them, the graph of so-called Autonomous Systems such as Internet Service Providers, the graph of webpages with hyperlinks, graphs of social networks: acquaintances, co-publications, the spreading of certain diseases, etc. These networks are formed by random processes, but their properties are quite different from the traditional Erdős-Rényi random graphs. They tend to be clustered, the neighborhoods of their nodes are denser than the average edge density, etc. Several models of those so called "scale-free" random graphs have been proposed and studied. For rigorous work, see [**13, 14**] for undirected models, [**53**] for "copying models", [**10**] for a directed model, [**9**] for the spread of viruses on these networks, and [**11**] for a survey of rigorous work with more complete references. For an extensive introduction to large networks, problems appearing there, and the connections with graph limits which are considered in this thesis, we strongly recommend the recent monograph by Lovász [**54**].

One of the very active areas of research related with large networks is graph property testing. In its most restricted form, it studies properties of very large graphs that can be tested by studying a randomly chosen induced subgraph of bounded size. For example we may ask what is the typical degree of a node, is the graph planar, does it contain a triangle, is it 3-colorable, etc. In general we have an enormously large amount of input data, so it is not possible to process with an exact answer, but we can provide an estimate based on a small sample. This is usually enough in the real world applications. For a precise definition of property testing and early work see [**34**], [**1**], and [**4**]. It is important that we can treat property testing in terms of the theory of graph limits [**16, 57, 58**]. In the space of graphons (limits of graphs) many problems and constructions have a simpler and clearer formulation (see [**18**] for a survey). This leads to a new characterization of testable properties by Lovász and Szegedy [**59**]. The theory can be also applied for testing permutations [**45, 49**]. A very similar issue is parameter testing, where we want to approximate some graph parameter of a very large graph by looking only at small samples. We say that a graph parameter $\mathcal{P}$ is testable if there exists a randomized algorithm that estimates the parameter $\mathcal{P}$ within the additive error ε based on a sample of size $f(\varepsilon)$ with probability $\geq 1-\varepsilon$. This area has a very natural treatment in the framework of graph limits [**15, 16**], since $\mathcal{P}$ is testable if and only if $\mathcal{P}$ is continuous on the graphon space.

Another source of motivation for studying extremal problems lies in possible consequences for some famous challenges in other disciplines. For instance, a seemingly slight extension of the celebrated result of Razborov [**66**] (giving a superpolynomial lower bound for monotone circuit complexity of the clique function) would imply that P $\neq$ NP. It is not surprising therefore that in recent years we are facing a rapid development of this area resulting in a variety of deep results, as well as new sophisticated methods, often based on tools from other areas of mathematics.

The aim of this thesis is to present new method based on algebraic and analytic tools – the celebrated method of flag algebras invented by Razborov [**67**]. This method provides a uniform framework for standard counting techniques used in extremal combinatorics. It is inspired by the theory of dense graph limits, on which we focus in Chapter 4. Despite the fact that the method is quite new, it has been successfully applied to various problems in extremal combinatorics, giving solutions to many long open-standing problems. In particular in Turán-type problems in graphs [**23, 35, 39, 41, 61, 63, 64, 70, 74, 76**], 3-graphs [**7, 27, 28, 32, 62, 69**], and hypercubes [**5, 8**], Caccetta-Häggkvist conjecture [**42, 71**], extremal problems in a colored setting [**6, 22, 38, 50**], and in geometry [**51**]. More details on these applications can be found in a recent survey of Razborov [**68**].

In Chapter 1 we give a tame introduction to this method illustrating basic ideas on the simplest example of the problem of triangles in graphs. Razborov introduced the method for an arbitrary universal first-order logic theory without constants or function symbols. For a better understanding, and in order to get an intuition, we restrict our attention to graphs.

In Chapter 2 we present an application of the method of flag algebras on graphs, solving the following problem of Erdős [**26**]. He conjectured that any triangle-free graph on n vertices has at most $(n/5)^5$ pentagons. We proved this conjecture in [**35**] using flag algebras supported by a semidefinite method (the proof is included in Chapter 2). The same result has been obtained independently in [**39**] by a slightly different, more elaborate approach.

In order to that the method can be utilized in a wider context, in Chapter 3 we apply flag algebras to specially defined combinatorial structures. We consider Seymour second neighborhood conjecture, which states that in every oriented graph there is a vertex with second neighborhood at least as big as the first neighborhood. In the class of Eulerian graphs we prove a statement, which implies the conjecture with an additive error.

The last part of the thesis concerns closely related field of graphons (limits of graphs) and permutons (limits of permutations). As described by Razborov [**73**], graph limits and flag algebras provide semantical and syntactical approaches (respectively) to the same class of objects (limits of sequences of combinatorial structures). The matter of interest are analytic properties of limits associated with convergent sequences of combinatorial structures. This line of research was initiated by the theory of limits of dense graphs [**15, 16, 17, 57**], followed by limits of sparse graphs [**12, 24, 25**], permutations [**33, 43, 44**], partial orders [**46**], etc. As mentioned earlier, it is strongly related with large networks, property testing, parameter testing, and many other areas of computer science. A recent monograph of Lovász [**54**] contains an excellent exposition of this exciting topic.

In Chapter 4 we focus on a question when the limit analytic object is uniquely determined by finitely many densities of substructures. This phenomenon is known as finite forcibility. Such graphons are related to uniqueness of extremal configurations in extremal graph theory as well as to other problems. A systematic study of finitely forcible graphons and permutons, which was started by Lovász and Szegedy in [**56**] and continued in [**30, 31, 33**], was motivated by a possibility of a better understanding of extremal configurations for problems in extremal graph theory. Trivially, for every finitely forcible graphon, there exists an extremal graph theory problem, for which this graphon is its unique solution. Lovász and Szegedy conjectured that this statement can be reversed – every extremal problem has a finitely forcible optimal solution. If an extremal problem has a unique solution then clearly the graphon corresponding to the solution is finitely forcible. However, the conjecture for general problems in extremal graph theory remains open. In Chapter 4, based on [**30**], we give some sufficient conditions for finite forcibility of permutons. We also show that all permutons that can be expressed as a finite combination of monotone permutons and quasirandom permutons are finitely forcible, which is the permuton counterpart of the result of Lovász and Sós [**55**] for graphons. Last but not least, we demonstrate that, somewhat surprisingly, finite forcibility of permutons is not preserved by the associated graphons (determined by the related permutation graphs). In particular, we find permutons that are finitely forcible but the associated graphons are not.

CHAPTER 1

Flag algebras

One of the earliest results in extremal graph theory is Mantel's theorem from 1907 [**60**] stating that a triangle-free graph on n vertices has at most $n^2/4$ edges. However, a similar question posed by Turán in 1941 [**77**] remains open: what is the maximum number of edges in a 3-uniform hypergraph with no tetrahedron? Here, we concentrate on questions with a similar flavor – what is the maximal number of some subgraphs (e.g., edges, pentagons) assuming there are no other fixed subgraphs (e.g., triangles). Usually, we are not interested in the exact number, but in the asymptotical behavior. Flag algebras drastically improve our ability to solve or to get approximate results to such problems.

In this chapter we we provide a brief introduction to the method of flag algebras. Our working example will be a classical theorem of Mantel. In subsequent sections we give several proofs of this result to explain the basic idea of the technique and only necessary formalism.

1.1. Basic definitions

We follow the standard graph theory notation. A *graph* is a pair $G = (V(G), E(G))$, where $V(G)$ is the set of *vertices* and $E(G)$ is a family of 2-element subsets of $V(G)$, called *edges*. We define the *edge density* of a graph G to be

$$d(G) = \frac{|E(G)|}{\binom{n}{2}},$$

where n is the size of $V(G)$.

Given a family $\mathcal{F}$ of graphs we say that a graph G is *$\mathcal{F}$-free* if G does not contain a subgraph isomorphic to any member of $\mathcal{F}$. For any integer $n \geq 1$ we define the *Turán number* of $\mathcal{F}$ to be

$$\mathrm{ex}(n, \mathcal{F}) = \max\{|E(G)| : G \text{ is } \mathcal{F}\text{-free}, |V(G)| = n\}.$$

The *Turán density* of $\mathcal{F}$ is defined to be the following limit (it always exists)

$$\pi(\mathcal{F}) = \lim_{n\to\infty} \frac{\mathrm{ex}(n,\mathcal{F})}{\binom{n}{2}}.$$

We can generalize these definitions as follows. Let A be a given graph on k vertices ($k \geq 2$). We define a set $C_A(G)$ consisting of all k-element subsets of $V(G)$ inducing a subgraph isomorphic to A. Now, we can define a generalized *density* of a graph G as

$$d(A,G) = \frac{|C_A(G)|}{\binom{n}{k}},$$

which is just the probability that a random $|V(A)|$-element set of vertices from G induces a graph isomorphic to A. If A is just a single edge, then we just get the typical definition of edge density. When A is a triangle, we get so-called triangle density.

We also generalize the definition of the *Turán number* of $\mathcal{F}$ to be

$$\mathrm{ex}_A(n,\mathcal{F}) = \max\{|C_A(G)| : G \text{ is } \mathcal{F}\text{-free}, |V(G)| = n\}.$$

Then, the *Turán density* of $\mathcal{F}$ becomes

$$\pi_A(\mathcal{F}) = \lim_{n\to\infty} \frac{\mathrm{ex}_A(n,\mathcal{F})}{\binom{n}{k}}.$$

This limit always exists because the sequence in the definition forms a decreasing sequence of real numbers in $[0,1]$. Determining the Turán density is equivalent to obtaining an asymptotic result $\mathrm{ex}_A(n,\mathcal{F}) \approx \pi_A(\mathcal{F})\binom{n}{k}$, provided that we are in the so-called 'non-degenerate' case when $\pi_A(\mathcal{F}) > 0$.

1.2. Intuitions

Flag algebras give us a systematic approach to finding counting arguments. They will be rigorously defined in the following sections, for the moment we wish to focus on presenting intuitions. We will focus on Turán density associated to some fixed graph A. Let $\mathcal{F}$ be a family of forbidden graphs whose Turán density we wish to compute (or at least approximate).

Let us fix some really large graph G and some small $l \geq 2$. Instead of counting appearances of A in graph G, we can count them in each possible graph H on l vertices ($l \geq |V(A)|$), and then, count the number of appearances of H in G. Thus we can write

$$d(A,G) = \sum_{|V(H)|=l} d(H,G)d(A,H).$$

Now, let let $\mathcal{F}_l^0$ be a family of all $\mathcal{F}$-free graphs on l vertices up to isomorphism (the reason why we place 0 here will be revealed later). If G is $\mathcal{F}$-free, then $d(H, G) = 0$ for H not in $\mathcal{F}_l^0$, so

$$d(A, G) = \sum_{H \in \mathcal{F}_l^0} d(H, G) d(A, H). \tag{1.1}$$

In particular we have an upper bound

$$d(A, G) \le \max_{H \in \mathcal{F}_l^0} d(A, H).$$

If l is sufficiently small we can explicitly determine $d(A, H)$ for all $H \in \mathcal{F}_l^0$ by computer search. However, unfortunately this bound on $d(A, G)$ in general is rather poor. The idea of the method is to generate further inequalities on the probabilities $d(H, G)$ that improve this bound. If we have a linear inequality

$$\sum_{H \in \mathcal{F}_l^0} c_H d(H, G) \ge 0,$$

then

$$d(A, G) \le \sum_{H \in \mathcal{F}_l^0} d(H, G)(d(A, H) + c_H) \le \max_{H \in \mathcal{F}_l^0} (d(A, H) + c_H),$$

which may actually be an improvement if some coefficients c_H are negative. Moreover, the inequalities we need can be of the form

$$\sum_{H \in \mathcal{F}_l^0} c_H d(H, G) + o(1) \ge 0, \tag{1.2}$$

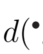
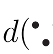
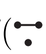
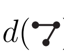
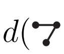
where $o(1)$ is taken with respect to $|V(G)|$. By using such an inequality we get

$$d(A, G) \le \max_{H \in \mathcal{F}_l^0} (d(A, H) + c_H) + o(1).$$

Thus

$$\pi_A(\mathcal{F}) \le \max_{H \in \mathcal{F}_l^0} (d(A, H) + c_H).$$

Let us consider an easy example. We will prove Mantel's theorem stating that triangle-free graph on n vertices has at most $n^2/4$ edges. It is easy to see that it is enough to show that $d(G) \le 1/2 + o(1)$ for every triangle-free graph G, where $o(1)$ is taken with respect to $n = |V(G)|$. We will consider $l = 3$. There are only 3 triangle-free graphs on 3 vertices – ,  and . For every G on at least 3 vertices, from (1.1) we get

$d(G) = d($$)d($$, G) + d($$)d($$, G) + d($$)d($$, G),$

and so

$$d(G) = \frac{1}{3}d(\;, G) + \frac{2}{3}d(\;, G).$$

If we manage to prove the inequality

$$\frac{1}{2}d(\;, G) - \frac{1}{6}d(\;, G) - \frac{1}{6}d(\;, G) + o(1) \geq 0, \tag{1.3}$$

then adding it to the previous equation, we will get

$$d(G) \leq \frac{1}{2}d(\;, G) + \frac{1}{6}d(\;, G) + \frac{1}{2}d(\;, G) + o(1) \leq \frac{1}{2} + o(1).$$

This is exactly the inequality we would like to prove. So, the only thing we need when we are using this technique, is to know how to get inequalities like (1.3).

Let us focus on one particular vertex in graph G (on pictures we will denote it by unfilled circle). We can define the density of a graph with one vertex fixed in a similar way as before – as the probability of finding an induced copy of this graph with one vertex already fixed. Let G° be a graph G with some vertex fixed. For example, from this definition density $d(\;, G^\circ)$ is equal to number of vertices connected to fixed vertex divided by $n-1$.

We can write the inequality

$$\left(d(\;, G^\circ) - d(\;, G^\circ)\right)^2 \geq 0$$

and so

$$d(\;, G^\circ)d(\;, G^\circ) - 2d(\;, G^\circ)d(\;, G^\circ) + d(\;, G^\circ)d(\;, G^\circ) \geq 0.$$

The product $d(\;, G^\circ)d(\;, G^\circ)$ is the probability that two vertices chosen at random (we may also choose two times one vertex) are not connected to the fixed vertex. On the other hand, the sum $d(\;, G^\circ) + d(\;, G^\circ)$ represents the same probability, but we assume that we will not choose the same vertex two times. Probability of this event is going to 0 as the size of graph G increases. Hence, we can write

$$d(\;, G^\circ)d(\;, G^\circ) = d(\;, G^\circ) + d(\;, G^\circ) + o(1).$$

In a similar way, we can prove

$$d(\;, G^\circ)d(\;, G^\circ) = \frac{1}{2}\left(d(\;, G^\circ) + d(\;, G^\circ)\right) + o(1),$$

$$d(\;, G^\circ)d(\;, G^\circ) = d(\;, G^\circ) + o(1).$$

Using these relations, we get

$$d(\;, G^\circ) + d(\;, G^\circ) - d(\;, G^\circ) - d(\;, G^\circ) + d(\;, G^\circ) + o(1) \geq 0.$$

Now, we can average over all possible choices of the fixed vertex. For example, averaging $d(\ , G^\circ)$ we will asymptotically get $\frac{1}{3}d(\ , G)$, because only in one case out of three possibilities of choosing a vertex in graph we will get graph . Thus, we can write

$$d(\ , G) + \frac{1}{3}d(\ , G) - \frac{2}{3}d(\ , G) - \frac{2}{3}d(\ , G) + \frac{1}{3}d(\ , G) + o(1) \geq 0$$

and so

$$d(\ , G) - \frac{1}{3}d(\ , G) - \frac{1}{3}d(\ , G) + o(1) \geq 0.$$

Dividing this equality by 2, we get (1.3), which is what we wanted to prove.

Summarizing, we started with some non-negative quadratic inequality on densities with some vertices fixed (in this example – inequality (1.2) with one vertex fixed). Then we changed multiplication of densities into densities of bigger graphs, averaged over all possible choices of fixed vertices, and we obtained wanted inequality of the form (1.2). The question is, how to get the starting non-negative quadratic inequality containing unknown coefficients and multiplications of densities. Such inequality can be considered as non-negativity of the product of unknown matrix of coefficients by vector of densities (from both sides). If we assume that the unknown matrix is positive semidefinite we will get non-negativity. Thus, we can consider the semidefinite programming problem – minimization of the upper bound with condition that matrix of variables is positive semidefinite. Of course, we can use more than one such inequality.

Flag algebras gives us a language to quickly do manipulations (like multiplication or averaging) on densities, like we did in the above example. The idea of the method is clear – we assume non-negativity of some inequalities on densities with some unknown variables and then we use semidefinite programming to find the best coefficients. These inequalities can be of any form – for example a quadratic one, like in the above example (this will be fully described in the Section 1.3), taken from Cauchy-Schwarz inequality, or from differentiating. More examples will be presented in the Section 1.4.

1.3. Semidefinite method

In this section we present one of the systematic approaches to Turán problems, but it can be used in many other types of extremal problems as well. It give us a computer algorithm to count or approximate densities of subgraphs.

We define a *type* σ to be an $\mathcal{F}$-free graph on s vertices ($s \geq 0$) together with a bijective labeling function $\theta : [s] = \{1, 2, \dots, s\} \longrightarrow V(\sigma)$. Then we define a *$\sigma$-flag* F to be an $\mathcal{F}$-free graph containing an induced copy of σ labeled by θ. We define the order of the flag $|F|$ to be $|V(F)|$. In other words, if we have are given a family $\mathcal{F}$ and a type σ (a graph with all vertices labeled by consecutive numbers from 1 to s) a σ-flag of order m is just an $\mathcal{F}$-free graph on m vertices, which has s labeled vertices inducing σ.

Given a graph G, let us fix a type σ on s vertices, and integers $l > s$, and $m \leq (l+s)/2$. This bound on m ensures that a graph on l vertices can contain two subgraphs on m vertices overlapping in exactly s vertices. Let $\mathcal{F}_m^\sigma$ be the set of all σ-flags of order m, up to isomorphism. Let Θ be the set of all injections from $[s]$ to $V(G)$. Given $F \in \mathcal{F}_m^\sigma$ and $\theta \in \Theta$, we define *induced density* of a flag $d(F, G; \theta)$ to be the probability that an m-element set V' chosen uniformly at random from $V(G)$ with $\mathrm{im}(\theta) \subseteq V'$, induces a σ-flag that is isomorphic to F. When $s = 0$, that is, if θ is the empty mapping and F is a usual graph, definition of $d(F, G; \theta)$ coincides with original definition of density $d(F, G)$.

If $F_a, F_b \in \mathcal{F}_m^\sigma$ and $\theta \in \Theta$, we can define $d(F_a, F_b, G; \theta)$ to be the probability that if we choose a random m-element set $V_a \subset V(G)$ with $\mathrm{im}(\theta) \subset V_a$ (so we are choosing only $m - s$ elements), and then we choose a random m-element set $V_b \subset V(G)$ such that $\mathrm{im}(\theta) = V_a \cap V_b$, then induced σ-flags are isomorphic to F_a and F_b, respectively. There is a difference between $d(F_a, F_b, G; \theta)$ and the product $d(F_a, G; \theta)d(F_b, G; \theta)$, because we assume that we cannot choose the same vertices during the second choice. But, when G is large, this probability is negligible, as the following lemma tells us.

LEMMA 1.1 (Razborov [**67**]). *For any F_a, $F_b \in \mathcal{F}_m^\sigma$ and $\theta \in \Theta$,*

$$d(F_a, G; \theta)d(F_b, G; \theta) = d(F_a, F_b, G; \theta) + o(1),$$

where the o(1) term tends to 0 as $|V(G)|$ tends to infinity.

Let us consider σ-flags $F_i \in \mathcal{F}_m^\sigma$. Assign some real coefficients a_i to these flags, and for fixed $\theta : [s] \longrightarrow V(G)$ consider the inequality

$$\left(\sum_{F_i \in \mathcal{F}_m^\sigma} a_i d(F_i, G; \theta) \right)^2 \geq 0.$$

Expanding the square we have

$$\sum_{F_i, F_j \in \mathcal{F}_m^\sigma} a_i a_j d(F_i, G; \theta) d(F_j, G; \theta) \geq 0.$$

We can also consider coefficients q_{ij} instead of products $a_i a_j$. If we assume that the matrix $Q = (q_{ij})_{F_i, F_j \in \mathcal{F}_m^\sigma}$ is positive semidefinite, then we will get the inequality

$$\sum_{F_i, F_j \in \mathcal{F}_m^\sigma} q_{ij} d(F_i, G; \theta) d(F_j, G; \theta) \geq 0.$$

Using the above lemma we obtain

$$\sum_{F_i, F_j \in \mathcal{F}_m^\sigma} q_{ij} d(F_i, F_j, G; \theta) + o(1) \geq 0.$$

Now, we average over a uniformly random choice of $\theta \in \Theta$, and using the linearity of expectation we get

$$\sum_{F_i, F_j \in \mathcal{F}_m^\sigma} q_{ij} \mathbb{E}_{\theta \in \Theta}[d(F_i, F_j, G; \theta)] + o(1) \geq 0.$$

This expectation can be computed by averaging over all l-vertex subgraphs of G. Let us denote Θ_H as the set of all injective mappings $\theta : [s] \longrightarrow V(H)$ and recall that $\mathcal{F}_l^0$ is the family of all $\mathcal{F}$-free graphs on l vertices, up to isomorphism. Thus, we have

$$\mathbb{E}_{\theta \in \Theta}[d(F_a, F_b, G; \theta)] = \sum_{H \in \mathcal{F}_l^0} \mathbb{E}_{\theta \in \Theta_H}[d(F_a, F_b, H; \theta)] d(H, G).$$

Hence

$$\sum_{H \in \mathcal{F}_l^0} \sum_{F_i, F_j \in \mathcal{F}_m^\sigma} q_{ij} \mathbb{E}_{\theta \in \Theta_H}[d(F_a, F_b, H; \theta)] d(H, G) + o(1) \geq 0.$$

Defining $c_H(\sigma, m, Q) = \sum_{F_i, F_j \in \mathcal{F}_m^\sigma} q_{ij} \mathbb{E}_{\theta \in \Theta_H}[d(F_a, F_b, H; \theta)]$, we get

$$\sum_{H \in \mathcal{F}_l^0} c_H(\sigma, m, Q) d(H, G) + o(1) \geq 0,$$

which is exactly inequality of the form (1.2), which can be used to get better upper bound for Turán density.

Furthermore, we can consider t choices of (σ_i, m_i, Q_i), where each σ_i is a type, each $m_i \leq (l + |\sigma_i|)/2$ is an integer, and each Q_i is a positive semidefinite matrix of dimension $|\mathcal{F}_{m_i}^{\sigma_i}| \times |\mathcal{F}_{m_i}^{\sigma_i}|$. For $H \in \mathcal{F}_l^0$ define

$$c_H = \sum_{i=1}^{t} c_H(\sigma_i, m_i, Q_i).$$

Since each Q_i is positive semidefinite matrix, we will get

$$\sum_{H \in \mathcal{F}_l^0} c_H d(H, G) + o(1) \geq 0.$$

If we combine this with

$$d(A, G) = \sum_{H \in \mathcal{F}_l^0} d(A, H) d(H, G),$$

then we will get

$$d(A, G) \leq \sum_{H \in \mathcal{F}_l^0} (d(A, H) + c_H) d(H, G) + o(1).$$

Hence

$$\pi_A(\mathcal{F}) \leq \max_{H \in \mathcal{F}_l^0} (d(A, H) + c_H). \tag{1.4}$$

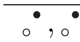
Since some of the c_H may be negative, for an appropriate choice of the (σ_i, m_i, Q_i), this bound may be significantly better than the averaging bound given before. Note that we now have a semidefinite programming problem: given any particular choice of the (σ_i, m_i) find positive semidefinite matrices Q_i which minimize the bound for $\pi_A(\mathcal{F})$ given by (1.4).

Summarizing, the method amounts to the following algorithm. If we want to bound $\pi_A(\mathcal{F})$, we should:

- pick some (not very large) l;
- determine $\mathcal{F}_l^0$ – a set of all $\mathcal{F}$-free graphs on l vertices;
- for each $H \in \mathcal{F}_l^0$ compute $d(A, H)$;
- pick some σ (a graph with labeled vertices);
- pick some integer $m \leq (l + |\sigma|)/2$;
- determine $\mathcal{F}_m^\sigma$ – a set of all $\mathcal{F}$-free σ-flags F_i (graphs containing labeled σ, other vertices are not labeled) on m vertices;
- compute $\mathbb{E}_{\theta \in \Theta_H}[d(F_i, F_j, H; \theta)]$ for each $H \in \mathcal{F}_l^0$;
- determine functions $c_H = \sum_{F_i, F_j \in \mathcal{F}_m^\sigma} q_{ij} \mathbb{E}_{\theta \in \Theta_H}[d(F_i, F_j, H; \theta)]$ over variables q_{ij} forming a matrix Q of dimension $|\mathcal{F}_m^\sigma| \times |\mathcal{F}_m^\sigma|$;
- minimize $\max_{H \in \mathcal{F}_l^0}(d(A, H) + c_H)$ using semidefinite programming assuming that Q is a positive semidefinite matrix.

This minimum is the bound for $\pi_A(\mathcal{F})$. To get a better bound we can take many triples (σ_i, m_i, Q_i), and for the functions c_H take the sums of the functions $c_H(\sigma_i, m_i, Q_i)$.

As an example, we can consider once more Mantel's theorem. We take $l = 3$. There are three triangle-free graphs on 3 vertices – , , and . Edge densities of them are equal to 0, 1/3, and 2/3, respectively. We will take type σ consisting of one vertex labeled by 1 and $m = 2$. There are two σ-flags of size 2 –  and  (unfilled vertex is labeled by 1). Required expectations are in the following table.

			
	1	1/3	0
	0	2/3	2/3
	0	0	1/3

For an example of this calculation we will calculate the middle column. In graph  we can place the label in three possible ways. There are only 2 vertices left, so when we are choosing at random two different vertices, we will just pick these two. And so, in one of the possibilities of placing the label, we will have pair of two isomorphic flags . In the remaining two possibilities, we will get pair  and . There is no possibility to get pair of two graphs , so there is 0 in the bottom row.

Now, we take variables q_{11}, q_{12}, q_{22} forming symmetric matrix Q. We need to minimize the expression

$$\max\left(q_{11}, \frac{1}{3} + \frac{1}{3}q_{11} + \frac{2}{3}q_{12}, \frac{2}{3} + \frac{2}{3}q_{12} + \frac{1}{3}q_{22}\right),$$

where Q is positive semidefinite. It can be easily seen that it is minimized when

$$Q = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}$$

and the minimum is exactly 1/2, what we wanted to prove.

The same reasoning was used in [**7**] for some hypergraph Turán problem, and in [**35**] to prove that the maximum number of C_5's in a triangle-free graph on n vertices is equal to $(n/5)^5$, which was conjectured by Erdős. This method was also used in many other papers, but with some modifications – it will be more explained in the next section.

1.4. The algebra

In this section, we will present formal definitions of flag algebra, operations in the algebra, and also give some useful methods using flag algebras.

In the previous section we defined and used only a type and a flag associated with $\mathcal{F}$-free graphs for some forbidden family $\mathcal{F}$. Similar definitions can be made in many other cases, for example for directed graphs or for arbitrary hypergraphs. In general, it can be defined in any universal theory in a first-order language without constants or function symbols. For more model theoretical approach see [**71**]. Here, we will present this theory for graphs only (for simplicity), but it is worth stressing that everything can be generalized. In the Chapter 3 we will present an application of this method for other structures.

The aim when applying flag algebras to extremal problems is to generate useful inequalities of the form $\sum_{F_i \in \mathcal{F}_l^\sigma} b_i d(F_i, G) + o(1) \geq 0$, valid for every graph $G \in \mathcal{F}^\sigma$ (in particular, when $s = 0$, we have inequalities for graphs). We can consider $\sum b_i d(F_i, G)$ in a little bit different way. Let us take these coefficients b_i and consider formal sum $\sum b_i F_i$ in $\mathbb{R}\mathcal{F}^\sigma$, which is the space of all formal finite linear combinations of σ-flags. We can think of any graph $G \in \mathcal{F}^\sigma$ as acting on $\mathbb{R}\mathcal{F}^\sigma$ via the mapping $\sum b_i F_i \longrightarrow \sum b_i d(F_i, G)$. So, let us identify such a mapping with G. We can think of every $\mathcal{F}$-free graph as an appropriate mapping on $\mathbb{R}\mathcal{F}^\sigma$. Notice that, by equality $d(\tilde{F}, G) = \sum_{F \in \mathcal{F}_l^\sigma} d(\tilde{F}, F) d(F, G)$, the linear combination $\tilde{F} - \sum_{F \in \mathcal{F}_l^\sigma} d(\tilde{F}, F) F$ is mapped to zero by G. So, we should factor it out. Let $\mathcal{K}^\sigma$ be a linear subspace of $\mathbb{R}\mathcal{F}^\sigma$ generated by all elements of the form

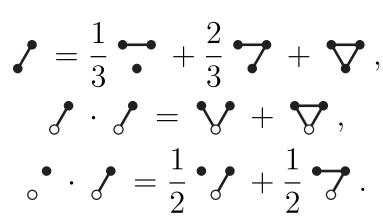
$$\tilde{F} - \sum_{F \in \mathcal{F}_l^\sigma} d(\tilde{F}, F) F, \tag{1.5}$$

where $\tilde{F} \in \mathcal{F}_{\tilde{l}}^\sigma$ and $s \leq \tilde{l} \leq l$. Let $\mathcal{A}^\sigma = \mathbb{R}\mathcal{F}^\sigma / \mathcal{K}^\sigma$ be the quotient space. The only remaining task to create an algebra is to define a multiplication operation. For any $F_1 \in \mathcal{F}_{l_1}^\sigma$ and $F_2 \in \mathcal{F}_{l_2}^\sigma$ we choose arbitrary $l \geq l_1 + l_2 - s$, set

$$F_1 \cdot F_2 = \sum_{F \in \mathcal{F}_l^\sigma} d(F_1, F_2, F) F,$$

and expand it by linearity. It can be proved (see [**67**]) that it is well-defined in $\mathcal{A}^\sigma$ (not in $\mathbb{R}\mathcal{F}^\sigma$), it doesn't depend on the choice of l and gives the structure of a commutative algebra in $\mathcal{A}^\sigma$. We know that $d(F_1, F_2, G) = d(F_1, G) d(F_2, G) + o(1)$, so when $|V(G)|$ is large the mapping G is 'approximate homomorphism' from $\mathcal{A}^\sigma$ to $\mathbb{R}$.

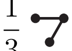
To get better understanding of these definitions, we present some self-explaining examples (as usual, unfilled circle represent the labeled vertex):



When we are proving something, we are often showing that some inequality holds for particular vertices (for example vertices connected by an edge). And then we are averaging over all possible choices of vertices to get inequality valid for densities, not dependent on any particular vertices. In flag algebras formalism this operation is described as linear operator from $\mathcal{A}^\sigma$ to $\mathcal{A}^0$, which are unlabeled graphs. We define averaging operator of some $F \in \mathcal{F}^\sigma$ to be

$$[\![F]\!] = q_\sigma(F) \cdot F',$$

where F' is an unlabeled version of F, and $q_\sigma(F)$ is the probability that an injective mapping $\theta : [s] \longrightarrow V(F')$ (chosen uniformly at random) defines an induced embedding of σ in F', with the resulting σ-flag isomorphic to F. Next, we extend the operator $[\![\cdot]\!]$ from $\mathcal{F}^\sigma$ to $\mathcal{A}^\sigma$ by linearity. Some self-explaining examples are displayed below:

$[\![$  $]\!] =$ , $\quad [\![$  $]\!] = \frac{1}{3}$ .

Another example – let D be a path P_3 on 3 edges with first vertex labeled by 1 and adjacent vertex labeled by 2. Then $[\![D]\!] = \frac{1}{6}P_3$, because we can choose an ordered pair of vertices in 12 ways and only 2 of them give a flag isomorphic to D.

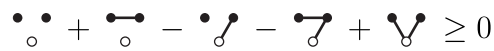
Now, the crucial point of the flag algebras theory goes as follows. With a given σ-flag G we identify the mapping $\mathbb{R}\mathcal{F}^\sigma \longrightarrow \mathbb{R}$ defined before. We can also identify this mapping with infinite vector of densities $(d(F,G))_{F \in \mathcal{F}^\sigma} \in [0,1]^{\mathcal{F}^\sigma}$; and vice versa. The space $[0,1]^{\mathcal{F}^\sigma}$ is compact in the product topology, so any sequence contains a convergent subsequence. Let $\mathrm{Hom}^+(\mathcal{A}^\sigma, \mathbb{R})$ be the set of all homomorphisms ϕ from $\mathcal{A}^\sigma$ to $\mathbb{R}$ such that $\phi(F) \geq 0$ for every $F \in \mathcal{F}^\sigma$. As we noticed, any σ-flag can be identified with such positive homomorphism. It can be proved (see [**67**]) that for any convergent sequence of σ-flags in $\mathcal{F}^\sigma$ the limit is in $\mathrm{Hom}^+(\mathcal{A}^\sigma, \mathbb{R})$; conversely, any element of $\mathrm{Hom}^+(\mathcal{A}^\sigma, \mathbb{R})$ is the limit of some sequence of σ-flags. This result gives us a correspondence between the final world inequalities $\sum_{F_i \in \mathcal{F}_l^\sigma} b_i d(F_i, G) + o(1) \geq 0$ for σ-flags G (in particular unlabeled graphs) and inequalities $\phi(\sum_{F_i \in \mathcal{F}_l^\sigma} b_i F_i) \geq 0$ for $\phi \in \mathrm{Hom}^+(\mathcal{A}^\sigma, \mathbb{R})$. In particular, the Turán density $\pi_A(\mathcal{F}) = \limsup_{G \in \mathcal{F}^0} d(A, G)$ can be rewritten as $\pi_A(\mathcal{F}) = \max_{\phi \in \mathrm{Hom}^+(\mathcal{A}^0, \mathbb{R})} \phi(A)$. From compactness, this maximum is achieved by some extremal homomorphism. In other words, positive homomorphisms are precisely those corresponding to the limits of convergent graph sequences. We will write $F \geq 0$ for σ-flag $F \in \mathcal{A}^\sigma$ if $\phi(F) \geq 0$ for any positive homomorphism ϕ. One can think of the value of $\phi([\![F]\!])$ as the expected value of $\phi(F)$, thus, if $\phi(F) \geq 0$ with probability one, then $\phi([\![F]\!]) \geq 0$.

Let us see how it works on an example. We will prove Mantel's theorem exactly in the same way as in the Section 1.2, but in the language of flag algebras. We start from the relation

$($  $-$  $)^2 \geq 0$

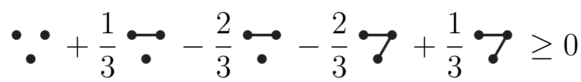
and so

Applying the definition of multiplication, we obtain

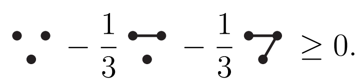


The same holds after using averaging operator. It is linear, so

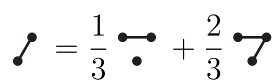
This, after applying the definition of averaging operator, gives



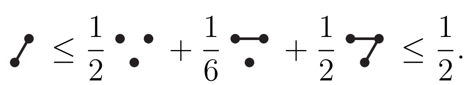
and so



Dividing the last inequality by 2 and adding to equation



(valid in flag algebra of triangle-free graphs) we get



This means that edge density of any triangle-free graph G is at most $1/2 + o(1)$, so $\pi(K_3) \leq 1/2$.

The big 'source' for inequalities on flag algebras are the Cauchy-Schwarz inequalities.

THEOREM 1.1 (Razborov [**67**]). *For any f, $g \in \mathcal{A}^\sigma$*

$$[\![f^2]\!] \cdot [\![g^2]\!] \geq [\![fg]\!]^2.$$

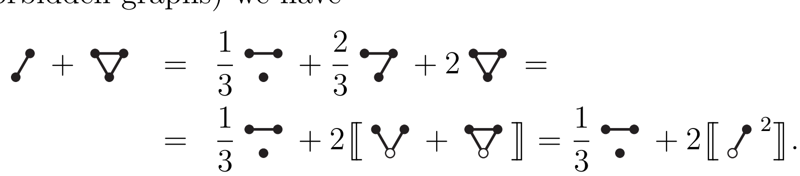
In particular

$$[\![f^2]\!] \cdot [\![\sigma]\!] \geq [\![f]\!]^2,$$

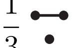
which implies

$$[\![f^2]\!] \geq 0.$$

As an easy example, we will very quickly show Mantel's theorem one more time. Let us notice that in the flag algebra of graphs (without any forbidden graphs) we have



From the theorem $[\![$  $^2]\!] \geq [\![$  $]\!]^2 =$  ^{2}, so

 $+$  $\geq \frac{1}{3}$  $+ 2$  $^2 \geq 2$  ^{2}.

This means that  ≥ 2  $^2 -$ . In particular, if the edge density is greater than $1/2$, then the triangle density is greater than 0, and therefore, there must be at least one triangle.

We get it more quickly than before, and it is also a stronger result. Of course, it is possible to translate this proof to 'normal' language, and instead of using operations on flags, to make some counting arguments. But, in practice, many proofs based on flag algebra use so many inequalities that after translation, it would be unreadable.

Moreover, using flag algebras we have a systematic approach to Turán problems. To bound some $\pi_A(\mathcal{F})$ we can choose some type σ (or few different types), pick some not very large l and make a computer program to generate all possible Cauchy-Schwarz inequalities for flags up to l vertices. Then expand each inequality and, using averaging operator, express them as linear inequalities on graphs on l vertices. After that, formulate a semidefinite programming problem to calculate numerically the best bound. If we are lucky, we can get good bound. After rationalization our proof is numerically stable. This method was used in many papers to obtain new results (see for example [**39, 75**]).

Flag algebras have also been used to obtain results in a different setting than Turán problems. For example to get new results on the Caccetta-Häggkvist conjecture for triangles (see [**42, 71**]), where we have assumption about minimal outdegree, in extremal problems in a colored setting [**6, 22, 38, 50**], or on the problem of selecting heavily covered points [**51**]. More applications can be found in a recent survey of Razborov [**68**].

It is worth mentioning that flag algebras permit 'differential methods' (see [**67**]). Maximum value of $\pi_A(\mathcal{F})$ is achieved for some extremal positive homomorphism ϕ, so any small perturbation of ϕ must reduce $\phi(A)$. Perturbation with respect to a single vertex is analogous to some deletion arguments, but general perturbation do not have any obvious analogue in final setting. That is why flag algebras may turn out to be very powerful tool for a wider collections of problems in combinatorics.

CHAPTER 2

The Erdős problem on pentagons in triangle-free graphs

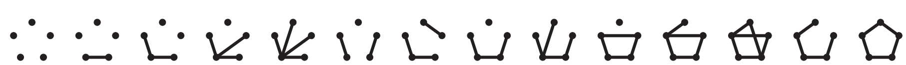
In [**26**] Erdős conjectured that the number of cycles of length 5 in a triangle-free graph of order n is at most $(n/5)^5$. Moreover, he suspected that this bound is attained in the case when n is divisible by 5 only by the blow-up of a pentagon (i.e., five sets of $n/5$ independent vertices; vertices from different sets are connected according to the edges in pentagon). Győri [**36**] showed that a triangle-free graph of order n contains no more than $c\left(\frac{n+1}{5}\right)^5$ pentagons, where $c = \frac{16875}{16384} < 1.03$. Recently, this bound has been further improved by Füredi (personal communication). In this chapter we present a proof of the Erdős conjecture using flag algebras. The result has been published in [**35**] and independently in [**39**].

According to the notation from the previous chapter we are interested in finding the value of $\mathrm{ex}_{C_5}(n, K_3)$, where K_3 states for a triangle and C_5 for a pentagon. As usual, our first task is to achieve the asymptotic value, i.e., the Turán density $\pi_{C_5}(K_3)$.

Theorem 2.1. $\pi_{C_5}(K_3) \leq \frac{24}{625}$.

Proof. We use the algorithm described in the Section 1.3.

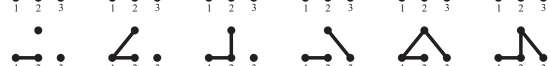
Let us consider family $\mathcal{H}$ of all triangle-free graphs on $l = 5$ vertices, up to isomorphism:



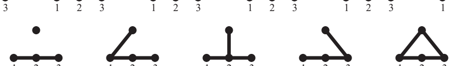
and three types on 3 vertices – σ_0 stands for a graph with no edges, the type σ_1 has one edge and σ_2 has two. In each case, we consider $m = 4$. There are 8 admissible σ_0-flags (the corresponding variables to these flags form the matrix P, say), 6 admissible σ_1-flags (we denote the corresponding matrix by Q) and 5 admissible σ_2-flags (matrix R).

σ_0-flags:

σ_1-flags:



σ_2-flags:



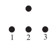
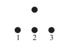
According to the described algorithm, now we need to find for each $H \in \mathcal{H}$ the functions $c_H(\sigma_0, 4, P)$, $c_H(\sigma_1, 4, Q)$, and $c_H(\sigma_2, 4, R)$ by computing all appearances of each pair of flags (on 4 vertices with 3 labeled ones) in each graph H. And then use the inequality

$$\pi_{C_5}(K_3) \leq \max_{H \in \mathcal{H}} (d(C_5, H) + c_H(\sigma_0, 4, P) + c_H(\sigma_1, 4, Q) + c_H(\sigma_2, 4, R)).$$

After all required calculations, we obtain

$$\begin{aligned}
\pi_{C_5}(K_3) \leq \frac{1}{120} \max\{&120p_{11}, 12p_{11} + 24p_{12} + 24p_{13} + 24p_{15} + 12q_{11},\\
&8p_{12} + 8p_{13} + 8p_{14} + 8p_{15} + 8p_{16} + 8p_{17} + 4p_{22} + 4p_{33} + 4p_{55} +\\
&\quad + 8q_{12} + 8q_{13} + 4r_{11},\\
&12p_{14} + 12p_{16} + 12p_{17} + 12p_{18} + 6q_{22} + 6q_{33} + 12r_{13},\\
&48p_{18} + 24r_{33},\\
&16p_{23} + 16p_{25} + 16p_{35} + 8q_{11} + 16q_{14},\\
&8p_{27} + 8p_{36} + 8p_{45} + 8q_{14} + 8q_{24} + 8q_{34} + 4q_{44} + 4r_{11},\\
&4p_{23} + 4p_{24} + 4p_{25} + 4p_{26} + 4p_{34} + 4p_{35} + 4p_{37} + 4p_{56} + 4p_{57} +\\
&\quad + 4q_{12} + 4q_{13} + 4q_{15} + 4q_{16} + 4q_{23} + 4r_{12} + 4r_{14},\\
&4p_{27} + 4p_{28} + 4p_{36} + 4p_{38} + 4p_{45} + 4p_{58} + 4q_{15} + 4q_{16} + 4q_{25} +\\
&\quad + 4q_{36} + 4r_{13} + 2r_{22} + 4r_{23} + 4r_{34} + 2r_{44},\\
&8p_{44} + 8p_{66} + 8p_{77} + 16q_{23} + 16r_{15},\\
&4p_{48} + 4p_{68} + 4p_{78} + 4q_{26} + 4q_{35} + 2q_{55} + 2q_{66} + 4r_{15} + 4r_{23} +\\
&\quad + 4r_{25} + 4r_{34} + 4r_{35} + 4r_{45},\\
&12p_{88} + 24r_{35} + 12r_{55},\\
&4p_{46} + 4p_{47} + 4p_{67} + 4q_{24} + 4q_{26} + 4q_{34} + 4q_{35} + 4q_{45} + 4q_{46} +\\
&\quad + 4r_{12} + 4r_{14} + 4r_{24},\\
&20q_{56} + 20r_{24} + 120\},
\end{aligned}$$

where the maximum is taken over all possible coefficients p_{ij}, q_{ij}, r_{ij} such that all of the respective matrices P, Q, and R are positive semi-definite.

For an explanation of the calculations, we will consider one example – graph , and count appearances of each pair of flags in it. We need to consider all 120 possibilities of placing 5 labels – three for a type and one additional for each vertex left in the two flags. Let us consider all those possibilities by looking which labels are connected by an edge. If labels 1 and 2 are connected by an edge (there are 12 such possibilities), we always get two flags , yielding coefficient $12q_{11}$. If labels 1 and 3 or 2 and 3 are connected by an edge, we get pairs of flags which are not under our consideration (we do not consider such type). If labels 1 and 4 or 1 and 5 are connected by an edge, we get flag  and flag , yielding coefficient $24p_{12}$. Similarly, we get coefficients $24p_{13}$ (2 and 4 or 2 and 5 are connected) and $24p_{15}$ (3 and 4 or 3 and 5 are connected). The only possibilities left are those when 4 and 5 are connected by an edge. In those situations we get two flags , yielding coefficient $12p_{11}$. Hence, we get

$$c_{\cdot\cdot\cdot} = \frac{1}{120}(12p_{11} + 24p_{12} + 24p_{13} + 24p_{15} + 12q_{11}).$$

Now, we can run the semidefinite programming, to obtain matrices P, Q, and R giving us the best bound.

Here, we present one of the choices of the matrices, which give us the required bound. Let us take P, Q, and R to be the matrices

$$P = \frac{1}{625}\begin{pmatrix} 24 & -36 & -36 & 24 & -36 & 24 & 24 & -36 \\ -36 & 277 & 97 & -79 & 97 & -79 & -259 & 54 \\ -36 & 97 & 277 & -79 & 97 & -259 & -79 & 54 \\ 24 & -79 & -79 & 247 & -259 & 67 & 67 & -36 \\ -36 & 97 & 97 & -259 & 277 & -79 & -79 & 54 \\ 24 & -79 & -259 & 67 & -79 & 247 & 67 & -36 \\ 24 & -259 & -79 & 67 & -79 & 67 & 247 & -36 \\ -36 & 54 & 54 & -36 & 54 & -36 & -36 & 54 \end{pmatrix},$$

$$Q = \frac{1}{2500}\begin{pmatrix} 1728 & -1551 & -1551 & -1308 & 687 & 687 \\ -1551 & 2336 & 742 & 908 & 2557 & -4084 \\ -1551 & 742 & 2336 & 908 & -4084 & 2557 \\ -1308 & 908 & 908 & 1728 & -254 & -254 \\ 687 & 2557 & -4084 & -254 & 15264 & -14424 \\ 687 & -4084 & 2557 & -254 & -14424 & 15264 \end{pmatrix},$$

$$R = \frac{1}{625}\begin{pmatrix} 1512 & 568 & -380 & 568 & -376 \\ 568 & 475 & -191 & 0 & -93 \\ -380 & -191 & 192 & -191 & -2 \\ 568 & 0 & -191 & 475 & -93 \\ -376 & -93 & -2 & -93 & 190 \end{pmatrix}.$$

It can be checked by any program for mathematical calculations (e.g., Mathematica, Maple) that matrix P multiplied by 625 has characteristic polynomial

$$x^4(x-360)^2(x^2-930x+53766),$$

and so it has eigenvalues 0, 0, 0, 0, ≈ 62, 360, 360 and ≈ 868, matrix Q multiplied by 2500 has characteristic polynomial

$$x(x^2-31282x+3219791)(x^3-7374x^2+7536419x-324955440)$$

and eigenvalues 0, ≈ 45, ≈ 103, ≈ 1170, ≈ 6159, ≈ 31179, and matrix R multiplied by 625 has characteristic polynomial

$$-x^2(x-475)(x^2-2369x+492426)$$

and eigenvalues 0, 0, ≈ 230, 475 and ≈ 2139. Thus, P, Q, and R are all positive semidefinite.

Hence, for an upper bound on $\pi_{C_5}(K_3)$ we get

$$\begin{aligned}\pi_{C_5}(K_3) \quad &\leq \quad \max\left\{\frac{24}{625},\frac{24}{625},\frac{24}{625},\frac{24}{625},\frac{24}{625},\frac{24}{625},\frac{322}{9375},\frac{2355}{62500},\frac{24}{625},\right.\\ &\left.\frac{24}{625},\frac{24}{625},\frac{24}{625},-\frac{126}{6250},\frac{24}{625}\right\} = \frac{24}{625}.\end{aligned}$$

□

The Erdős conjecture is a straightforward consequence of the above result.

Theorem 2.2. *The number of pentagons in a triangle-free graph of order n is at most $\left(\frac{n}{5}\right)^5$.*

Proof. Suppose that there is a triangle-free graph G on n vertices which has at least $\left(\frac{n}{5}\right)^5+\varepsilon$ cycles C_5, where $\varepsilon>0$. Then, we can construct triangle-free graphs G_{nN} consisting of n sets of N independent vertices and all edges between vertices in different sets according to the edges in G.

The graph G_{nN} has nN vertices and at least $\left(\left(\frac{n}{5}\right)^5+\varepsilon\right)N^5$ cycles C_5. Thus, the Turán density is at least

$$\pi_{C_5}(K_3) \geq \lim_{N\to\infty}\frac{\left(\frac{nN}{5}\right)^5+\varepsilon N^5}{\binom{nN}{5}} = \frac{24}{625}+\frac{120\varepsilon}{n^5} > \frac{24}{625},$$

which contradicts Theorem 2.1. □

Since multiplying a positive semidefinite matrix by a vector from both sides can be written as a sum of squares, instead of using the matrices, we could also just write the wanted inequality as sum of some non-negative terms. We actually use this other approach in the next section, but it is worth noting that the two are mathematically equivalent.

CHAPTER 3

Seymour second neighborhood conjecture

In this chapter we we introduce the notion of applying flag algebras to combinatorial structures other than graphs. Our general technique to deal with an open problem were the typical flag algebraic approaches do not work is to define and work with a special structure. The considered problem is so called Seymour second neighborhood conjecture. We will prove the conjecture for Eulerian graphs with an additive error.

In every oriented graph (directed simple graph) we define the *first neighborhood* of a vertex v as the set of all the vertices w to which we can go from v in one step, i.e., there is an edge (v, w). Its size we denote by $d^+(v)$ and call it outdegree of a vertex v. Similarly, we define indegree $d^-(v)$ as the number of vertices w having an edge (w, v). We define the *second neighborhood* of a vertex v as the set of all the vertices w to which we can go from v in two steps and not in one, i.e, they are not in the first neighborhood and there exists a vertex u and edges (v, u) and (u, w). Its size we denote by $d^{++}(v)$.

The main conjecture is

Conjecture 3.1 (Seymour second neighborhood conjecture). *In every oriented graph there exists a vertex v such that $d^{++}(v) \geq d^+(v)$.*

The special case when the considered graph is a tournament was conjectured by Dean and proved by Fisher [**29**] and later, using different method, by Havet and Thomassé[**40**]. Kaneko and Locke [**47**] proved this conjecture for oriented graphs having a vertex with outdegree at most 6. Godbole, Cohn, and Wright [**21**] have proved that the conjecture holds for almost all oriented graphs.

An interesting approach was introduced by Chen, Shen, and Yuster [**19**]. They proved the conjecture with a multiplicative error, i.e., there exist a vertex v such that $d^{++}(v) \geq ad^+(v)$, where a is the solution of $2x^3 + x^2 = 1$, which is ≈ 0.657.

Using presented in this chapter approach with flag algebras, it is possible to prove similar result, but with an additive error in the class of Eulerian graphs (oriented graph for which every vertex has the outdegree equal to the indegree).

THEOREM 3.1. *For every Eulerian graph on n vertices*

$$\sum_v (d^{++}(v) - d^{+}(v) + 0.075507n) \geq 0$$

In particular, there exists a vertex v such that

$$d^{++}(v) \geq d^{+}(v) - 0.075507n.$$

Since the purpose of this chapter is to present the method, not necessarily the best possible bound, for the sake of simplicity, we explicitly show the full proof only for the error $0.2n$. In order to achieve the bound $0.075507n$ more computations are required and it is impossible is to write them in the paper.

It is worth mentioning that Conjecture 3.1 implies a special case of the Caccetta-Häggkvist conjecture – every oriented graph on n vertices with minimal outdegree and indegree at least $n/3$, contains a directed triangle. Proving it with multiplicative or additive errors, will also provide a proof for the Caccetta-Häggkvist conjecture. Strictly speaking, proving the Seymour second neighborhood conjecture with multiplicative error a and additive error cn, will prove that every oriented graph with minimal outdegree and indegree at least $\frac{1+c}{2+a}n$, contains a directed triangle.

3.1. Flag algebra setting

We cannot directly use flag algebras to this problem, since it is impossible to express second neighborhood of a vertex using just densities of some subgraphs. Let us present different framework, proposed by Oleg Pikhurko, which made possible to use flag algebras in this case.

For every oriented graph one can construct a new structure – directed graph with two types of edges – by adding additional *dotted* edges representing second neighborhood, i.e., whenever w is in the second neighborhood of v we add dotted directed edge (v, w). There might be dotted edges (v, w) and (w, v). Also dotted edge (v, w) and edge (w, v) may appear.

We will now work in the world of this type of structures. Let us consider a set $\mathcal{G}$ of directed graphs with two types of edges – edges and dotted edges. We forbid loops, dotted loops, double edges, double dotted edges, edges making 2-cycles, edge (v, w) and dotted edge (v, w), and if there are edges (v, u), (u, w), and no edge (v, w), then there must be a dotted edge (v, w). The last condition give us the fact that every graph in $\mathcal{G}$ is obtained from an oriented graph by adding dotted edges according to the second neighborhood, and, possibly, some additional dotted edges. Since we are interested in proving lower bounds for the second neighborhood, additional dotted edges are not influencing the obtained bounds.

Now, the Conjecture 3.1 can be rewritten as follows – in every graph in $\mathcal{G}$ there exists a vertex with dotted outdegree at least as big as outdegree. It implies the conjecture for oriented graphs from the latter fact.

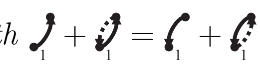
In $\mathcal{G}$ we can now use the flag algebras framework.

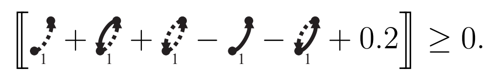
3.2. Main proof

In order to prove that for every Eulerian graph on n vertices

$$\sum_v (d^{++}(v) - d^{+}(v) + 0.2n) \geq 0$$

it is enough to show the following lemma.

LEMMA 3.1. *Every graph in $\mathcal{G}$ with*  *satisfies*

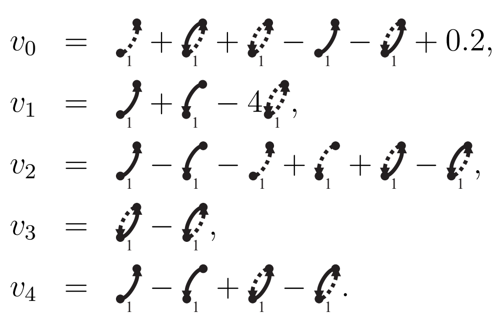


It is so, since existence of a counterexample, by a similar argument as in the end of the Chapter 2, provides a convergent sequence of counterexamples, and results with a counterexample to the above lemma.

PROOF. It is enough to show the inequality

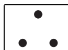
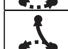
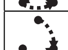
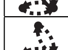
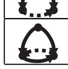
$$\left[\!\left[v_0 \right]\!\right] \geq \frac{3}{40}\left[\!\left[v_1^2 \right]\!\right] + \frac{24}{40}\left[\!\left[v_2^2 \right]\!\right] + \frac{37}{40}\left[\!\left[v_3^2 \right]\!\right] - \frac{69}{40}\left[\!\left[v_4^2 \right]\!\right], \tag{3.1}$$

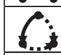
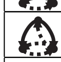
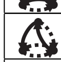
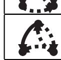
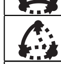
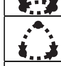
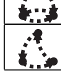
where

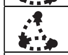
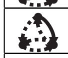
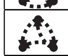
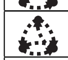
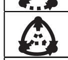
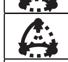
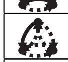
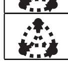


This inequality implies the lemma, since $v_4 = 0$ from the assumption, and the other right-hand side terms are non-negative.

We represent both sides of the inequality (3.1) in the space of graphs from $\mathcal{G}$ on 3 unlabeled vertices. There are 85 such graphs in $\mathcal{G}$. All components from the inequality (3.1) multiplied by 120 are presented in the following table. The difference between the left-hand side and the right-hand side of the inequality is also shown.

	$120\llbracket v_0 \rrbracket$	$9\llbracket v_1^2 \rrbracket$	$72\llbracket v_2^2 \rrbracket$	$114\llbracket v_3^2 \rrbracket$	$-207\llbracket v_4^2 \rrbracket$	diff.
	24	0	0	0	0	24
	4	0	0	0	0	4
	44	0	0	0	0	44
	24	0	0	0	0	24
	64	0	0	0	0	64
	−16	3	24	0	−69	26
	24	0	−24	0	0	48
	24	0	24	0	0	0
	4	0	24	0	−69	49
	44	−12	0	0	0	56
	−16	3	24	0	−69	26
	24	0	24	0	0	0
	24	0	−24	0	0	48
	4	0	24	0	−69	49
	44	−12	0	0	0	56
	64	0	24	0	0	40
	64	0	−24	0	0	88
	44	0	−24	0	0	68
	44	0	24	0	0	20
	84	0	0	0	0	84
	64	0	24	0	0	40
	44	0	24	0	0	20
	44	0	−24	0	0	68
	84	0	0	0	0	84
	24	0	24	38	−69	31
	64	0	0	0	0	64
	24	0	24	38	−69	31
	64	0	0	0	0	64
	104	48	0	0	0	56
	−36	9	24	0	−69	0
	4	3	24	0	−69	46
	−16	3	24	0	−69	26

	24	−21	24	0	−69	90
	4	3	−72	0	69	4
	−16	3	24	0	−69	26
	24	−21	−24	0	69	0
	4	3	24	0	−69	46
	44	0	24	0	0	20
	44	0	−72	0	0	116
	24	0	−72	0	69	27
	24	0	24	0	−69	69
	64	−12	−24	0	0	100
	44	0	24	0	0	20
	44	0	24	0	0	20
	24	0	24	0	−69	69
	64	−12	24	0	0	52
	−16	3	24	0	−69	26
	24	0	24	0	−69	69
	24	0	24	0	−69	69
	4	0	24	38	−69	11
	4	0	24	−38	−69	87
	44	−12	24	0	−69	101
	24	0	−72	0	69	27
	4	0	24	38	−69	11
	44	−12	−24	0	69	11
	24	−21	24	0	−69	90
	64	−12	24	0	0	52
	64	−12	−24	0	0	100
	44	−12	−24	0	69	11
	44	−12	24	0	−69	101
	84	24	0	0	0	60
	84	0	24	0	0	60
	64	0	24	0	0	40
	104	0	24	0	0	80
	84	0	−72	0	0	156

	64	0	24	0	0	40
	64	0	−72	0	0	136
	104	0	−24	0	0	128
	64	0	24	0	0	40
	44	0	24	38	−69	51
	44	0	−72	−38	69	85
	84	0	−24	0	0	108
	44	0	24	38	−69	51
	84	0	24	0	0	60
	104	0	24	0	0	80
	84	0	24	0	0	60
	84	0	−24	0	0	108
	124	48	0	0	0	76
	24	0	24	38	−69	31
	64	0	24	38	−69	71
	24	0	−72	−114	207	3
	64	0	−24	−38	69	57
	64	0	24	38	−69	71
	104	48	0	0	0	56
	144	144	0	0	0	0

For every graph the difference is non-negative, thus the inequality 3.1 holds. □

Remark 3.1. *Using the same method, but considering graphs in* $\mathcal{G}$ *on 4 vertices (there are 7101 of them), we can improve the constant* 0.2 *in the Lemma 3.1 to approx. 0.075507, but the number of calculations would be too big to write in a paper.*

CHAPTER 4

Finitely forcible graphons and permutons

The results of this chapter has been obtained in a joint paper [**30**]. We focus on a question when the limit analytic object is uniquely determined by finitely many densities of substructures. We consider this property, known as finite forcibility, for permutons (permutation limits) and the related graphons (graph limits) via permutation graphs. In fact, questions of this kind are closely related to quasirandomness and they were studied well before the theory of limits of combinatorial objects emerged. For example, the classical result of Chung, Graham and Wilson [**20**] says that the homomorphic densities of K_2 and C_4 guarantee that densities of all subgraphs behave as in the random graph $G_{n,1/2}$. In the language of graphons, this result asserts that the graphon identically equal to $1/2$ is finitely forcible by densities of 4-vertex subgraphs. A similar result on permutations, which was originally raised as a question by Graham, was proven by Král' and Pikhurko [**52**] who exploited the analytic view of permutation limits.

Let us now give motivation for the concepts. The result of Chung, Graham, and Wilson [**20**] on finite forcibility of the graphon identically equal to $1/2$ was generalized by Lovász and Sós [**55**] who proved that any stepwise graphon is finitely forcible. These results were further extended by Lovász and Szegedy [**56**] who also gave several conditions when a graphon is not finitely forcible.

We start this chapter with proving an analogue of the result of Lovász and Sós [**55**] for permutons, which is stated as Corollary 4.1. We then focus on finite forcibility of permutons with infinite recursive structure, and on the interplay between finite forcibility of permutons and graphons, partly motivated by Question 11 from [**56**].

A graph can be associated with a permutation in the following way: the vertices of the graph correspond to the elements of the permutation and two of them are joined by an edge if they form an inversion. Along the same lines, a graphon can be associated with a permuton. We show that, surprisingly, there exist finitely forcible permutons such that the associated graphons are not finitely forcible. We proved that permutons with infinite recursive structures – union of monotone permutations (Section 4.3) and union of random permutations (Section 4.4) are finitely forcible. Then, we show that the associated graphons are not finitely forcible in Sections 4.5 and 4.6, where we also prove the stronger statement, that all graphons with infinite recursive structure are not finitely forcible. Let us also remark that the methods we use in this chapter were subsequently extended by Glebov, Král', and Volec [**33**] to resolve Conjecture 9 from [**56**] on compactness of finitely forcible graphons.

4.1. Notation

In this section, we introduce concepts related to graphs and permutations and their limits used in the sequel. We start with the slightly simpler notion of permutation limits.

4.1.1. Permutations and permutons. The theory of permutation limits was built by Hoppen, Kohayakawa, Moreira, Ráth, and Sampaio in [**43, 44**]. Here, we follow the analytic view of the limit as used in [**52**], which also appeared in an earlier work of Presutti and Stromquist [**65**].

A *permutation of order* n is a bijective mapping from $[n]$ to $[n]$, where $[n]$ denotes the set of integers from 1 to n. The order of a permutation π is also denoted by $|\pi|$. The set of all permutations of order n is denoted by S_n. In what follows, we identify a sequence of n different integers $a_1 \dots a_n$ between 1 and n with a permutation π by setting $\pi(i) = a_i$. For example, the identity permutation of order 4 is denoted by 1234.

If π is a permutation of order n, a *subpermutation* induced by $1 \le i_1 < \dots < i_k \le n$ in π is a permutation σ of order k such that $\sigma(j) < \sigma(j')$ if and only if $\pi(i_j) < \pi(i_{j'})$. For example, the subpermutation of 7126354 induced by $3, 4, 6$ is 132. Subpermutations are more commonly referred to as *patterns* but we decided to use the term subpermutation in the paper to be consistent with the analogous concept for graphs as well as with previous work on permutation limits. A *density* $d(\sigma, \pi)$ of a permutation σ of order k in a permutation π of order n is the number of k-tuples inducing σ in π divided by $\binom{n}{k}$. Conveniently, we set $d(\sigma, \pi) = 0$ if $k > n$.

An infinite sequence $(\pi_i)_{i\in\mathbb{N}}$ of permutations with $|\pi_i| \to \infty$ is *convergent* if $d(\sigma, \pi_i)$ converges for every permutation σ. We see that one can associate with every convergent sequence of permutations the following analytic object: a *permuton* is a probability measure μ on the σ-algebra $\mathring{A}$ of Borel sets of the unit square $[0, 1]^2$ such that μ has *uniform marginals*, i.e., $\mu([\alpha,\beta]\times[0,1]) = \mu([0,1]\times[\alpha,\beta]) = \beta-\alpha$ for every $0 \le \alpha \le \beta \le 1$. In what follows, we use λ for the uniform measure on $\mathring{A}$. More generally, if $A \subseteq [0,1]^2$ is a non-trivial convex polygon, i.e., a convex polygon different from a point (however, which can be a segment), we define λ_A to be the unique probability measure on $\mathring{A}$ with support A and mass uniformly distributed inside A. In particular, $\lambda_{[0,1]^2} = \lambda$.

We now describe the relation between permutons and convergent sequences of permutations. Let μ be a permuton. For an integer n, one can sample n points $(x_1,y_1),\ldots,(x_n,y_n)$ in $[0,1]^2$ randomly based on μ. Because μ has uniform marginals, the x-coordinates of all these points are mutually different with probability one. The same holds for their y-coordinates. Assume that this is indeed the case. One can then define a permutation π of order n based on the n points $(x_1,y_1),\ldots,(x_n,y_n)$ as follows: let $i_1,\ldots,i_n \in [n]$ be such that $x_{i_1} < x_{i_2} < \cdots < x_{i_n}$ and define π to be the unique bijective mapping from $[n]$ to $[n]$ satisfying that $\pi(j) < \pi(j')$ if and only if $y_{i_j} < y_{i_{j'}}$. We say that a permutation π of order n obtained in the just described way is a *μ-random permutation* of order n. A *uniformly random permutation* is a λ-random permutation, i.e., each permutation of order n is chosen with probability $1/n!$ at random.

If μ is a permuton and σ is a permutation of order n, then $d(\sigma,\mu)$ is the probability that a μ-random permutation of order n is σ. We now recall the core results from [**43, 44**]. For every convergent sequence $(\pi_i)_{i\in\mathbb{N}}$ of permutations, there exists a unique permuton μ such that

$$d(\sigma,\mu) = \lim_{i\to\infty} d(\sigma,\pi_i) \text{ for every permutation } \sigma.$$

This permuton is the *limit* of the sequence $(\pi_i)_{i\in\mathbb{N}}$. On the other hand, if μ is a permuton and π_i is a μ-random permutation of order i, then with probability one the sequence $(\pi_i)_{i\in\mathbb{N}}$ is convergent and μ is its limit.

We now give four examples of the just defined notions (the corresponding permutons are depicted in Figure 1). Let us consider a sequence $(\pi_i^1)_{i\in\mathbb{N}}$ such that π_i^1 is the identity permutation of order i, i.e., $\pi_i^1(k) = k$ for $k \in [i]$. This sequence is convergent and its limit is the measure λ_A where $A = \{(x,x), x\in[0,1]\}$. Similarly, the limit of a sequence $(\pi_i^2)_{i\in\mathbb{N}}$, where π_i^2 is the permutation of order i defined as $\pi_i^2(k) = i+1-k$ for $k\in[i]$, is λ_B where $B = \{(x,1-x), x\in[0,1]\}$. A little bit more complicated example is the following: the sequence

$(\pi_i^3)_{i\in\mathbb{N}}$, where π_i^3 is the permutation of order $2i$ defined as

$$\pi_i^3(k) = \begin{cases} 2k-1 & \text{if } k \in [i], \\ 2(k-i) & \text{otherwise} \end{cases}$$

is convergent and its limit is the measure $\frac{1}{2}\lambda_C + \frac{1}{2}\lambda_D$, where $C = \{(x/2, x), x \in [0,1]\}$ and $D = \{((x+1)/2, x), x \in [0,1]\}$. Next, consider a sequence $(\pi_i^4)_{i\in\mathbb{N}}$ such that π_i^4 is a random permutation. This sequence is convergent with probability one and its limit is the measure λ.

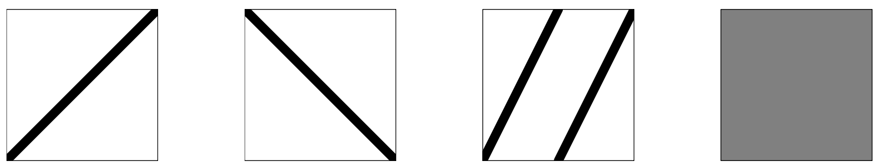


FIGURE 1. The limits of sequences $(\pi_i^1)_{i\in\mathbb{N}}$, $(\pi_i^2)_{i\in\mathbb{N}}$, $(\pi_i^3)_{i\in\mathbb{N}}$ and $(\pi_i^4)_{i\in\mathbb{N}}$.

A permuton μ is *finitely forcible* if there exists a finite set S of permutations such that every permuton μ' satisfying $d(\sigma,\mu) = d(\sigma,\mu')$ for every $\sigma \in S$ is equal to μ. For example, the following result from [**52**] asserts that the random permuton is finitely forcible with $S = S_4$.

THEOREM 4.1. *Let μ be a permuton. It holds that $d(\sigma,\mu) = 1/24$ for every $\sigma \in S_4$ if and only if $\mu = \lambda$.*

If μ is a permuton, then F_μ is the function from $[0,1]^2$ to $[0,1]$ defined as $F_\mu(x,y) = \mu([0,x]\times[0,y])$. For example, if $\mu = \lambda$, then $F_\mu(x,y) = xy$. Observe that F_μ is always a continuous function satisfying $F_\mu(\xi,1) = F_\mu(1,\xi) = \xi$ for every $\xi \in [0,1]$. Furthermore, notice that $\mu \neq \mu'$ implies $F_\mu \neq F_{\mu'}$, that is, the function F_μ determines the permuton μ.

The next theorem was implicitly proven in [**52**]. We include its proof for completeness.

THEOREM 4.2. *Let $p(x,y)$ be a polynomial and k a non-negative integer. There exist a finite set S of permutations and coefficients γ_σ, $\sigma \in S$, such that*

$$\int_{[0,1]^2} p(x,y) F_\mu^k(x,y)\,\mathrm{d}\lambda = \sum_{\sigma\in S} \gamma_\sigma d(\sigma,\mu) \tag{4.1}$$

for every permuton μ.

PROOF. By additivity, it suffices to consider the case $p(x,y)=x^\alpha y^\beta$ for non-negative integers α and β. Fix a permuton μ. Since μ has uniform marginals, the product $x^\alpha y^\beta F_\mu^k(x,y)$ for $(x,y)\in[0,1]^2$ is equal to the probability that out of $\alpha+\beta+k$ points are chosen randomly independently based on μ, the first α points belong to $[0,x]\times[0,1]$, the next β points belong to $[0,1]\times[0,y]$, and the last k points belong to $[0,x]\times[0,y]$. So, the integral in (4.1) is equal to the probability that the above holds for a uniform choice of a point (x,y) in $[0,1]^2$.

Since μ is a measure with uniform marginals, a point (x,y) uniformly distributed in $[0,1]^2$ can be obtained by sampling two points randomly independently based on μ and setting x to be the first coordinate of the first of these two points and y to be the second coordinate of the second point. Thus, we can consider the following random event. Let us choose $\alpha+\beta+k+2$ points independently at random based on μ and denote by x the first coordinate of the last but one point, and by y is the second coordinate of the last point. Then the integral on the left hand side of (4.1) is equal to the probability that the first α points belong to $[0,x]\times[0,1]$, the next β points belong to $[0,1]\times[0,y]$, and the following k points belong to $[0,x]\times[0,y]$. We conclude that the equation (4.1) holds with $S=S_{\alpha+\beta+k+2}$ and γ_σ equal to the probability that the following holds for a random permutation π of order $\alpha+\beta+k+2$: $\pi(i)\le\pi(\alpha+\beta+k+1)$ for $i\le\alpha$ and for $\alpha+\beta+1\le i\le\alpha+\beta+k$, and $\sigma(\pi(i))\le\sigma(\pi(\alpha+\beta+k+2))$ for $\alpha+1\le i\le\alpha+\beta+k$. □

Instead of sampling two additional points to get a random point with respect to the uniform measure λ, we can also sample just a single point, which is a random point with respect to μ. This gives the following.

THEOREM 4.3. *Let $p(x,y)$ be a polynomial and k a non-negative integer. There exist a finite set S of permutations and coefficients γ_σ, $\sigma\in S$, such that*

$$\int_{[0,1]^2} p(x,y)F_\mu^k(x,y)\mathrm{d}\mu=\sum_{\sigma\in S}\gamma_\sigma d(\sigma,\mu) \tag{4.2}$$

for every permuton μ.

Let now $\overline{S}_k$ be the set of permutations of order k with one distinguished element; we call such permutations *rooted*. To denote rooted permutations, we add a bar above the distinguished element: e.g., if the second element of the permutation 2341 is distinguished, we write $2\overline{3}41$. Note that $\left|\overline{S}_k\right|=k!\cdot k$. If $\sigma\in\overline{S}_k$, then $F_\mu^\sigma(x,y)$ is the probability that the point (x,y) and $k-1$ points randomly independently chosen based on μ induce the permutation σ with the distinguished element corresponding to the point (x, y). Observe that $F_\mu(x,y) = F_\mu^{1\bar{2}}(x,y)$, $F_\mu^{1\bar{2}}(x,y) + F_\mu^{2\bar{1}}(x,y) = x$ and $F_\mu^{1\bar{2}}(x,y) + F_\mu^{\bar{2}1}(x,y) = y$. This notation comes from the notion of 1-labeled flags introduced in Chapter 1.

Similarly to Theorem 4.3, the following is true. We omit the proof as it is completely analogous to that of Theorem 4.2.

THEOREM 4.4. *Let Σ be a multiset of rooted permutations. There exist a finite set S of permutations and coefficients γ_σ, $\sigma \in S$, such that*

$$\int_{[0,1]^2} \prod_{\sigma \in \Sigma} F_\mu^\sigma(x,y) \mathrm{d}\mu = \sum_{\sigma \in S} \gamma_\sigma d(\sigma, \mu) \tag{4.3}$$

for every permuton μ.

4.1.2. Graphs and graphons. The other limit structure we consider is limits of graphs. If G and G' are graphs, then by $G \cup G'$ we denote the disjoint union of G and G' and by $G + G'$ the graph obtained from $G \cup G'$ by adding all edges between G and G'. If G is a graph and U is a subset of its vertices, then let $G \setminus U$ be the graph obtained from G by removing the vertices of U and all edges containing at least one vertex from U. We recall that the density $d(H, G)$ of H in G is the probability that $|H|$ randomly chosen vertices of G induce a subgraph isomorphic to H. If $|H| > |G|$, we set $d(H, G) = 0$.

We now survey basic results related to the theory of dense graph limits as developed in [**15, 16, 17, 57**]. A sequence of graphs $(G_i)_{i\in\mathbb{N}}$ is *convergent* if the limit $d(H, G_i)$ exists for every H. The associated limit object is called a *graphon*: it is a symmetric λ-measurable function from $[0, 1]^2$ to $[0, 1]$. Here, symmetric stands for the property that $W(x, y) = W(y, x)$ for every $x, y \in [0, 1]$. If W is a graphon, then a W*-random graph* of order k is obtained by sampling k random points $x_1, \ldots, x_k \in [0, 1]$ uniformly and independently and joining the i-th and the j-th vertex by an edge with probability $W(x_i, x_j)$. As in the case of permutations, we write $d(H, W)$ for the probability that a W-random graph of order $|H|$ is isomorphic to H. For every convergent sequence $(G_i)_{i\in\mathbb{N}}$ of graphs, there exists a graphon W such that $d(H, W) = \lim_{i\to\infty} d(H, G_i)$ for every graph H. We call such a graphon W a *limit* of $(G_i)_{i\in\mathbb{N}}$. On the other hand, if W is a graphon, then with probability one the sequence $(G_i)_{i\in\mathbb{N}}$ where G_i is a W-random graph of order i is convergent and its limit is W.

Unlike in the case of permutations, the limit of a convergent sequence of graphs is not unique. For example, if W is a limit of $(G_i)_{i\in\mathbb{N}}$ and $\varphi : [0, 1] \to [0, 1]$ is a measure preserving transformation, then the graphon $W' := W(\varphi(x), \varphi(y))$ is also a limit of $(G_i)_{i\in\mathbb{N}}$. Let us introduce the following definition of equivalence of graphons: two graphons W and W' are *weakly isomorphic* if $d(H, W) = d(H, W')$ for every graph H.

Finally, a graphon W is called *finitely forcible* if there exist graphs $H_1, \ldots, H_k$ such that any graphon W' satisfying $d(H_i, W) = d(H_i, W')$ for $i \in [k]$ is weakly isomorphic to W.

The densities of graphs in a graphon W can be expressed as integrals using W. If W is a graphon and H is a graph of order k with vertices $v_1, \ldots, v_k$ and edge set E, then

$$d(H,W) = \frac{k!}{|\mathrm{Aut}(H)|} \int\limits_{[0,1]^k} \prod_{v_iv_j\in E} W(x_i,x_j) \prod_{v_iv_j\notin E} (1-W(x_i,x_j))\mathrm{d}x_1 \ldots \mathrm{d}x_k$$

where $\mathrm{Aut}(H)$ is the automorphism group of H.

A permutation π of order k can be associated with a graph G_π of order k as follows. The vertices of G_π are the integers between 1 and k and ij is an edge of G if and only if either $i < j$ and $\pi(i) > \pi(j)$, or $i > j$ and $\pi(i) < \pi(j)$. If $(\pi_i)_{i\in\mathbb{N}}$ is a convergent sequence of permutations, then the sequence of graphs $(G_{\pi_i})_{i\in\mathbb{N}}$ is also convergent. Moreover, if two convergent sequences of permutations have the same limit, then the graphons associated with the two corresponding (convergent) sequences of graphs are weakly isomorphic. In this way, we may associate each permuton μ with a graphon W_μ, which is unique up to a weak isomorphism (see Figure 2 for examples).

FIGURE 2. The graphons associated with the first three permutons depicted in Figure 1, where the point (0,0) is in the lower left corner.

4.2. Permutons with finite structure

In this section, we give a sufficient condition on a permuton to be finitely forcible. A function $f : [0,1]^2 \to \mathbb{R}$ is called piecewise polynomial if there exist finitely many polynomials $p_1, \ldots, p_k$ such that $f(x, y) \in \{p_1(x, y), \ldots, p_k(x, y)\}$ for every $(x, y) \in [0, 1]^2$.

THEOREM 4.5. *Every permuton μ such that F_μ is piecewise polynomial is finitely forcible.*

PROOF. Let μ be a permuton such that F_μ is piecewise polynomial, that is, there exist polynomials $p_1, \dots, p_k$ such that $F_\mu(x,y) \in \{p_1(x,y), \dots, p_k(x,y)\}$ for every $(x,y) \in [0,1]^2$. Let $\mathcal{F}$ be the set of all continuous functions f on $[0,1]^2$ such that for every $(x,y) \in [0,1]^2$ $f(x,y) \in \{p_1(x,y), \dots, p_k(x,y)\}$. The set $\mathcal{F}$ is finite. Indeed, let

$$q(x,y) = \prod_{1\le i<j\le k} (p_j(x,y) - p_i(x,y))$$

and let Q be the set of all points $(x,y) \in \mathbb{R}^2$ such that $q(x,y) = 0$. By Harnack's curve theorem, the set Q has finitely many connected components. Bézout's theorem implies that the number of branching points in each of these components is finite and these points have finite degrees. Consequently, $\mathbb{R}^2 \setminus Q$ has finitely many components. If $A_1, \dots, A_\ell$ are all the connected components of $[0,1]^2 \setminus Q$, then each function $f \in \mathcal{F}$ coincides with one of the k polynomials $p_1, \dots, p_k$ on every A_i. So, $|\mathcal{F}| \le k^\ell$.

Observe that the function $F_\mu(x,y)$ is continuous since the measure μ has uniform marginals. By the Stone-Weierstrass theorem, there exist a polynomial $p(x,y)$ and $\varepsilon > 0$ such that

$$\int_{[0,1]^2} (F_\mu(x,y) - p(x,y))^2 \,\mathrm{d}\lambda \;<\; \varepsilon \text{ , and} \tag{4.4}$$

$$\int_{[0,1]^2} (f(x,y) - p(x,y))^2 \,\mathrm{d}\lambda \;>\; \varepsilon \text{ for every } f \in \mathcal{F},\ f \ne F_\mu. \tag{4.5}$$

Let ε_0 be the value of the left hand side of (4.4). We claim that the unique permuton μ' satisfying

$$\int_{[0,1]^2} \prod_{i=1}^{k} (F_{\mu'}(x,y) - p_i(x,y))^2 \,\mathrm{d}\lambda \;=\; 0 \text{ , and} \tag{4.6}$$

$$\int_{[0,1]^2} (F_{\mu'}(x,y) - p(x,y))^2 \,\mathrm{d}\lambda \;=\; \varepsilon_0 \tag{4.7}$$

is μ. Assume that μ' is a permuton satisfying both (4.6) and (4.7). The equation (4.6) implies that $F_{\mu'} \in \mathcal{F}$. Next, (4.5), (4.7), and (4.4) yield that $F_{\mu'} \ne f$ for every $f \in \mathcal{F}$, $f \ne F_\mu$. We conclude that $F_{\mu'} = F_\mu$ and thus $\mu' = \mu$.

By Theorem 4.2, the left hand sides of (4.6) and (4.7) can be expressed as finite linear combinations of densities $d(\sigma, \mu)$. Let S be the set of all permutations appearing in these linear combinations. Any permuton μ' with $d(\sigma, \mu') = d(\sigma, \mu)$ for every $\sigma \in S$ satisfies both (4.6) and (4.7) and thus it must be equal to μ. This shows that μ is finitely forcible. □

We immediately obtain the following corollary.

COROLLARY 4.1. *If μ is a permuton such that there exist non-negative reals $\alpha_1, \dots, \alpha_k$ and non-trivial polygons $A_1, \dots, A_k \subseteq [0,1]^2$ satisfying $\mu = \sum_{i=1}^{k} \alpha_i \lambda_{A_i}$, then μ is finitely forcible.*

PROOF. Let F_i, $i \in [k]$, be the function from $[0,1]^2$ to $[0,1]$ defined as $F_i(x,y) = \lambda_{A_i}([0,x] \times [0,y])$. Clearly, each function F_i is piecewise polynomial. Since $F_\mu = \sum_{i=1}^{k} \alpha_i F_i$, the finite forcibility of μ follows from Theorem 4.5. □

A particular case of permutons that are finitely forcible by Corollary 4.1 is the following. If k is an integer, $z_1, \dots, z_k \in [0,1]$ are reals such that $z_1 + \cdots + z_k = 1$ and M is a square matrix of order k with entries being non-negative reals summing to z_i in the i-th row and in the i-th column, we can define a permuton μ_M to be the sum

$$\mu_M = \sum_{i,j=1}^{k} M_{ij} \mu_{A_{ij}} \ ,$$

where $A_{ij} = [s_{i-1}, s_i] \times [s_{j-1}, s_j]$, $i,j \in [k]$ and $s_i = z_1 + \cdots + z_i$ (in particular, $s_0 = 0$ and $s_k = 1$). For instance, if $z_1 = z_2 = z_3 = 1/3$ and

$$M = \begin{pmatrix} 0 & 0 & 1/3 \\ 2/9 & 1/9 & 0 \\ 1/9 & 2/9 & 0 \end{pmatrix} ,$$

we get the permuton depicted in Figure 3.

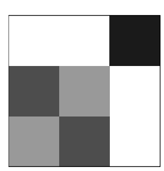


FIGURE 3. The permuton μ_M constructed as an example at the end of Section 4.2. The gray area in the picture is the support of the measure and different shades correspond to the density of the measure.

4.3. Union of monotone permutations

In this section we show that permutons with infinite recursive structure – permutons related with union of monotone permutations are finitely forcible. Such permutons appears often in extremal problems.

For $\alpha \in (0,1)$, define μ^m_α to be the permuton

$$\mu^m_\alpha = \sum_{i=1}^{\infty} (1-\alpha)\alpha^{i-1} \lambda_{I(1-\alpha^{i-1},1-\alpha^i)} \ ,$$

where $I(z,z') = \{(x, z'+z-x), x \in [z,z']\}$. Examples of the just defined permutons can be found in Figure 4. We next show that all permutons μ^m_α are finitely forcible.

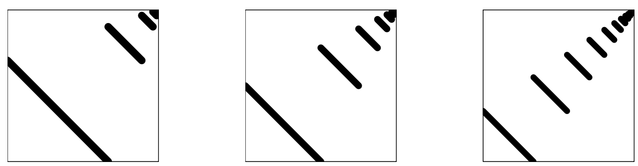


FIGURE 4. The permutons $\mu^m_{1/3}$, $\mu^m_{1/2}$, and $\mu^m_{2/3}$.

THEOREM 4.6. *For every $\alpha \in (0,1)$, the permuton μ^m_α is finitely forcible.*

PROOF. We claim that any permuton μ satisfying

$$d(231,\mu) + d(312,\mu) = 0 \ , \tag{4.8}$$

$$d(21,\mu) = (1-\alpha)^2 \sum_{i=0}^{\infty} \alpha^{2i} \ , \text{ and} \tag{4.9}$$

$$\int_{[0,1]^2} \left(1 - x - y + F_\mu(x,y) - \tfrac{\alpha}{1-\alpha}\left(x + y - 2F_\mu(x,y)\right)\right)^2 \mathrm{d}\mu = 0 \tag{4.10}$$

is equal to μ^m_α. This would prove the finite forcibility of μ^m_α by Theorem 4.3. Note that the permuton μ^m_α satisfies (4.8), (4.9), and (4.10).

Assume that a permuton μ satisfies (4.8), (4.9), and (4.10). Let X be the support of μ and consider the binary relation R defined on the support of μ such that $(x,y)R(x',y')$ if

- $x = x'$ and $y = y'$, or
- $x < x'$ and $y > y'$, or
- $x > x'$ and $y < y'$.

The relation R is an equivalence relation. Indeed, the reflexivity and symmetry is clear. To prove transitivity, consider three points (x,y), (x',y') and (x'',y'') such that $(x,y)R(x',y')$ and $(x',y')R(x'',y'')$

but it does not hold that $(x,y)R(x'',y'')$. By the definition of R, either $x < x'$ and $x'' < x'$, or $x > x'$ and $x'' > x'$. If $x < x'$ and $x'' < x'$, then we obtain that $d(231,\mu) > 0$ unless $x = x''$ (recall that R is defined on the support of μ). We can now assume that $x = x''$ and $y < y''$. Since μ has uniform marginals, the support of μ intersects at least one of the open rectangles $(0,x)\times(y,y'')$, $(x,x')\times(y,y'')$ and $(x',1)\times(y,y'')$. However, this yields that $d(231,\mu) > 0$ in the first two cases and $d(312,\mu) > 0$ in the last case. The case $x > x'$ and $x'' > x'$ is handled in an analogous way.

Let $\mathcal{R}$ be the set of equivalence classes of R. If $A \in \mathcal{R}$, let A_x and A_y be the projections of A on the x and y axes. It is not hard to show that A_x is a closed interval for each $A \in \mathcal{R}$ and these intervals are internally disjoint for different choices of $A \in \mathcal{R}$. The same holds for the projections on the y axis. Since μ has uniform marginals, the intervals A_x and A_y must have the same length for every $A \in \mathcal{R}$. Moreover, the definition of R implies that if A_x precedes A'_x, then A_y also precedes A'_y for any $A, A' \in \mathcal{R}$. We conclude that there exists a set $\mathcal{I}$ of internally disjoint closed intervals such that

$$\bigcup_{[z,z']\in\mathcal{I}} [z,z'] = [0,1] \text{ and}$$

the support of μ is equal to (because the density of subpermutations 231 and 312 is zero)

$$\bigcup_{[z,z']\in\mathcal{I}} \{(x, z'-x+z), x \in [z,z']\} \ .$$

Note that some intervals contained in $\mathcal{I}$ may be formed by single points. Let $\mathcal{I}_0$ be the subset of $\mathcal{I}$ containing the intervals of positive length.

Let $[z,z'] \in \mathcal{I}_0$ and let $I = \{(x, z'-x+z), x \in [z,z']\}$. Since $\mu([0,x]\times[0,y]) = \mu([0,z]\times[0,z])$ and the measure μ has uniform marginals, it follows that $F_\mu(x,y) = z$. The equality (4.10) implies that the (continuous) function integrated in (4.10) is zero for every $(x,y) \in I$. Substituting $x + y = z + z'$ and $F_\mu(x,y) = z$ into this function implies

$$z' = z + (1-\alpha)(1-z) \ . \tag{4.11}$$

Let Z be the set formed by the left end points of intervals in $\mathcal{I}_0$. Define z_1 to be the minimum elements of Z, and in general z_i to be the minimum element of $Z \setminus \bigcup_{j<i}\{z_j\}$. The existence of these elements follows from (4.11) and the fact that the intervals in $\mathcal{I}_0$ are internally disjoint. If Z is finite, we set $z_k = 1$ for $k > |Z|$. We derive from the definition of Z and from (4.11) that

$$\mathcal{I}_0 = \left\{[z_i, z_i + (1-\alpha)(1-z_i)],\ i \in \mathbb{N}^+\right\} \setminus \{[1,1]\} \ .$$

Consequently, we obtain

$$d(21,\mu) = \sum_{i=1}^{\infty}(1-\alpha)^2(1-z_i)^2 = (1-\alpha)^2\sum_{i=1}^{\infty}(1-z_i)^2 \ . \tag{4.12}$$

For $j \in \mathbb{N}$, we define $\beta_j \in [0,1]$ as follows:

$$\beta_j = \begin{cases} 1-z_1 & \text{for } j=0, \\ \frac{1-z_j}{\alpha(1-z_{j-1})} & \text{if } z_j \neq 1 \text{ and } j>0, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

The equation (4.12) can now be rewritten as

$$d(21,\mu) = (1-\alpha)^2\sum_{i=1}^{\infty}\alpha^{2(i-1)}\prod_{j=1}^{i}\beta_j^2 \ . \tag{4.13}$$

Hence, the equality (4.9) can hold only if $\beta_j = 1$ for every j which implies that $z_i = 1-\alpha^{i-1}$. Consequently, the permutons μ and μ^m_α are identical. □

4.4. Union of random permutations

We now show finite forcibility of another class of permutons with infinite recursive structure – permutons related with union of random permutations. Its structure is similar to that of μ^m_α. The proof proceeds along similar lines as the proof of Theorem 4.6 but we have to overcome several new technical difficulties.

For $\alpha \in (0,1)$, define μ^r_α to be the permuton

$$\mu^r_\alpha = \sum_{i=1}^{\infty}(1-\alpha)\alpha^{i-1}\lambda_{[1-\alpha^{i-1},1-\alpha^i]\times[1-\alpha^{i-1},1-\alpha^i]} \ .$$

See Figure 5 for examples. Our goal is to show that all permutons μ^r_α are finitely forcible.

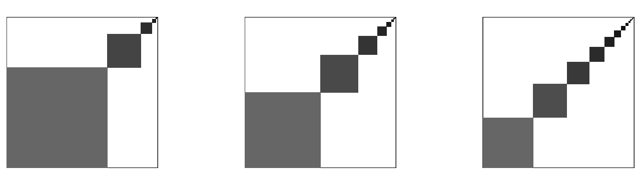


FIGURE 5. The permutons $\mu^r_{1/3}$, $\mu^r_{1/2}$, and $\mu^r_{2/3}$.

We start by proving an auxiliary lemma.

LEMMA 4.1. *There exist a finite set S of permutations and reals γ_σ, $\sigma \in S$, such that the following is equivalent for every permuton μ:*

- $\sum_{\sigma\in S} \gamma_\sigma d(\sigma,\mu) = 0$,
- *μ restricted to $[x_1,x_2]\times[y_2,y_1]$ is a (possibly zero) multiple of $\lambda_{[x_1,x_2]\times[y_2,y_1]}$ for any two points (x_1,y_1) and (x_2,y_2) of the support of μ with $x_1 < x_2$ and $y_1 > y_2$.*

PROOF. The proof technique is similar to that used in [**52**]. Let $\tilde\lambda_A(X) = \lambda(X\cap A)$, i.e., $\tilde\lambda_A(X) = \lambda(A)\cdot\lambda_A(X)$. Let (x_1,y_1) and (x_2,y_2) be two points of the support of μ with $x_1 < x_2$ and $y_1 > y_2$. By Cauchy-Schwartz inequality, the measure μ restricted $[x_1,x_2]\times[y_2,y_1]$ is a multiple of $\tilde\lambda_{[x_1,x_2]\times[y_2,y_1]}$ if and only if it holds that

$$\left(\int\limits_{(x,y)} \mu([x_1,x]\times[y_2,y])^2 \,\mathrm{d}\tilde\lambda_{[x_1,x_2]\times[y_2,y_1]}\right)\times \left(\int\limits_{(x,y)} (x-x_1)^2(y-y_2)^2 \,\mathrm{d}\tilde\lambda_{[x_1,x_2]\times[y_2,y_1]}\right) - \left(\int\limits_{(x,y)} (x-x_1)(y-y_2)\mu([x_1,x]\times[y_2,y]) \,\mathrm{d}\tilde\lambda_{[x_1,x_2]\times[y_2,y_1]}\right)^2 = 0\,. \tag{4.14}$$

Since the left hand side of (4.14) cannot be negative, we obtain that the second statement in the lemma is equivalent to

$$\int\limits_{(x_1,y_1)}\int\limits_{(x_2,y_2)}\int\limits_{(x,y)}\int\limits_{(x',y')} (x'-x_1)^2(y'-y_2)^2\cdot\mu\left([x_1,x]\times[y_2,y]\right)^2 - (x-x_1)(y-y_2)\cdot\mu\left([x_1,x]\times[y_2,y]\right)\cdot(x'-x_1)(y'-y_2)\cdot \mu([x_1,y_2]\times[x',y'])\,\mathrm{d}\tilde\lambda_{[x_1,x_2]\times[y_2,y_1]}\,\mathrm{d}\tilde\lambda_{[x_1,x_2]\times[y_2,y_1]}\,\mathrm{d}\mu\,\mathrm{d}\mu \;=\; 0\,. \tag{4.15}$$

In the rest of the proof, we show that the left hand side of (4.15) can be expressed as a linear combination of finitely many subpermutation densities. Since this argument follows the lines of the proofs of Theorems 4.2–4.4, we only briefly explain the main steps.

The left hand side of (4.15) is equal to the expected value of the integrated function in (4.15) for two points (x_1,y_1) and (x_2,y_2) randomly chosen in $[0,1]^2$ based on μ and two points (x,y) and (x',y') randomly chosen in $[0,1]^2$ based on λ when treating the value of the integrated function to be zero if $x_1 \geq x_2$, $y_1 \geq y_2$, $x \notin [x_1, x_2]$, $x' \notin [x_1, x_2]$, $y \notin [y_1, y_2]$, or $y' \notin [y_1, y_2]$. Such points (x_1, y_1), (x_2, y_2), (x, y), and (x', y') can be obtained by sampling six random points from $[0,1]^2$ based on μ since μ has uniform marginals (see the proof of Theorem 4.2 for more details). When the four points (x_1, y_1), (x_2, y_2), (x, y), and (x', y') are sampled, any of the quantities x_1, y_2, x, y, x', y', $\mu([x_1, y_2] \times [x, y])$, and $\mu([x_1, y_2] \times [x', y'])$ appearing in the product is equal to the probability that a point randomly chosen in $[0,1]^2$ based on μ has a certain property in a permutation determined by the sampled points. Since we need to sample six additional points to be able to determine each of the products appearing in (4.14), the left hand side of (4.14) is equal to a linear combination of densities of 12-element permutations with appropriate coefficients. We conclude that the lemma holds with $S = S_{12}$. $\square$

Analogously, one can prove the following lemma. Since the proof follows the lines of the proof of Lemma 4.1, we omit further details.

LEMMA 4.2. *There exist a finite set S of permutations and reals γ_σ, $\sigma \in S$ such that the following is equivalent for every permuton μ:*

- $\sum_{\sigma \in S} \gamma_\sigma d(\sigma, \mu) = 0$,
- *if (x_1, y_1), (x_2, y_2), and (x_3, y_3) are three points of the support of μ with $x_1 < x_2 < x_3$ and $y_2 < y_3 < y_1$, then μ restricted $[x_2, x_3] \times [y_2, y_3]$ is a (possibly zero) multiple of $\lambda_{[x_2,x_3]\times[y_2,y_3]}$.*

We are now ready to show that each permuton μ^r_α, $\alpha \in (0,1)$, is finitely forcible.

THEOREM 4.7. *For every $\alpha \in (0,1)$, the permuton μ^r_α is finitely forcible.*

PROOF. Let S_0 be the union of the two sets of permutations from Lemmas 4.1 and 4.2. Next, consider the following eight functions:

$$\begin{array}{rclrcl}
F^{\nwarrow}_\mu(x,y) & = & F^{2\bar{1}}_\mu(x,y)\,, & f^{\nwarrow}_\mu(x,y) & = & F^{23\bar{1}}_\mu(x,y) + F^{32\bar{1}}_\mu(x,y)\,,\\
F^{\nearrow}_\mu(x,y) & = & F^{\bar{1}2}_\mu(x,y)\,, & f^{\nearrow}_\mu(x,y) & = & F^{\bar{2}31}_\mu(x,y)\,,\\
F^{\swarrow}_\mu(x,y) & = & F^{1\bar{2}}_\mu(x,y)\,, & f^{\swarrow}_\mu(x,y) & = & F^{31\bar{2}}_\mu(x,y)\,,\\
F^{\searrow}_\mu(x,y) & = & F^{\bar{2}1}_\mu(x,y)\,, & f^{\searrow}_\mu(x,y) & = & F^{\bar{3}12}_\mu(x,y) + F^{\bar{3}21}_\mu(x,y)\,.
\end{array}$$

To save space in what follows, we often omit parameters when no confusion can arise, e.g., we write $F^{\searrow}_\mu$ for the value $F^{\searrow}_\mu(x,y)$ if x and y are clear from the context.

We claim that any permuton satisfying the following three conditions is equal to μ^r_α:

$$d(\sigma, \mu) = d(\sigma, \mu^r_\alpha) \text{ for every } \sigma \in S_0, \tag{4.16}$$

$$\int\limits_{[0,1]^2} \Big((1-\alpha)\left(F_\mu^{\nearrow} f_\mu^{\searrow} - F_\mu^{\searrow} f_\mu^{\nearrow}\right) f_\mu^{\nwarrow} - \\ \alpha\big(F_\mu^{\nwarrow} f_\mu^{\nwarrow} f_\mu^{\searrow} + F_\mu^{\searrow} f_\mu^{\nwarrow} f_\mu^{\searrow} + F_\mu^{\nwarrow} f_\mu^{\swarrow} f_\mu^{\searrow} + F_\mu^{\searrow} f_\mu^{\nwarrow} f_\mu^{\nearrow}\big)\Big)^2 \,\mathrm{d}\mu = 0\ , \tag{4.17}$$

and

$$d(21,\mu) = \frac{(1-\alpha)^2}{2} \sum_{i=0}^{\infty} \alpha^{2i}\ . \tag{4.18}$$

This would prove the finite forcibility of μ_α^r by Theorem 4.4.

Suppose that a permuton μ satisfies (4.16), (4.17), and (4.18). Let X be the support of μ and consider the binary relation R defined on the support of μ such that $(x,y)R(x',y')$ if

- $x = x'$ and $y = y'$,
- $x < x'$ and $y > y'$, or
- $x > x'$ and $y < y'$.

Unlike in the proof of Theorem 4.6, the relation R need not be an equivalence relation. Instead, we consider the transitive closure R_0 of R and let $\mathcal{R}$ be the set of the equivalence classes of R_0.

We define $\rho((x,y),(x',y'))$, where (x,y) and (x',y') are two points of the support of μ such that $(x,y)R(x',y')$, as follows

$$\rho\left((x,y),(x',y')\right) = \begin{cases} \frac{\mu([x,x']\times[y',y])}{(x'-x)(y-y')} & \text{if } x < x' \text{ and } y > y', \\ \frac{\mu([x',x]\times[y,y'])}{(x-x')(y'-y)} & \text{if } x > x' \text{ and } y < y', \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Since μ satisfies (4.16), Lemma 4.1 implies that any three points (x,y), (x',y') and (x'',y'') of the support of μ such that $(x,y)R(x',y')$ and $(x',y')R(x'',y'')$ satisfy $\rho((x,y),(x',y')) = \rho((x',y'),(x'',y''))$. In particular, the quantity $\rho((x,y),(x',y'))$ is the same for all pairs of points (x,y) and (x',y') with $(x,y)R(x',y')$ lying in the same equivalence class of R_0. So, we may define $\rho(A)$ to be this common value for each equivalence class $A \in \mathcal{R}$ or for a closure of such class.

As in the proof of Theorem 4.6, we define A_x and A_y to be the projections of an equivalence class $A \in \mathcal{R}$ on the x and y axes. The definition of R yields that A_x and A_y are closed intervals for all $A \in \mathcal{R}$ and these intervals are internally disjoint for different choices of $A \in \mathcal{R}$. Since μ has uniform marginals, the intervals A_x and A_y must have the same length for every $A \in \mathcal{R}$. As in the proof of Theorem 4.6, we conclude that there exists a set $\mathcal{I}$ of internally disjoint closed intervals such that

$$\bigcup_{[z,z']\in\mathcal{I}} [z,z'] = [0,1] ,$$

the support of μ is a subset of

$$\bigcup_{[z,z']\in\mathcal{I}} [z,z'] \times [z,z'] ,$$

and the interior of each of these squares intersects at most one class $A \in \mathcal{R}$. Since some intervals contained in $\mathcal{I}$ may be formed by single points, we define $\mathcal{I}_0$ to be the subset of $\mathcal{I}$ containing the intervals of positive length.

Let $[z,z'] \in \mathcal{I}_0$ and let A be the closure of the corresponding equivalence class from $\mathcal{R}$. Let $f(x)$, $x \in [z,z']$, be the minimum y such that (x,y) belongs to A; similarly, $g(x)$ denotes the maximum such y.

Assume first that $\rho(A) > 0$. Since μ has uniform marginals, it must holds that $g(x) - f(x) = \rho(A)^{-1}$ for every $x \in (z,z')$. From (4.16) and Lemma 4.1 we see that the functions f and g are non-decreasing, and similarly (4.16) and Lemma 4.2 imply that f and g are non-increasing. We conclude that $A = ([z,z'] \times [z,z'])$ and $\rho(A) = (z'-z)^{-1}$.

Assume now that $\rho(A) = 0$. Lemma 4.2 and (4.16) imply that if $(x,y) \in (z,z') \times (z,z')$ belongs to the support of μ, then $\mu([x,z'] \times [z,y]) = 0$ (otherwise, $\rho(A) > 0$). But then (x,y) cannot be in relation R with another point of the support of μ. So, we conclude that the case $\rho(A) = 0$ cannot appear.

The just presented arguments show the support of the measure μ is equal to

$$\bigcup_{[z,z']\in\mathcal{I}} [z,z'] \times [z,z']$$

and the measure is uniformly distributed inside each square $[z,z'] \times [z,z']$, $[z,z'] \in \mathcal{I}_0$.

Let $[z,z']$ be one of the intervals from $\mathcal{I}_0$. Recall that we have argued that

$$\mu\big([0,z] \times [0,z] \cup [z,z'] \times [z,z'] \cup [z',1] \times [z',1]\big) = 1$$

and the measure μ is uniform inside the square $[z,z'] \times [z,z']$ (see Figure 6). By (4.17), the following holds for almost every point of the support of μ:

$$(4.19)\quad (1-\alpha)\left(F_\mu^{\nearrow} f_\mu^{\searrow} - F_\mu^{\searrow} f_\mu^{\nearrow}\right) f_\mu^{\nwarrow} = \\ \alpha\left(F_\mu^{\nwarrow} f_\mu^{\nwarrow} f_\mu^{\searrow} + F_\mu^{\searrow} f_\mu^{\nwarrow} f_\mu^{\searrow} + F_\mu^{\nwarrow} f_\mu^{\swarrow} f_\mu^{\searrow} + F_\mu^{\searrow} f_\mu^{\nwarrow} f_\mu^{\nearrow}\right) .$$

In particular, this holds for all points in $[z, z'] \times [z, z']$ since the functions appearing in (4.19) are continuous.

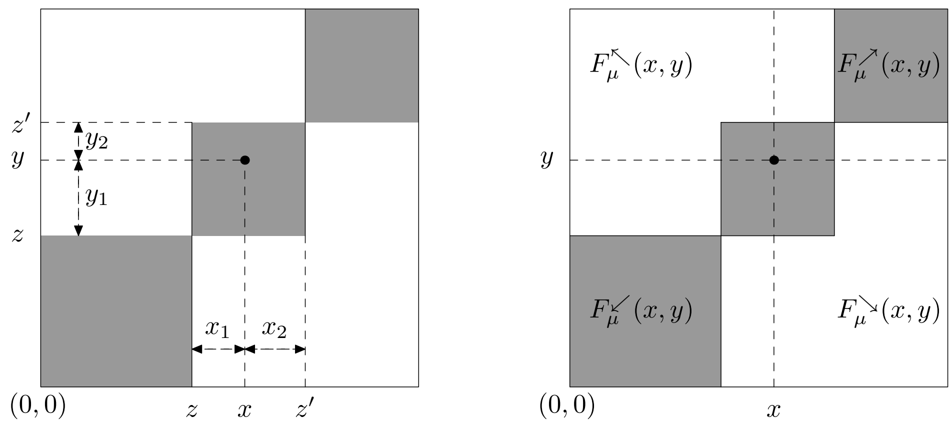


FIGURE 6. Notation used in equalities (4.20) and (4.21). Areas that can contain the support of μ are drawn in grey.

Let (x, y) be a point from $(z, z') \times (z, z')$. Let $x_1 = x - z$, $x_2 = z' - x$, $y_1 = y - z$, and $y_2 = z' - y$ (see Figure 6). Since all the quantities appearing in (4.19) are positive, we may rewrite (4.19) as

(4.20)
$$(1-\alpha)\left(F_\mu^{\nearrow} - F_\mu^{\searrow}\frac{f_\mu^{\nearrow}}{f_\mu^{\searrow}}\right) = \alpha\left(F_\mu^{\nwarrow} + F_\mu^{\searrow} + F_\mu^{\nwarrow}\frac{f_\mu^{\swarrow}}{f_\mu^{\nwarrow}} + F_\mu^{\searrow}\frac{f_\mu^{\nearrow}}{f_\mu^{\searrow}}\right) .$$

Observe that $F_\mu^{\nearrow}(x, y) = \mu([x, 1] \times [y, 1])$, $F_\mu^{\nwarrow}(x, y) = \mu([z, x] \times [y, z']) = \frac{x_1 y_2}{z'-z}$, and $F_\mu^{\searrow}(x, y) = \mu([x, z'] \times [z, y]) = \frac{x_2 y_1}{z'-z}$. Further observe that

$$\frac{f_\mu^{\nearrow}(x,y)}{f_\mu^{\searrow}(x,y)} = \frac{\frac{2x_2^2 y_1 y_2}{2(z'-z)^2}}{\frac{x_2^2 y_1^2}{(z'-z)^2}} = \frac{y_2}{y_1} \quad \text{and} \quad \frac{f_\mu^{\swarrow}(x,y)}{f_\mu^{\nwarrow}(x,y)} = \frac{\frac{2x_1^2 y_1 y_2}{2(z'-z)^2}}{\frac{x_1^2 y_2^2}{(z'-z)^2}} = \frac{y_1}{y_2} .$$

Plugging these observations in (4.20), we obtain that

(4.21)
$$(1-\alpha)\left(\mu([x,1]\times[y,1]) - \frac{x_2 y_2}{z'-z}\right) = \alpha\frac{x_1 y_2 + x_2 y_1 + x_1 y_1 + x_2 y_2}{z'-z} .$$

Since $x_1 + x_2 = y_1 + y_2 = z' - z$ and $\frac{x_2 y_2}{z'-z} = \mu([x, z'] \times [y, z'])$, we obtain from (4.21) that

$$(1-\alpha)\mu([z',1]\times[z',1]) = \alpha\frac{(z'-z)^2}{z'-z} = \alpha(z'-z) . \tag{4.22}$$

Finally, we substitute $1-z'$ for $\mu([z',1]\times[z',1])$ in (4.22) and get the following:

$$z' = z + (1-\alpha)(1-z)\ . \tag{4.23}$$

So, we conclude that the right end point of every interval in $\mathcal{I}_0$ is uniquely determined by its left end point.

Let Z be the set formed by the left end points of intervals in $\mathcal{I}_0$. As in the proof of Theorem 4.6, for a positive integer i, let z_i be the ith smallest element of Z. Notice that the existence of minimum elements follows from (4.23). If Z is finite, we set $z_k = 1$ for $k > |Z|$.

We derive from the definition of Z and from (4.23) that

$$\mathcal{I}_0 \;=\; \left\{[z_i, z_i + (1-\alpha)(1-z_i)],\ i \in \mathbb{N}^+\right\} \setminus \{[1,1]\}\ .$$

Consequently, we obtain

$$d(21,\mu) = \sum_{i=1}^{\infty} \frac{(1-\alpha)^2(1-z_i)^2}{2} = \frac{(1-\alpha)^2}{2}\sum_{i=1}^{\infty}(1-z_i)^2\ . \tag{4.24}$$

Analogously to the proof of Theorem 4.6, for $j \in \mathbb{N}$, we define $\beta_j \in [0,1]$ as follows:

$$\beta_j = \begin{cases} 1-z_1 & \text{for } j=0,\\ \frac{1-z_j}{\alpha(1-z_{j-1})} & \text{if } z_j \neq 1 \text{ and } j>0\text{, and}\\ 0 & \text{otherwise.} \end{cases}$$

The equation (4.24) can be rewritten as

$$d(21,\mu) = \frac{(1-\alpha)^2}{2}\sum_{i=1}^{\infty}\alpha^{2(i-1)}\prod_{j=1}^{i}\beta_j^2;\ . \tag{4.25}$$

Hence, the equality (4.18) can hold only if $\beta_j = 1$ for every j, i.e., $z_i = 1-\alpha^{i-1}$. This implies that the permutons μ and μ^r_α are identical. $\square$

4.5. Union of complete graphs

In this section, we prove that graphons corresponding to permutons from Section 4.3 are not finitely forcible. Any graph associated with a union of monotone permutations is a union of complete graphs. Such graphs can be characterized by not having as an induced subgraph a path on 3 vertices P_3. Thus, in this section we focus on graphons with $d(P_3, W) = 0$. We start with the following lemma, which seems to be of be of independent interest. Informally, the lemma asserts that any finitely forcible graphon with zero density of P_3 can be forced by finitely many densities of complete graphs.

LEMMA 4.3. *If W_0 is a finitely forcible graphon and $d(P_3, W_0) = 0$, then there exists an integer ℓ_0 such that any graphon W with $d(P_3, W) = 0$ and $d(K_\ell, W) = d(K_\ell, W_0)$ for $\ell \le \ell_0$ is weakly isomorphic to W_0.*

PROOF. To prove the statement of the lemma, it is enough to show the following claim: the density of any n-vertex graph G in a graphon W with $d(P_3, W) = 0$ can be expressed as a combination of densities of $K_1, \dots, K_n$ in W. We proceed by induction on $n + k$ where n and k are the numbers of vertices and components of G respectively. If $n = k = 1$, there exists only a single one-vertex graph K_1 and the claim holds.

Assume now that $n+k > 2$. If G is not a union of complete graphs, then $d(G, W) = 0$ since $d(P_3, W) = 0$. So, we assume that G is a union of k complete graphs $G_1, \dots, G_k$, i.e., $G = G_1 \cup \dots \cup G_k$. If $k = 1$, then $G = K_n$ and the claim clearly holds. So, we assume $k > 1$.

For $2 \le i \le k$, we denote

$$H_i = (G_1 + G_i) \cup \bigcup_{j \in [k] \setminus \{1,i\}} G_j \,.$$

Observe that the following holds:

$$\begin{aligned} & d(G_1, W)d(G_2 \cup \dots \cup G_k, W) = \\ & p_1 \cdot d\left(G_1 \cup \dots \cup G_k, W\right) + \sum_{i=2}^{k} p_i \cdot d\left(H_i, W\right) \end{aligned} \tag{4.26}$$

where p_1 is the probability that a set V of randomly chosen $|G_1|$ vertices of the graph $G_1 \cup \dots \cup G_k$ induces a complete graph and the graph $(G_1 \cup \dots \cup G_k) \setminus V$ is isomorphic to $G_2 \cup \dots \cup G_k$, and p_i, $i > 1$, is the probability that a set V of randomly chosen $|G_1|$ vertices of H_i induces a complete graph and the graph $H_i \setminus V$ is isomorphic to $G_2 \cup \dots \cup G_k$. To see (4.26), observe that the product $d(G_1, W)d(G_2 \cup \dots \cup G_k, W)$ is equal to the product of the probability that a W-random graph of order $|G_1|$ is isomorphic to G_1 and the probability that a W-random graph of order $|G_2| + \dots + |G_k|$ is isomorphic to $G_2 \cup \dots \cup G_k$. This is equal to the probability that randomly chosen $|G_1|$ vertices of a W-random graph of order $|G_1| + \dots + |G_k|$ induce a subgraph isomorphic to G_1 and the remaining vertices induce a subgraph isomorphic to $G_2 \cup \dots \cup G_k$. This probability is equal to the right hand side of (4.26).

By induction, $d(G_2 \cup \dots \cup G_k, W)$ and $d\left(H_i, W\right)$, $2 \le i \le k$, can be expressed as combinations of densities of complete graphs of order at most n in W. Rearranging the terms of (4.26), we obtain that $d(G_1 \cup \dots \cup G_k, W)$ is equal to a combination of densities of complete graphs of order at most n in W. $\square$

For a sequence of non-negative reals $\vec{a} = (a_i)_{i\in\mathbb{N}}$ such that $\sum_{i=1}^{\infty} a_i = 1$, define a graphon $W^c_{\vec{a}}$ such that $W(x,y) = 1$ if and only if there exists $j \in \mathbb{N}$ such that

$$\sum_{i=1}^{j-1} a_i \le x, y \le \sum_{i=1}^{j} a_i \ .$$

We consider a particular case of this graphon W^c_α for $\alpha \in (0,1)$: $W^c_\alpha = W^c_{\vec{a}}$ for $a_i = (1-\alpha)\alpha^{i-1}$. Observe that $W^c_\alpha = W_{\mu^m_\alpha}$. The main result of this section asserts that unlike the associated permuton μ^m_α, the graphon W^c_α is not finitely forcible. Although it immediately follows as a corollary of the more general Theorem 4.9 from Section 4.6, we give its proof here to increase the readability.

Theorem 4.8. *For every $\alpha \in (0,1)$, the graphon $W^c_\alpha = W_{\mu^m_\alpha}$ is not finitely forcible.*

Proof. Observe that $d(P_3, W^c_\alpha) = 0$. By Lemma 4.3, it is enough to show that W^c_α is not finitely forcible with $S = \{P_3, K_1, \dots, K_n\}$ for any $n \in \mathbb{N}$, i.e., by setting the densities of P_3 and the complete graphs of orders $1, \dots, n$. Suppose for the sake of contradiction that for some $n \in \mathbb{N}$ the graphon W^c_α is uniquely determined by the densities of P_3 and $K_1, \dots, K_n$. Let $a_i = (1-\alpha)\alpha^{i-1}$. Further, define functions $F_i(x_1, \dots, x_{n+1}) : \mathbb{R}^{n+1} \to \mathbb{R}$ for $i = 1, \dots, n$ as follows:

$$F_i(x_1, \dots, x_{n+1}) = \sum_{j=1}^{n+1} \left(x_j^i - a_j^i\right) \ . \tag{4.27}$$

Observe that if $x_1 + \dots + x_{n+1} = a_1 + \dots + a_{n+1}$, which is equivalent to $F_1(x_1, \dots, x_{n+1}) = 0$, then it holds that

$$d(K_i, W^c_{\vec{b}}) = d(K_i, W^c_\alpha) + F_i(x_1, \dots, x_{n+1}) \text{ for } i \in [n], \tag{4.28}$$

where $\vec{b}$ is the sequence with $b_i = x_i$ for $i \le n+1$ and $b_i = a_i$ for $i > n+1$. Hence, to obtain the desired contradiction, it suffices to prove that there exist functions $g_j(x_{n+1})$, $j \in [n]$, on some open neighborhood of a_{n+1} such that

$$F_i(g_1(x_{n+1}), \dots, g_n(x_{n+1}), x_{n+1}) = 0 \text{ for every } i \in [n]. \tag{4.29}$$

Indeed, if such functions $g_j(x_{n+1})$, $j \in [n]$, exist, then (4.28) yields that the densities of $K_1, \dots, K_n$ in the graphon $W^c_{\vec{b}}$ with $b_i = g_i(x_{n+1})$ for $i \le n$, $b_{n+1} = x_{n+1}$ and $b_i = a_i$ for $i > n+1$ equal their densities in the graphon $W^c_{\vec{a}}$. This implies that $W^c_{\vec{a}}$ is not forced by the densities of P_3 and $K_1, \dots, K_n$.

We now establish the existence of functions $g_1, \ldots, g_n$ satisfying (4.29) on some open neighborhood of a_{n+1}. Observe that

$$\frac{\partial F_i}{\partial x_j}(x_1, \ldots, x_{n+1}) = i \cdot x_j^{i-1} \ .$$

We consider the Jacobian matrix of the functions $F_1, \ldots, F_n$ with respect to $x_1, \ldots, x_n$. The determinant of the Jacobian matrix is equal to

$$n! \prod_{1 \le j < j' \le n} (x_{j'} - x_j) \ . \tag{4.30}$$

Substituting $x_j = a_j$ for $j = 1, \ldots, n$, we obtain that the Jacobian matrix has non-zero determinant. In particular, the Jacobian is non-zero. The implicit function theorem now implies the existence of the functions $g_1, \ldots, g_n$ satisfying (4.29). This concludes the proof. □

4.6. Union of graphs

In this section, we prove general theorem stating that every graphon with infinite recursive structure is not finitely forcible. In particular, in proves that graphons related to permutons μ_α^r from Section 4.4 are not finitely forcible.

Let W be a graphon. For a sequence of non-negative reals $\vec{a} = (a_i)_{i\in\mathbb{N}}$ such that $\sum_{i=1}^{\infty} a_i = 1$, define a graphon $W_{\to\vec{a}}$ as follows. Informally speaking, we take the graphon $W_{\vec{a}}^c$ and plant a copy of W on each of its "components". Formally, for $x, y \in [0, 1)$, let j_x and j_y be the integers such that

$$\sum_{i=1}^{j_x-1} a_i \le x < \sum_{i=1}^{j_x} a_i \quad \text{and}$$
$$\sum_{i=1}^{j_y-1} a_i \le y < \sum_{i=1}^{j_y} a_i \ .$$

If $j_x \neq j_y$, then $W_{\to\vec{a}}(x, y) = 0$. If $j_x = j_y$, then

$$W_{\to\vec{a}}(x, y) = W\left(\frac{x - \sum_{i=1}^{j_x-1} a_i}{a_{j_x}}, \frac{y - \sum_{i=1}^{j_y-1} a_i}{a_{j_y}}\right) \ .$$

The set of pairs (x, y) with one of the coordinates being equal to zero has mesure zero, so we can for example set the values of $W(x, y)$ for such pairs to be equal to zero. Similarly as before, we also define $W_{\to\alpha}$ for $\alpha \in (0, 1)$ as $W_{\to\alpha} = W_{\to\vec{a}}$, where $a_i = (1 - \alpha)\alpha^{i-1}$.

If W^0_ρ is the graphon identically equal to $\rho \in [0,1]$, then $W^0_{1,\to\vec{a}}$ is $W^c_{\vec{a}}$ (we put a comma to separate the two indices, the first referring to the density of edges inside connected components, the second determining the sizes of those). More generally, we use $W^r_{\rho,\alpha}$ for the graphon $W^0_{\rho,\to\vec{a}}$, where $a_i = (1-\alpha)\alpha^{i-1}$. Examples can be found in Figure 7.

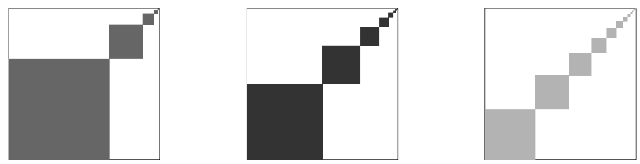


FIGURE 7. The graphons $W^r_{1/2,1/3}$, $W^r_{3/4,1/2}$, and $W^r_{1/4,2/3}$.

Our main theorem asserts that a graphon $W_{\to\alpha}$ is not finitely forcible unless it is W^0_0.

THEOREM 4.9. *For every $\alpha \in (0,1)$ and every graphon W, if the graphon $W_{\to\alpha}$ is finitely forcible, then W is weakly isomorphic to W^0_0, i.e., the graphon $W_{\to\alpha}$ is identically equal to zero up to a set of measure zero.*

PROOF. It is enough to show that for every n, there exists $\vec{b}$ different from $\vec{a}$, $a_i = (1-\alpha)\alpha^{i-1}$, such that

$$d\left(G, W_{\to\vec{b}}\right) = d\left(G, W_{\to\alpha}\right) \text{ for every graph } G \text{ with } |G| \leq n. \tag{4.31}$$

The proof of Theorem 4.8 yields that for every n, there exists such $\vec{b}$ different from $\vec{a}$ satisfying

$$d\left(G, W^c_{\vec{b}}\right) = d\left(G, W^c_\alpha\right) \text{ for every graph } G \text{ with } |G| \leq n. \tag{4.32}$$

We claim that this $\vec{b}$ also satisfies (4.31). Also note that (4.32) is non-zero only for graphs G that are disjoint union of cliques.

Let G be a graph with n vertices and let $G_1, \ldots, G_k$ be the connected components of G. Furthermore, let $\mathcal{F} = \{I_1, \ldots, I_\ell\}$ be the partition of $[k]$ according to the isomorphism classes of the graphs $G_1, \ldots, G_k$, i.e., for every $i, j \in [\ell]$ with $i \neq j$ and every $a_1, a_2 \in I_i$ and $a_3 \in I_j$, the graphs G_{a_1} and G_{a_2} are isomorphic, and the graphs G_{a_1} and G_{a_3} are not isomorphic.

Observe that

(4.33)

$$d\left(G, W_{\to\vec{b}}\right) = \sum_{f:[k]\to\mathbb{N}} c(f) \left(\prod_{i=1}^{\infty} d\left(\bigcup_{j\in f^{-1}(i)} G_j, W \right) b_i^{\left| \bigcup_{j\in f^{-1}(i)} G_j \right|} \right)$$

with the normalizing factor

$$c(f) = \prod_{m\in[\ell]} \frac{\prod_{i=1}^{\infty} |f^{-1}(i)\cap I_m|!}{|I_m|!},$$

where we set $0! = 1$ and the density $d(\emptyset, W)$ of the empty graph in the graphon W to 1.

We consider partitions of the set of connected components of G. If $Q = \{Q_1, \ldots, Q_k\}$ is such a partition, we slightly abuse the notation and identify Q_i with the subgraph of G induced by the components of Q_i. In particular, $|Q_i|$ denotes the number of vertices in this subgraph.. Furthermore, we always view a partition Q as a multiset, and also allow some of the Q_i's to be empty. Let $\mathcal{Q}$ be the set of all such partitions. The identity (4.33) can now be rewritten as follows:

$$d\left(G, W_{\to\vec{b}}\right) = \sum_{Q\in\mathcal{Q}} \prod_{i\in[k]} d(Q_i, W) d\left(\bigcup_{i\in[k]} K_{|Q_i|}, W^c_{\vec{b}} \right), \tag{4.34}$$

where K_0 is the empty graph. Since $\vec{b}$ satisfies (4.32), we obtain that it satisfies (4.34), and therefore also (4.31). □

We immediately obtain the following two corollaries.

Corollary 4.2. *For every $\alpha \in (0,1)$ and every $\rho \in (0,1]$, the graphon $W^r_{\rho,\alpha}$ is not finitely forcible.*

Corollary 4.3. *For every $\alpha \in (0,1)$, the graphon $W_{\mu^r_\alpha} = W_{\lambda,\to\alpha}$, which is associated with the permuton μ^r_α, is not finitely forcible.*

4.7. Conclusion

We have shown that graphons associated with finitely forcible permutons need not be finitely forcible. In [**56**], Question 11 asks whether there exists a "2-dimensional" finitely forcible graphon and such graphons naturally arise from permutons. Glebov, Král', and Volec [**33**] constructed a finitely forcible graphon where the Minkowski dimension of the associated topological space of typical points is two but the space is not connected. Our discussions with the authors of [**56**] led to the intuition that the graphon W_λ (as defined at the end of Section 4.1) is a good candidate for a finitely forcible graphon with the associated space being connected and having dimension two.

PROBLEM 4.1. *Is the graphon W_λ associated with the permuton λ finitely forcible?*

More generally, we suspect that all graphons associated with permutons μ_M constructed at the end of Section 4.2 might be finitely forcible.

PROBLEM 4.2. *Let M be a square matrix of order k with entries being non-negative reals such the sum of the entries in the i-th row is equal to that in the i-column and the sum of all the entries of M is one. Is the graphon W_{μ_M} associated with the permuton μ_M finitely forcible?*

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