Robust Markowitz Portfolio Selection in a Stochastic Factor Model

Dariusz Zawisza

Institute of Mathematics,
Jagiellonian University in Krakow, Poland

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Financial Market

Probability space \((\Omega, \mathcal{F}_T, P)\), \(T > 0\) is an investment horizon.

- A bank account
  \[ dP_t = rP_t \, dt \]

- A risky asset
  \[ dS_t = b(Y_t) \, dt + \sigma(Y_t) \, dB^1_t \]

- An untradable economic factor
  \[ dY_t = g(Y_t) \, dt + a(Y_t) \left( \rho \, dB^1_t + \sqrt{1 - \rho^2} \, dB^2_t \right) \]

- A market price of risk ratio is defined as
  \[ \lambda(y) := \frac{b(y) - r}{\sigma(y)}. \]
Portfolio Dynamics

- $\bar{\pi}$ - part of the wealth invested in the risky asset $S$
- The portfolio dynamics

$$d\bar{X}_t = r\bar{X}_t \ dt + (b(Y_t) - r)\bar{\pi}_t \ dt + \sigma(Y_t)\bar{\pi}_t \ dB^1_t$$

- Transformation of $\bar{X}_t$ and $\bar{\pi}_t$ to the $T$-forward values

$$X_t := e^{(T-t)r}\bar{X}_t, \quad \pi_t := e^{(T-t)r}\bar{\pi}_t$$

- Then

$$dX_t = \pi_t(b(Y_t) - r)dt + \pi_t\sigma(Y_t)dB^1_t$$

- Simplifying assumption: $r = 0$

$$dX_t = \pi_t b(Y_t) dt + \pi_t \sigma(Y_t) dB^1_t$$
Model Misspecification

The investor knows only that the correct measure belongs to a class of possible measures.

\[ Q := \left\{ Q \sim P \mid \frac{dQ}{dP} = \mathcal{E}\left( \int \eta_1 t \ dB_1^t + \eta_2 t \ dB_2^t \right)_T \right\}, \]

\( M \) denotes the set of all progressively measurable processes \( \eta = (\eta_1, \eta_2) \) taking values in a fixed compact convex set \( \Gamma \subset \mathbb{R}^2 \) (Schied [4]).
Robust Investor

- The worst-case scenario criterion:

$$\text{maximize } \inf_{Q \in Q} \mathbb{E}^Q_{x,y} U(X^T_\pi) \text{ over } \pi \in \mathcal{A}.$$
The worst-case scenario criterion:

$$\maximize \inf_{Q \in \mathcal{Q}} \mathbb{E}^{Q}_{X,\gamma} U(X_{T}^{\pi}) \text{ over } \pi \in \mathcal{A}.$$ 

This can be considered as a zero sum stochastic differential game, where investor is looking for a saddle point \((\pi^*, Q^*)\) such that

$$\mathbb{E}^{Q^*}_{X,\gamma} U(X_{T}^{\pi}) \leq \mathbb{E}^{Q^*}_{X,\gamma} U(X_{T}^{\pi^*}) \leq \mathbb{E}^{Q}_{X,\gamma} U(X_{T}^{\pi^*}).$$
**Markowitz Selection Problem**

Markowitz type investor is looking for a strategy $\pi^*$ such that

$$\text{Var}_{x,y}(X^{\pi^*}_T) \leq \text{Var}_{x,y}(X^{\pi}_T),$$

if $E_{x,y}(X^{\pi}_T) = A$. 
Markowitz Selection Problem

Markowitz type investor is looking for a strategy $\pi^*$ such that

$$\text{Var}_{x,y}(X_T^{\pi^*}) \leq \text{Var}_{x,y}(X_T^{\pi}),$$

if $E_{x,y}(X_T^{\pi}) = A$.

Robust Markowitz Selection Problem

Find a pair of controls $(\pi^*, Q^*)$ such that

$$\text{Var}_{x,y}^{Q^*}(X_T^{\pi^*}) \leq \text{Var}_{x,y}^{Q^*}(X_T^{\pi}),$$

if $E^{Q^*}(X_T^{\pi}) = A$, and

$$\text{Var}_{x,y}^{Q}(X_T^{\pi^*}) \leq \text{Var}_{x,y}^{Q^*}(X_T^{\pi^*}),$$

if $E^{Q}(X_T^{\pi^*}) = A$. 

Dariusz Zawisza
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Robust Markowitz Portfolio Selection in a Stochastic Factor Model
**Auxiliary Problem**

The auxiliary problem is constructed via Lagrange multipliers:

\[
J_{\theta}^{\pi,Q}(x, y, t) := \mathbb{E}_{x,y,t}(X_T^\pi - A)^2 - \theta(\mathbb{E}_{x,y,t}(X_T^\pi) - A) \\
= \mathbb{E}_{x,y,t}(X_T^\pi - D)^2 - (A + \frac{\theta}{2})^2 + A^2 + \theta A, \quad D = A + \frac{\theta}{2},
\]

\[
J_D^{\pi,Q}(x, y, t) := \mathbb{E}_{x,y,t}(X_T^\pi - D)^2.
\]
**Auxiliary Problem**

The auxiliary problem is constructed via Lagrange multipliers:

\[ J_{\theta}^{\pi, Q}(x, y, t) := \mathbb{E}_{x, y, t}^Q(X_T^\pi - A)^2 - \theta(\mathbb{E}_{x, y, t}^Q(X_T^\pi) - A) \]

\[ = \mathbb{E}_{x, y, t}^Q(X_T^\pi - D)^2 - (A + \frac{\theta}{2})^2 + A^2 + \theta A, \quad D = A + \frac{\theta}{2}, \]

\[ J_D^{\pi, Q}(x, y, t) := \mathbb{E}_{x, y, t}^Q(X_T^\pi - D)^2. \]

**Lemma**

Let \((\pi^*(D), Q^*(D))\) be a family of saddle points for \(J_D^{\pi, Q}\). If there exist \(D^*\) such that

\[ \mathbb{E}_{x, y}^{Q^*(D^*)}(X_T^{\pi^*(D^*)}) = A, \]

then \((\pi^*(D^*), Q^*(D^*))\) is a solution to the robust Markowitz selection problem.
Differential operator

\[ \mathcal{L}^{\pi, \eta} V(x, y, t) := V_t + \frac{1}{2} a^2(y) V_{yy} + \frac{1}{2} \pi^2 \sigma^2(y) V_{xx} \]
\[ + \pi \sigma(y) \rho a(y) V_{xy} + a(y) (\rho \eta_1 \rho + \bar{\rho} \eta_2) V_y \]
\[ + g(y) V_y + \pi (b(y) + \sigma(y) \eta_1) V_x \]
Differential operator

\[ \mathcal{L}^{\pi, \eta} V(x, y, t) := V_t + \frac{1}{2} a^2(y) V_{yy} + \frac{1}{2} \pi^2 \sigma^2(y) V_{xx} \]
\[ + \pi \sigma(y) \rho a(y) V_{xy} + a(y)(\rho \eta_1 \rho + \tilde{\rho} \eta_2) V_y \]
\[ + g(y) V_y + \pi (b(y) + \sigma(y) \eta_1) V_x \]

The first step is to find \( V \) such that

\[
\max_{\pi \in \mathbb{R}} \min_{\eta \in \Gamma} \mathcal{L}^{\pi, \eta} V(x, y, t) = \min_{\eta \in \Gamma} \max_{\pi \in \mathbb{R}} \mathcal{L}^{\pi, \eta} V(x, y, t) = 0,
\]

\[ V(x, y, T) = (x - D)^2 \]

and apply the following verification theorem, which is slight modification of Mataramvura and Øksendal [2] result.
Verification Theorem

Suppose there exist a nonnegative function $V \in \mathcal{C}^{2,2,1}(\mathbb{R}^2 \times [0, T]) \cap \mathcal{C}(\mathbb{R}^2 \times [0, T])$ and an admissible Markov control $(\pi^*(x, y, t), \eta^*(x, y, t))$ such that

$$\mathcal{L}^{\pi^*(x,y,t),\eta} V(x, y, t) \geq 0,$$

$$\mathcal{L}^{\pi,\eta^*(x,y,t)} V(x, y, t) \leq 0,$$

$$\mathcal{L}^{\pi^*(x,y,t),\eta^*(x,y,t)} V(x, y, t) = 0,$$

$$V(x, y, T) = (x - D)^2$$

for all $\eta \in \Gamma$, $\pi \in \mathbb{R}$, $(x, y, t) \in \mathbb{R}^2 \times [0, T)$, and

$$\mathbb{E}_{x, y, t}^Q \left( \sup_{t \leq s \leq T} | V(X_{s}^{\pi^*}, Y_{s}, s) | \right) < +\infty$$

for all $(x, y, t) \in \mathbb{R}^2 \times [0, T)$, $\pi \in \mathcal{A}$, $Q \in \mathcal{Q}$. Then $(\pi^*(x, y, t), \eta^*(x, y, t), V)$ is a solution to the auxiliary problem.
Applying standard minimax results to prove that

\[
\min_{\eta \in \Gamma} \max_{\pi \in \mathbb{R}} L^{\pi,\eta} V(x, y, t) = \max_{\pi \in \mathbb{R}} \min_{\eta \in \Gamma} L^{\pi,\eta} V(x, y, t),
\]

we can reduce the problem to solving only

\[
\min_{\eta \in \Gamma} \max_{\pi \in \mathbb{R}} L^{\pi,\eta} V(x, y, t) = 0,
\]

\[
V(x, y, T) = (x - D)^2.
\]
Applying standard minimax results to prove that

\[
\min_{\eta \in \Gamma} \max_{\pi \in \mathbb{R}} \mathcal{L}^{\pi,\eta} V(x, y, t) = \max_{\pi \in \mathbb{R}} \min_{\eta \in \Gamma} \mathcal{L}^{\pi,\eta} V(x, y, t),
\]

we can reduce the problem to solving only

\[
\min_{\eta \in \Gamma} \max_{\pi \in \mathbb{R}} \mathcal{L}^{\pi,\eta} V(x, y, t) = 0,
\]

\[
V(x, y, T) = (x - D)^2.
\]

The maximum with respect to \(\pi\) is reached at

\[
\pi^*(x, y, t, \eta) = -\frac{\rho a(y)}{\sigma(y)} \frac{V_{xy}}{V_{xx}} - \frac{b(y) + \eta_1(y, t)\sigma(y)}{\sigma^2(y)} \frac{V_x}{V_{xx}}.
\]
HJBI EQUATION

After substitution

\[ V_t + \frac{1}{2} a^2(y) V_{yy} - \frac{1}{2} \rho^2 a^2(y) V_{xy}^2 V_{xx} + g(y) V_y + \max_{\eta \in \Gamma} \left( -\frac{1}{2} \frac{(\eta_1 + \lambda(y))^2 V_x^2}{V_{xx}} \right) = 0, \]

\[ V(x, y, T) = (x - D)^2. \]
HJBI EQUATION

After substitution

\[ V_t + \frac{1}{2} a^2(y) V_{yy} - \frac{1}{2} \rho^2 a^2(y) \frac{V_{xy}^2}{V_{xx}} + g(y) V_y + \max_{\eta \in \Gamma} \left( -\frac{1}{2} \frac{(\eta_1 + \lambda(y))^2 V_x^2}{V_{xx}} \right) = 0, \]

\[ V(x, y, T) = (x - D)^2. \]

The solution should be of the form

\[ V(x, y, t) = (x - D)^2 F(y, t). \]
\[ F_t + \frac{1}{2} a^2(y) F_{yy} - \frac{\rho^2 a^2(y) F_y^2}{F} + (g(y) - 2\rho a(y)\lambda(y))F_y \]

\[ + \max_{\eta \in \Gamma} \left( -\eta_1 \rho a(y) F_y + \sqrt{1 - \rho^2 a(y)\eta_2} F_y - (\eta_1 + \lambda(y))^2 F \right) = 0 \]
**Hopf-Cole Transformation**

\[
F_t + \frac{1}{2} a^2(y) F_{yy} - \frac{\rho^2 a^2(y) F_y^2}{F} + (g(y) - 2\rho a(y) \lambda(y)) F_y \\
+ \max_{\eta \in \Gamma} \left( -\eta_1 \rho a(y) F_y + \sqrt{1 - \rho^2 a(y) \eta_2 F_y} - (\eta_1 + \lambda(y))^2 F \right) = 0
\]

To remove the nonlinear term \( \frac{\rho^2 a^2(y) F_y^2}{F} \) the Hopf-Cole type transformation is used:

**Case I:** \( \rho^2 \neq \frac{1}{2} \)

\[
F(y, t) = \left( \alpha(y, t) \right)^\delta, \quad \delta = \frac{1}{1 - 2\rho^2}.
\]
Assuming that $\rho^2 < \frac{1}{2}$, we have

$$\alpha_t + \frac{1}{2} a^2(y) \alpha_{yy} + (g(y) - 2\rho a(y) \lambda(y)) \alpha_y + \max_{\eta \in \Gamma} \left(-\eta_1 \rho a(y) \alpha_y + \sqrt{1 - \rho^2 a(y) \eta_2} \alpha_y - (1 - 2\rho^2)(\eta_1 + \lambda(y))^2 \alpha\right) = 0.$$
Assuming that $\rho^2 < \frac{1}{2}$, we have

$$\alpha_t + \frac{1}{2} a^2(y) \alpha_{yy} + (g(y) - 2\rho a(y) \lambda(y)) \alpha_y + \max_{\eta \in \Gamma} \left( -\eta_1 \rho a(y) \alpha_y + \sqrt{1 - \rho^2} \eta_2 a(y) \alpha_y - (\eta_1 + \lambda(y))^2 \right) = 0.$$

**Case II: $\rho^2 = \frac{1}{2}$**

$$F(y, t) = e^{\alpha(y,t)}$$

We get

$$\alpha_t + \frac{1}{2} a^2(y) \alpha_{yy} + (g(y) - 2\rho a(y) \lambda(y)) \alpha_y$$

$$+ \max_{\eta \in \Gamma} \left( -\eta_1 \rho a(y) \alpha_y + \sqrt{1 - \rho^2} \eta_2 a(y) \alpha_y - (\eta_1 + \lambda(y))^2 \right) = 0.$$
Existence of HJBI Solutions

**Theorem**

If $a$ is bounded, Lipschitz continuous and uniformly elliptic, $\lambda$, $g$ are bounded and Lipschitz continuous, then there exist a solution of the equation

\[
F_t + \frac{1}{2} a^2(y) F_{yy} - \rho^2 a^2(y) \frac{F^2_y}{F} + (g(y) - 2\rho a_y \lambda(y)) F_y
\]

\[
+ \max_{\eta \in \Gamma} \left( -\eta_1 \rho a(y) F_y + \sqrt{1 - \rho^2 a(y) \eta_2} F_y - (\eta_1 + \lambda(y))^2 F \right) = 0
\]

bounded together with the derivative $F_y$. 

Solution to the Auxiliary Problem

**Theorem**

If $a$ is bounded, Lipschitz continuous and uniformly elliptic, $\lambda$, $g$ are bounded and Lipschitz continuous, then

$$
\sup_{\pi \in \mathcal{A}_t} \inf_{Q \in \mathcal{Q}} J^{\pi, Q}(x, y, t) = \inf_{Q \in \mathcal{Q}} \sup_{\pi \in \mathcal{A}_t} J^{\pi, Q}(x, y, t) = (x - D)^2 F(y, t),
$$

and there exists the optimal pair of controls $(\eta_1^*(y, t), \eta_2^*(y, t))$ such that $\eta^* = (\eta_1^*(y, t), \eta_2^*(y, t))$ realizes maximum in equation \((\star)\), and the optimal strategy is given by

$$
\pi^*(y, x, t) = -\frac{\rho a(y)(x - D)}{\sigma(y)} \frac{F_y}{F} - \frac{(\lambda(y) + \eta_1^*(y, t))(x - D)}{\sigma(y)}.
$$

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It is sufficient to find $D$ such that
\[ \mathbb{E}_{x_0, y_0}^{Q^*}(X_T^{\pi^*(D)}) = A, \]
\[ dX_t = (X_t - D) \zeta(Y_t, t)(b(Y_t) + \eta^*_1(Y_t, t)\sigma(Y_t))dt \]
\[ + (X_t - D) \zeta(Y_t, t)\sigma(Y_t)dB_{\eta^*_1}^t, \]
\[ \zeta(y, t) := -\frac{\rho a(y) F_y}{\sigma(y) F} - \left(\lambda(y) + \eta^*_1(y, t)\right) \]
\[ - \frac{\eta^*_1(y, t)}{\sigma(y)}. \]
Back to the Robust Markowitz Problem

It is sufficient to find $D$ such that

$$\mathbb{E}^{Q^*}_{x_0, y_0}(X_T^{\pi^*(D)}) = A,$$

$$dX_t = (X_t - D)\zeta(Y_t, t)(b(Y_t) + \eta_1^*(Y_t, t)\sigma(Y_t))dt$$

$$+ (X_t - D)\zeta(Y_t, t)\sigma(Y_t)dB_t^{\eta_1^*},$$

$$\zeta(y, t) := -\frac{\rho a(y) F_y}{\sigma(y)} - \frac{(\lambda(y) + \eta_1^*(y, t))}{\sigma(y)}.$$

After simple transformations we get

$$A = D + (x_0 - D)\mathbb{E}^{Q^*}_{y_0} \exp\left(\int_0^T \zeta(Y_s)(b(Y_s) + \eta_1^*(Y_s, s)\sigma(Y_s))dsight)$$

$$- \frac{1}{2} \int_0^T \zeta^2(Y_s, s)\sigma^2(Y_s)ds + \int_0^T \zeta(Y_s, s)\sigma(Y_s)dB_s^{\eta_1^*}.$$
The Black-Scholes Model

\[
\begin{cases}
    dS_t = bS_t dt + \sigma S_t dB_t, \\
    \Gamma = [-R, R], \\
    \lambda > R.
\end{cases}
\]

\[
F_t + \max_{\eta_1 \in \Gamma} \left( -(\eta_1 + \lambda)^2 F \right) = 0,
\]

\[
\eta_1^* = -R, \quad \pi^*(x) = -\frac{(\lambda - R)(x - D^*)}{\sigma}, \quad D^* = \frac{A - x_0 e^{-(\lambda - R)^2 T}}{1 - e^{-(\lambda - R)^2 T}}.
\]
References I


References II


Thank you for your attention.