Topic VII

Series.

1. Prove that if the series $\sum a_n$ converges to a real limite then $\lim\limits_{n\to\infty} a_n = 0$.

2. Calculate, assuming that $|x| < 1$, $a \neq 0$:

$$\text{a) } \sum_{n=1}^{\infty} ax^n \qquad \text{b) } \sum_{n=1}^{\infty} anx^n \qquad \text{c) } \sum_{n=1}^{\infty} an^2x^n.$$

3. Examine the convergence of series $\sum a_n$, where :

$$\text{a) } a_n = \frac{3n+2}{5n^2-n+1}, \quad \text{b) } a_n = \frac{\sqrt{n}-1}{2\sqrt{n^2+1}}, \quad \text{c) } a_n = \frac{1}{n(\sqrt{n^2+n}-n)},$$

$$\text{d) } a_n = \frac{\sqrt{n+2}-\sqrt{n}}{\sqrt{n+1}}, \quad \text{e) } a_n = \frac{n^{10}}{10^n}, \quad \text{f) } a_n = \frac{1000^n}{n!},$$

$$\text{g) } a_n = \frac{(n!)^2}{(2n)!}, \quad \text{h) } a_n = \frac{(n!)^2 \cdot 3^n}{(2n)!}, \quad \text{i) } a_n = \frac{2^n \cdot n!}{n^n},$$

$$\text{j) } a_n = \left(\frac{2n+2}{3n+2}\right)^{\frac{n}{4}}, \quad \text{k) } a_n = \frac{\left(\frac{n+1}{n}\right)^{n^2}}{4^n}, \quad \text{l) } a_n = \left(\frac{2+(-1)^n}{\pi}\right)^n.$$

4. Show that $\forall\, x > 0$ the following series is convergent $\sum\limits_{n=0}^{\infty} \frac{x^n}{n!}$.

5. Find series $\sum a_n$ with all $a_n > 0$ such that $\sum a_n < +\infty$ and $\sum a_n \ln n = +\infty$.

6. Find series $\sum a_n$ with all $a_n > 0$ such that $\sum a_n = +\infty$ and $\sum\limits_{n=2}^{\infty} \frac{a_n}{\ln(\ln n)} < +\infty$.

7. Examine the convergence of $\sum\limits_{n=2}^{\infty} a_n$:

$$\text{a) } a_n = \frac{n}{10^{\ln n}}, \quad \text{c) } a_n = \frac{1}{\ln(n!)}, \quad \text{d) } a_n = \frac{1}{n \cdot 10^{\ln(\ln n)}}.$$

8. Examine the convergence of the following series:

a) $-\frac{1}{2} + \frac{1}{6} - \frac{1}{10} + \frac{1}{14} - \frac{1}{18} + \cdots$

b) $-\frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} - \frac{1}{7} + \cdots$

c) $-\frac{1}{2} + \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{1}{32} + \frac{1}{64} - \cdots$

d) $-\frac{1}{2} + \frac{1}{5} + \frac{1}{6} - \frac{1}{4} + \frac{1}{9} + \frac{1}{10} - \frac{1}{8} + \frac{1}{17} + \frac{1}{18} - \frac{1}{16} + \frac{1}{33} + \frac{1}{34} - \cdots$

e) $1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} - \frac{1}{9} + \cdots$

f) $1 + \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \frac{1}{5} + \frac{1}{6} + \frac{1}{7} - \frac{1}{8} - \frac{1}{9} - \frac{1}{10} + \frac{1}{11} + \frac{1}{12} - \frac{1}{13} - \frac{1}{14} - \frac{1}{15} + \cdots$

g) $1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \frac{1}{7} - \frac{1}{14} - \frac{1}{16} + \cdots$

h) $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \frac{1}{13} + \frac{1}{15} - \frac{1}{8} + \cdots$

i) $\frac{1}{2^1} + \frac{2}{2^2} - \frac{3}{2^3} + \frac{4}{2^4} + \frac{5}{2^5} - \frac{6}{2^6} + \frac{7}{2^7} + \frac{8}{2^8} - \frac{9}{2^9} + \cdots$

j) $\frac{1}{1^2} + \frac{1}{3^2} - \frac{1}{2^2} + \frac{1}{5^2} + \frac{1}{7^2} - \frac{1}{4^2} + \frac{1}{9^2} + \frac{1}{11^2} - \frac{1}{6^2} + \cdots$

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