Temat X

Continuity continued

1. Where possible, find the upper and lower bounds of the following functions. Indicate which functions actually attain the bounds.

a) $f(x) = \dfrac{1}{1+x^2}$; $D_f = \mathbb{R}$.

b) $f(x) = \dfrac{2}{e^x + e^{-x}}$; $D_f = \mathbb{R}$.

c) $f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$; $D_f = \mathbb{R}$.

d) $f(x) = x\sqrt{1-x^2}$; $D_f = (-1, 1)$.

e) $f(x) = \dfrac{2 + x - x^2}{(x-1)^2}$; $D_f = (1, +\infty)$.

f) $f(x) = \dfrac{x^2 + x - 1}{x - 1}$; $D_f = (1, +\infty)$.

g) $f(x) = \dfrac{x^2 + x - 1}{x - 1}$; $D_f = \mathbb{R} \setminus \{1\}$.

h) $f(x) = \cos \frac{1}{x}$; $D_f = \mathbb{R} \setminus \{0\}$.

i) $f(x) = e^{-x} \cos x$; $D_f = (0, +\infty)$.

j) $f(x) = e^{-x} \cos x$; $D_f = [0, +\infty)$.

2. Determine if the following functions have an inverse. In the cases that they do, give the domain of the inverse function and determine if it is continuous.

a) $f(x) = x^2$; $D_f = [-2, 1]$.

b) $f(x) = x^2$; $D_f = [-2, -1]$.

c) $f(x) = x^3$; $D_f = \mathbb{R}$.

d) $f(x) = 2^{x^3}$; $D_f = \mathbb{R}$.

e) $f(x) = \operatorname{ctg} x$; $D_f = (-2\pi, -\pi)$.

f) $f(x) = \dfrac{ax + b}{cx + d}$, $c \neq 0$; $D_f = \mathbb{R} \setminus \{-\frac{d}{c}\}$.

g) $f(x) = \dfrac{e^x - e^{-x}}{2}$; $D_f = \mathbb{R}$.

h) $f(x) = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$; $D_f = \mathbb{R} \setminus \{0\}$.

3. a) Show that the function $f(x) = \frac{1}{x}$ satisfies a Lipschitz condition on $(\varepsilon, +\infty)$,$\varepsilon > 0$, i.e., there is a positive constant $L_\varepsilon(f)$ such

$$|f(x) - f(y)| \leq L_\varepsilon(f)|x - y|$$

for all $x, y \in (\varepsilon, +\infty)$. Conversely, show that no Lipschitz condition is possible for f on $(0, +\infty.)$

b) Show that $f(x) = x^2$ satisfies a Lipschitz condition on any bounded interval but does not on the open intervals $(-\infty, 0)$, $(0, \infty)$ and $(-\infty, \infty)$

c) Show that a Lipschitz function with bounded domain is a bounded function.

4. Give an example of a continuous function $f(x)$, with domain $D_f = [0, 1)$, such that the range= $f(D_f)$ is the interval:

a) $(0, 1]$, b) $[0, 1]$, c) $(0, 1)$;

d) $[0, +\infty)$, e) $(0, +\infty)$, f) $(-\infty, +\infty)$.

5. a) Explain why among the triangles inscribed in a circle of radius R there is one with maximal area.

b) A mountain climber climbs a mountain with a stop watch timing the ascent. At the top, the climber immediately turns around, resets the timer to zero, and begins to descend along the same path. Show that there is a point on the path where the travel time up is the same as the travel time down.

Kristofer Abraśki i Waldemar PaŁuna