Topic II

School Mathematics. Extension.

1. Show that:

a) for any $x, y, z \in \mathbb{R}$ we have $|x+y+z| \leq |x|+|y|+|z|$;

b) for any $x, y \in \mathbb{R}$ we have $||x|-|y|| \leq |x-y|$;

c) for any $x, y \in \mathbb{R}$ we have $\max\{x, y\} = \frac{1}{2}(x+y+|x-y|)$. Find the analogous formula for $\min\{x, y\}$.

2. Calculate:

$(5^{\frac{1}{5}}-4^{\frac{1}{4}})(5^{\frac{1}{5}}-3^{\frac{1}{3}})(5^{\frac{1}{5}}-2^{\frac{1}{2}})(5^{\frac{1}{5}}-1)(4^{\frac{1}{4}}-3^{\frac{1}{3}})(4^{\frac{1}{4}}-2^{\frac{1}{2}})(4^{\frac{1}{4}}-1)(3^{\frac{1}{3}}-2^{\frac{1}{2}})(3^{\frac{1}{3}}-1)(2^{\frac{1}{2}}-1).$

3. Calculate:

$$\left(\frac{1}{\frac{1}{\frac{1}{\sqrt{2}-1}-2}-2}-1\right)^2.$$

4. Lest $f(x)=\sqrt{1+x^2+x^4}$. Find $f(-x)$, $f(\frac{1}{x})$, $f(\sqrt{x})$ i $f(x^2)$.

5. Find the maximal domain (the subset of $\mathbb{R}$) on which the functions below are well defined:

a) $f(x)=\sqrt{x^2}$; b) $f(x)=(\sqrt{x})^2$; c) $f(x)=\sqrt{4-\sqrt{x}}$;

d) $f(x)=\sqrt[3]{4-\sqrt{x}}$; e) $f(x)=\frac{x}{|x|}$; f) $f(x)=\sqrt{\frac{x+1}{x-1}}$.

6. a) Find the equation for the bisector of the line seqment between the points $(1,-5)$ i $(3,-1)$.

b) Find the center and the radius of the circle given by the equation $2x^2+2y^2+2x-2y=1$.

c) Find the vertex of the parabola given by the equation $2y=x^2-4x+8$.

7. a) Solve the equation $|x-2|+|x-8|=1$.

b) Solve the inequality $||2x-3|-3|>2$.

c) Solve the system of equations:

$$\begin{cases} |x+1|+|y-1|=5, \\ |x+1|=4y-4. \end{cases}$$

d) Solve the system of equations:

$$\begin{cases} x^2-x-6>0, \\ 2x \geq 15-x^2. \end{cases}$$

8. Calculate:

a) $\sqrt{15-6\sqrt{6}}+\sqrt{15+6\sqrt{6}}$;

b) $\sqrt[3]{20+14\sqrt{2}}+\sqrt[3]{20-14\sqrt{2}}$.

c) Calcute $x^3-\frac{1}{x^3}$, under the assumption that $x-\frac{1}{x}=4$.

9. In Cartesian coordinate system, draw the plot of the function f:

a) $y=f(x)=|1-x|-|x-2|-|x-3|$;

b) $y=f(x)=|x-1|\,(|x|-1)$.

10. Find the vaule of the parameter $m \in \mathbb{R}$ under the assumption that the one of the solution of the equation
$4x^2-15x+4m^2=0$ equals to the square of the other one.

11. Find the value of the parameter $m \in \mathbb{R}$ in such a way that the solutions x_1, x_2 of the equation
$x^2 + mx + 7 = 0$ satisfy $|x_1 - x_2| = 1$.

12. Let a_n be an arithmetic sequence, given by $a_{n+1} = a_n + \frac{1}{2}$. We know that $a_1 a_2 a_3 = \frac{495}{4}$. Find $a_1 + a_2 + a_3$

13. Show that the polynomial $W(x)$, such that all the coeficients are integers and such that: $W(13) = 3$ and $W(17) = 5$, cannot exist.

14. The biker covered a distance 96 km in time 2 hours less than he planned to. In every hour, he covered a distance 1 km bigger than he planned to cover in 75 minutes. What was the speed of the biker? (We assume that the speed was constant)

15. Show that the plot of any polynomial of the degree 2 has the axe of symmetry. Is the analogous sentence true for any polynomial with an even degree?

16. Solve the inequalities:
a) $x^3 + 2x^2 - 5x - 6 < 0$; b) $\frac{14}{x^2-5x+6} < \frac{10}{2-x} - 3$.

17. Solve the equation:

$$x^2 + 4x - 16\sqrt{2x} + 20 = 0.$$

18. Show that for any $x \in \mathbb{R}$ the following inequalities are true:
a) $x^4 - 6x^3 + 10x^2 - 4x + 4 > 0$; b) $x^2 - x + \frac{1}{2} > 0$; c) $x^4 - x + 1 > 0$.

19. Solve the equation $\sin x + \cos 2x = 0$.

20. Solve the system of equations:

$$\begin{cases} x + y = \frac{\pi}{2}; \\ \cos^2 y - \cos^2 x = 1. \end{cases}$$

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