Temat V

Sequences I.

1. Find the limit of the following sequences if they converge, otherwise prove that they diverge :
a) $a_n = \sin \frac{\pi}{n}$,

b) $a_n = \sin n$,

c) $a_n = \frac{1-n}{1-2\sqrt{n}}$,

d) $a_n = \frac{1+\sqrt[99]{n}}{1+\sqrt[100]{n}}$.

2. Find the limit of

$$a_n = \frac{n^3 + 4n^2 + n - 1}{n^4 + n^3 + 2n^2 - 2n - 1}.$$

3. Give a lower bound for each the follwing sequences:
a) $a_n = n^2 - 5n + 6$,

b) $a_n = |n^2 - 5n + 6|$,

c) $a_n = |n^2 - 5n + 1|$,

d) $a_n = |n^2 - 13| - 2000n$,

e) $a_n = \sqrt{n^2 - 3n + 6}$.

4. Determine the smallet n_0 such that for all $n \geq n_0$, either $a_{n+1} > a_n$ or $a_{n+1} < a_n$, if:
a) $a_n = n^2 - 8n + 7$,

b) $a_n = -\frac{1}{3n+17}$,

c) $a_n = \left(1 - \frac{5}{n}\right)^n$,

d) $a_n = \sqrt[n]{n}$.

5. Determine if the following sequences are monotone:
a) $a_n = \frac{2^n + 3^{n+1}}{2^{n+1} + 3^n}$,

b) $a_1 = 1$, $a_{n+1} = -\frac{1}{2}a_n^2 + 2a_n$,

c) $a_1 = 3$, $a_{n+1} = -\frac{1}{2}a_n^2 + 2a_n$,

d) $a_1 = 5$, $a_{n+1} = -\frac{1}{2}a_n^2 + 2a_n$.

6. Prove that the sequence

$$\begin{aligned} a_1 &= \sqrt{5}, \\ a_2 &= \sqrt{5 + \sqrt{5}}, \\ a_3 &= \sqrt{5 + \sqrt{5 + \sqrt{5}}}, \\ &\vdots \\ a_n &= \underbrace{\sqrt{5 + \sqrt{5 + \sqrt{5 + \dots}}}} \end{aligned}$$

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is increasing and bounded.

7. The Fibonacci sequence (F_n), is defined by the following recurrence relation: $F_1 = 1,\ F_2 = 1,\ F_n = F_{n-1} + F_{n-2}$ for $n = 3, 4, 5, \ldots$.
Use induction to show that:

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right].$$

Is the sequence $a_n = \frac{F_{n+1}}{F_n}$ monotone?

Are $b_n = \frac{F_{2n}}{F_{2n-1}}$ and $c_n = \frac{F_{2n+1}}{F_{2n}}$ monotone?

8. Find a function f such that $a_n = f(n)$ for the following recurrence relations:
a) $a_{n+2} = a_{n+1} + a_n + n^2$, $a_1 = 1$, $a_2 = 2$;
b) $a_{n+2} = 4a_{n+1} + 4a_n$, $a_1 = 1$, $a_2 = 2$;
c) $a_{n+2} = 4a_{n+1} + 4a_n + 2^n$, $a_1 = 1$, $a_2 = 2$;
d) $a_{n+2} = 2a_{n+1} - a_n$, $a_1 = 1$, $a_2 = 2$;
e) $a_{n+2} = 2a_{n+1} - a_n + n$, $a_1 = 1$, $a_2 = 2$.
f) $a_{n+2} = a_n$, $a_1 = 1$, $a_2 = 2$.
g) $a_{n+2} = a_n + n$, $a_1 = 1$, $a_2 = 2$.
h) $a_{n+2} = a_n + (-1)^n$, $a_1 = 1$, $a_2 = 2$.
i) $a_{n+2} = \frac{1}{2}a_{n+1} + \frac{1}{2}a_n$, $a_1 = 0$, $a_2 = 1$.

9. Using the results of questions $4, 5, 8$ and 9 from Temat 3, calculate the following limits:

$$\text{a)}\ \lim_{n\to\infty} \frac{1+2+3+\cdots+n}{n^2},$$

$$\text{b)}\ \lim_{n\to\infty} \frac{1^2+2^2+3^2+\cdots+n^2}{n^3},$$

$$\text{c)}\ \lim_{n\to\infty} \frac{1^3+2^3+3^3+\cdots+n^3}{n^4},$$

$$\text{d)}\ \lim_{n\to\infty} \frac{1^k+2^k+3^k+\cdots+n^k}{n^{k+1}}.$$

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