Topic IX

Functions. Continuity of functions.

0. Find „left–side" and „right–side" limits of functions:
a) $f(x) = 2^{\frac{1}{x-1}}$, at point $x = 1$,
b) $f(x) = \dfrac{2^{\frac{1}{x}} + 3}{3^{\frac{1}{x}} + 2}$, at point $x = 0$.

1. Find functions $f(g(x))$ and $g(f(x))$, where:
a) $f(x) = 1 - x^2, \quad g(x) = 2x + 3$;
b) $f(x) = -17, \quad g(x) = |x|$;
c) $f(x) = \sqrt{x^2 - 3}, \quad g(x) = x^2 + 3$;
d) $f(x) = x^2 + 1, \quad g(x) = \frac{1}{x^2+1}$;
e) $f(x) = x^3 - 4, \quad g(x) = \sqrt[3]{x + 4}$.

2. Find a function of the form $f(x) = x^k$ (k does not have to be an integer) and a function $g(x)$ in such a way that $f(g(x)) = h(x)$, where:
a) $h(x) = \dfrac{1}{1 + x^2}$;
b) $h(x) = \dfrac{1}{\sqrt{x + 10}}$;
c) $h(x) = \dfrac{1}{(1 + x + x^2)^3}$.

3. Let$f(x) = 1 + x^2$. Find a function $g(x)$, to have

$$f(g(x)) = 1 + x^2 - 2x^3 + x^4.$$

4. Let $g(x) = 1 + \sqrt{x}$.Find a function$f(x)$, to have

$$f(g(x)) = 3 + 2\sqrt{x} + x.$$

5. Find a function $g(x)$ to have $f(g(x)) = h(x)$, where

$$f(x) = x^2, \qquad h(x) = x^4 + 1.$$

6. a) Is this possible to choose the value $f(1)$, in such a way that the function defined for $x \neq 1$ by the formula

$$f(x) = \frac{|x - 1|}{(x - 1)^3}$$

is continuous on $\mathbb{R}$?
b) Is this possible to choose the values $f(-2)$, $f(3)$, in such a way that the function defined for $x \in \mathbb{R} \setminus \{-2, 3\}$ as

$$f(x) = \frac{x + 1}{x^2 - x - 6}$$

is continuous on $\mathbb{R}$?
c) Is this possible to choose the values $f(-1)$, $f(1)$, in such a way that the function defined for $x \in \mathbb{R} \setminus \{-1, 1\}$ as

$$f(x) = \frac{|x^2 - 1|}{x^2 - 1}$$

is continuous on $\mathbb{R}$?

7. a) Show that equation $x^4 + 2x - 1 = 0$ is satisfied for some $x \in [0, 1]$.
b) Show that equation $x^5 - 5x^3 + 3 = 0$ is satisfied for some $x \in [-3, 2]$.
c) Show that equation $x^3 - 4x + 1 = 0$ has three different real valued solutions.
d) Show that there is an x between $\frac{\pi}{2}$ and π such that $\operatorname{tg} x = -x$.

8. At which points the following function is continuous

$$f(x) = \begin{cases} 0, & x \in \mathbb{Q}, \\ x^2, & x \notin \mathbb{Q}? \end{cases}$$

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