Temat III

Induction, benomial Theorem, inequalities.

1. Show that for any $n \geq 5$ we have

$$2^n > n^2 .$$

2. Show that for any $n \in \mathbb{N} \setminus \{1\}$ we have $7|(n^7 - n)$.

3. Show that for any $n \in \mathbb{N} \setminus \{1\}$ we have $42|(n^7 - n)$

4. Show that for any $n \in \mathbb{N}$ we have

$$1 + 2 + 3 + \cdots + n = \frac{n\,(n+1)}{2} .$$

5. Show that for any $n \in \mathbb{N}$ we have

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n\,(n+1)\,(2n+1)}{6} .$$

6. Find a general formula for the sum:

$$3 \cdot 7 \,+\, 7 \cdot 11 \,+\, 11 \cdot 15 \,+\, \cdots + (4n-1)(4n+3) .$$

7. Show that for any $n \in \mathbb{N}$ we have

$$\frac{1}{3 \cdot 7} + \frac{1}{7 \cdot 11} + \frac{1}{11 \cdot 15} + \cdots + \frac{1}{(4n-1)\,(4n+3)} = \frac{n}{3\,(4n+3)} .$$

8. Using the binomial theorem find a general formulas for:

$$\begin{aligned}
(1+1)^4 &= \cdots \\
(2+1)^4 &= \cdots \\
(3+1)^4 &= \cdots \\
&\vdots \\
(n+1)^4 &= \cdots
\end{aligned}$$

and next, using the results of exercices 4 i 5 find a general patern for:

$$1^3 + 2^3 + 3^3 + \cdots + n^3 .$$

9. Assuming that we already know the general paterns for the following sums:

$$\begin{aligned}
S^1(n) &\overset{def}{=} 1^1 + 2^1 + 3^1 + \cdots + n^1 \\
S^2(n) &\overset{def}{=} 1^2 + 2^2 + 3^2 + \cdots + n^2 \\
S^3(n) &\overset{def}{=} 1^3 + 2^3 + 3^3 + \cdots + n^3 \\
&\vdots \\
S^k(n) &\overset{def}{=} 1^k + 2^k + 3^k + \cdots + n^k
\end{aligned}$$

find a general formula for the sum:

$$S^{k+1}(n) \overset{def}{=} 1^{k+1} + 2^{k+1} + 3^{k+1} + \cdots n^{k+1}.$$

10. Show that for any $n \in \mathbb{N}$ we have

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n \,.$$

11. Show that for any $n \in \mathbb{N}$ we have

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0 \,.$$

12. For any $n \in \mathbb{N}$ calculate:

$$-1\binom{n}{1} + 2\binom{n}{2} - 3\binom{n}{3} + \cdots + (-1)^{n-1}(n-1)\binom{n}{n-1} + (-1)^n(n)\binom{n}{n} \,.$$

13. Using the binomial theorem show that for any $n \in \mathbb{N}$ the Bernoulli inequality with n as a power and $x \geq 0$, is true.
14. Using the binomial theorem show that for any $n \in \mathbb{N}$ the Bernoulli inequality with n as a power and $x > -1$, is true.
Hint: divide the interval $(-1, 0)$ into two subintervals and consider 2 cases: $x \in (-1, a)$ and $x \in [a, 0)$. How should you choose the value of a ?

15. Show that for any nonnegative real numbers$a$, b we have

$$\frac{a+b}{2} \geq \sqrt{ab} \,.$$

16. Prove the inequality between an „arithmetic mean" and a „geometric mean" for any number n of nonnegative real numbers .

17. Show that for any $x > 0$ and $y > 0$ the inequality

$$(x+y)^n \leq 2^{n-1}\left(x^n + y^n\right)$$

holds true for any $n \in \mathbb{N}$.

18. Show that for any $n \in \mathbb{N}$ we have

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n \cdot (n+1)} < 1 \,.$$

19. Show that for any $n \in \mathbb{N}$ we have

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 2 \,.$$

20. Show that for any $n \in \mathbb{N}$ we have

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n} \,.$$

21. Show that for any $n \in \mathbb{N}$ we have

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{2n-1}{2n} < \frac{1}{\sqrt{2n}} \,.$$

Krzysztof Barański i Waldemar Pałuba