COMPUTATIONAL HYPERBOLICITY

Marcin Mazur

Jagiellonian University
Faculty of Mathematics and Computer Science
Institute of Mathematics
Grota Roweckiego 6, 30-348 Kraków, Poland

Jacek Tabor

Jagiellonian University
Faculty of Mathematics and Computer Science
Institute of Computer Science
Grota Roweckiego 6, 30-348 Kraków, Poland

Abstract. Using semihyperbolicity as a basic tool, we provide a general computer assisted method for verifying hyperbolicity of a given set. As a consequence we obtain that the Hénon attractor is hyperbolic for some parameter values.

1. Introduction

Computer assisted methods play an important role in the theory of dynamical systems. They often allow to prove significant theorems which defy strictly analytical verification, see for example [18]. Moreover, the growing power of computers and the use of more advanced methods allows to deal with more and more complicated problems [7, 11].

One of the most important and interesting directions in this field is the computer assisted investigation of hyperbolicity and hyperbolicity-like conditions. The aim of this paper is to make another step in this direction.

Let us recall the main computer assisted results concerning the hyperbolicity of the Hénon map. Using the notion of cone field Hruska [12] proved hyperbolicity of the complex Hénon attractor for some parameter values. Another approach was made by Arai [3] who adapted the notion of quasi-hyperbolicity to numerical verification and proved the hyperbolicity of the chain recurrent set of the real Hénon map for a large set of parameter values. Following Arai, we focus our attention on the real case.

For the convenience of the reader and to establish notation we recall some basic information. Given $a, b \in \mathbb{R}$ by $H_{a,b} : \mathbb{R}^2 \to \mathbb{R}^2$ we denote the real Hénon map

$$H_{a,b} \colon \mathbb{R}^2 \ni (x, y) \mapsto (a - x^2 + by, x) \in \mathbb{R}^2.$$

We study the behavior of the function $H_{a,b}$ on the set of such points in $\mathbb{R}^2$ that have bounded orbits; this set is commonly called the Hénon attractor – we denote

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it by $\mathrm{inv}(H_{a,b})$. Devaney and Nitecki [8] showed in 1979 that

$$\mathrm{inv}(H_{a,b}) \subset P_{a,b} := \{(x,y) \in \mathbb{R}^2 : |x| \le R_{a,b}, |y| \le R_{a,b}\},$$

where $R(a,b) := \frac{1}{2}(1+|b|+\sqrt{(1+|b|)^2+4a})$. Arai [3, Lemma 4.1] proved that the same inclusion holds for the set of chain recurrent points, i.e.,

$$CR(H_{a,b}) \subset P_{a,b}.$$

Consequently, we directly obtain that

$$CR(H_{a,b}) \subset \mathrm{inv}(H_{a,b}).$$

The main result of Arai [3, Theorem 1.1] says that $H_{a,b}$ is quasi-hyperbolic on $P_{a,b}$ for a large set of parameter values (in particular for $a = 5.4$, $b = -1$). By [5] it guarantees hyperbolicity of $H_{a,b}$ on $CR(H_{a,b})$. This gives a partial support to the conjecture posed in [6][1] concerning hyperbolicity of $H_{a,b}$ for $a = 5.4$ and $b = -1$.

By applying another approach based on the notion of semihyperbolicity [1, 2, 9, 15], which occurs to be well-adapted to proving hyperbolicity, we partially improve Arai's result[2] and show the hyperbolicity of $\mathrm{inv}(H_{a,b})$ for $a = 5.4$ and $b = -1$.

The reader not accustomed to semihyperbolicity can understand it at first as a numerical condition which guarantees hyperbolicity. However, in fact it can be applied to the investigation of more general systems then hyperbolic ones as it does not require the map to be invertible or even differentiable and the set to be invariant.

For the convenience of the reader we would like to explain the leading idea of semihyperbolicity. Consider the operator $A\colon \mathbb{R}^2 \to \mathbb{R}^2$ given in the matrix form by

$$A := \begin{bmatrix} a_s & 0 \\ 0 & a_u \end{bmatrix}.$$

If $\lambda_s := |a_s| < 1$ and $\lambda_u := |a_u| > 1$ then A is hyperbolic. Now we can ask how much we can "perturb the zeros" in A to preserve its hyperbolicity. More precisely, let $\mu_s, \mu_u \ge 0$ be given. We are interested under what assumptions the matrix

$$\begin{bmatrix} a_s & b_s \\ b_u & a_u \end{bmatrix},$$

is hyperbolic, where $|b_s| \le \mu_s, |b_u| \le \mu_u$ are arbitrary. By direct computation one can obtain the following condition:

$$(\lambda_u - 1)(1-\lambda_s) > \mu_s\mu_u. \tag{1}$$

The above idea gave rise to the notion of semihyperbolicity. As we see, semihyperbolicity does not require the splitting of the space to be invariant. As a consequence, contrary to the hyperbolicity, semihyperbolicity can be directly and relatively easily verified with strict numerical computations.

Let us present a consequence of the main results of the paper, which gives us a finite number of numerically verifiable conditions jointly guaranteeing hyperbolicity. By $\mathcal{L}(E)$ we denote the space of bounded linear operators on a Banach space E.

Theorem. *Let U be an open subset of $\mathbb{R}^2$, $f\colon U \to \mathbb{R}^2$ a C^1 map, and G a finite family of compact subsets of $\mathbb{R}^2$ such that $\langle G\rangle := \bigcup G \subset U$.*

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[1] See Section 3, page 175: "We believe that for this case we have hyperbolicity and hence a conjugacy."

[2] Contrary to Arai we are able to prove hyperbolicity of the whole Hénon attractor, however since our approach has larger computational complexity we are able to deal with smaller number of parameter values.

Consider set-valued maps $F : G \rightrightarrows G$ and $DF : G \rightrightarrows \mathcal{L}(\mathbb{R}^2)$ such that

$$f(x) \in F(\sigma),\ D_x f \in DF(\sigma) \text{ for every } \sigma \in G, x \in \sigma.$$

Let $\lambda_s \in [0, 1)$, $\lambda_u > 1$, $\mu_s \geq 0$, $\mu_u \geq 0$ satisfying (1) be given. We assume for each $\sigma \in G$ we are given a base e^σ_s, e^σ_u of $\mathbb{R}^2$ such that for every $\sigma \in G$, $\tau \in F(\sigma)$

$$B_\tau \circ DF(\sigma) \circ (B_\sigma)^{-1} \subset \begin{bmatrix} [-\lambda_s, \lambda_s] & [-\mu_s, \mu_s] \\ [-\mu_u, \mu_u] & \mathbb{R} \setminus (-\lambda_u, \lambda_u) \end{bmatrix},$$

where $B_\sigma = [e^\sigma_s, e^\sigma_u]$. (Note that e^σ_s and e^σ_u are not multi-valued, unlike $F(\sigma)$.)

Then

- *the set $\langle G \rangle$ is semihyperbolic for f;*
- *if f is invertible then the invariant set*

$$S = \mathrm{inv}(\langle G \rangle, f) := \{x : f^k(x) \in \langle G \rangle, k \in \mathbb{Z}\}$$

is hyperbolic for f and at each point $x \in S$ the forward Lyapunov exponents $\chi^s(x)$, $\chi^u(x)$ satisfy

$$\chi^s(x) \leq \ln\left(\lambda_s + \frac{\mu_s \mu_u}{\lambda_u - 1}\right), \quad \chi^u(x) \geq \ln\left(\lambda_u - \frac{\mu_s \mu_u}{1 - \lambda_s}\right).$$

Remark 1.1. Following the main result of [15], we can easily strengthen the above inequalities and show that the Lyapunov exponents at every point $x \in S$ satisfy

$$\chi^s(x) \leq \ln\left(\frac{\lambda_s + \lambda_u}{2} - \frac{\sqrt{(\lambda_u - \lambda_s)^2 - 4\mu_s\mu_u}}{2}\right),$$

$$\chi^u(x) \geq \ln\left(\frac{\lambda_s + \lambda_u}{2} + \frac{\sqrt{(\lambda_u - \lambda_s)^2 - 4\mu_s\mu_u}}{2}\right).$$

However, to make this paper more independent and self-sufficient, we give another proof that does not allow us to obtain the latter estimations.

2. Semi-hyperbolicity versus hyperbolicity

In this section we rewrite the basic concepts of [15, 14] in terms of smooth maps in Euclidean spaces.

Here and later on U stands for an open subset of $\mathbb{R}^n$ and $\|\cdot\|$ denotes an original norm in $\mathbb{R}^n$. Let us first reformulate the notion of hyperbolicity.

Let $f : U \to f(U) \subset \mathbb{R}^n$ be a C^1 diffeomorphism.

If $\|\cdot\|_a$, $\|\cdot\|_b$ are given norms, then by $\|\cdot\|^b_a$ we denote the operator norm induced by $\|\cdot\|_a$ and $\|\cdot\|_b$.

Definition 2.1. An f-invariant compact subset K of U is said to be (λ_s, λ_u)-*hyperbolic*, where $\lambda_s < 1 < \lambda_u$, if for each $x \in K$ there exist a norm $\|\cdot\|_x$ on $\mathbb{R}^n$ and a splitting $\mathbb{R}^n = E^s_x \oplus E^u_x$ with corresponding projections P^s_x and $P^u_x = I - P^s_x$ such that

[H0] $\{\|\cdot\|_x\}_{x \in K}$ is an uniformly equivalent family of norms on $\mathbb{R}^n$, that is there exist $c > 0$ such that

$$c^{-1}\|v\| \leq \|v\|_x \leq c\|v\| \text{ for } x \in K,\ v \in \mathbb{R}^n;$$

[H1] the projections are uniformly bounded, that is

$$\sup_{x \in K}\{\|P^s_x\|, \|P^u_x\|\} < \infty;$$

[H2] the splitting is invariant (that is $D_xfE^{s,u}_x = E^{s,u}_{f(x)}$ for $x \in K$) and

$$\|(D_xf)|_{E^s_x}\|^{f(x)}_x \leq \lambda_s,\ \|(D_xf^{-1})|_{E^u_x}\|^x_{f(x)} \leq \lambda_u^{-1} \quad \text{for } x \in K.$$

As is known [20] from the above definition one can obtain the continuous dependence of projections P^s_x and P^u_x on $x \in K$.

Now let us proceed to the description of the notion of semihyperbolicity. For simplicity let us first restrict to the linear case.

Definition 2.2 (semihyperbolicity – linear case)**.** Let E, F be Banach spaces equipped with the norms $\|\cdot\|_E$ and $\|\cdot\|_F$, respectively, $A\colon E \to F$ be a bounded linear operator and let splittings $E = E^s \oplus E^u$, $F = F^s \oplus F^u$ be given. Then we can write A with respect to this splitting in the matrix form

$$A = \begin{bmatrix} A_s & B_s \\ B_u & A_u \end{bmatrix} : E^s \oplus E^u \to F^s \oplus F^u.$$

Let $\lambda_s, \lambda_u, \mu_s, \mu_u \in \mathbb{R}_+$. We say that A is $(\lambda_s, \lambda_u, \mu_s, \mu_u)$-*semihyperbolic* (with respect to the above norms and splitting) if

(2) $$\lambda_s < 1,\ \lambda_u > 1 \text{ and } (\lambda_u - 1)(1 - \lambda_s) > \mu_s\mu_u;$$

and

- $\|A_s\|^F_E \leq \lambda_s$, A_u is invertible and $\|A_u^{-1}\|^E_F \leq \lambda_u^{-1}$;
- $\|B_s\|^F_E \leq \mu_s$, $\|B_u\|^F_E \leq \mu_u$.

Note that from the invertibility of A_u the following equalities directly follows:

$$\dim E^s = \dim F^s,\ \ \dim E^u = \dim F^u.$$

Now we are ready to present the general definition.

Definition 2.3. Let $f : U \to \mathbb{R}^n$ be of class C^1 and let $\lambda_s, \lambda_u, \mu_s, \mu_u \in \mathbb{R}_+$ satisfy (2). Let K be a compact subset of U. We say that f is $(\lambda_s, \lambda_u, \mu_s, \mu_u)$-*semihyperbolic* on K if for each $x \in K$ there exist a norm $\|\cdot\|_x$ and a splitting $\mathbb{R}^n = E^s_x \oplus E^u_x$ with corresponding projections P^s_x and $P^u_x = I - P^s_x$ such that conditions [H0], [H1] and

[SH2] the operator

$$D_xf : (E^s_x \oplus E^u_x, \|\cdot\|_x) \to (E^s_{f(x)} \oplus E^u_{f(x)}, \|\cdot\|_{f(x)})$$

is $(\lambda_s, \lambda_u, \mu_s, \mu_u)$-semihyperbolic for all $x \in K$ such that $f(x) \in K$

hold.

Let us notice that the invariance of the splitting is not assumed in the above definition. Moreover, we do not assume the invertibility of f and the invariance of K. However, as shows the following theorem, under these additional assumptions the considered conditions are equivalent.

Theorem 2.4. *Let $f\colon U \to f(U) \subset \mathbb{R}^n$ be a C^1 diffeomorphism and K a compact invariant $(\lambda_s, \lambda_u, \mu_s, \mu_u)$-semihyperbolic subset of U. Take*

$$\varepsilon = \frac{c\mu_s}{1 - \lambda_s}$$

where c is any constant satisfying

$$1 < c < \frac{(1 - \lambda_s)(\lambda_u - 1)}{\mu_s\mu_u}.$$

Then the set K is $(\lambda_s + \varepsilon^{-1}\mu_s, \lambda_u - \varepsilon\mu_u)$-hyperbolic.

The above theorem is a consequence of the main result of [15]. Nevertheless, following [14] in this paper we give an independent proof that takes into account the present context (note, that f is defined not on a compact Riemannian manifold, but on some neighborhood of a compact invariant subset K of the Euclidean space $\mathbb{R}^n$ and we work with a family of norms that vary, even not necessarily continuously, in points of the set K).

Before proceeding further we should establish some notation.

Let ε be a positive constant, $\mathbb{R}^n = E^s \oplus E^u$ be a splitting with corresponding projections P^s, $P^u = I - P^s$ and $\|\cdot\|$ be a norm on $\mathbb{R}^n$. The set

$$C_\varepsilon(E^s, E^u, \|\cdot\|) = \{v \in \mathbb{R}^n \mid \|P^s v\| \leq \varepsilon \|P^u v\|\}$$

is called a cone in $\mathbb{R}^n$.

For a set $K \subset \mathbb{R}^n$ a collection of cones

$$\mathcal{C} = \{\mathcal{C}_x\}_{x \in K} = \{C_{\varepsilon(x)}(E^s_x, E^u_x, \|\cdot\|_x)\}_{x \in K},$$

determined by the splittings $\mathbb{R}^n = E^s_x \oplus E^u_x$, the norms $\|\cdot\|_x$ on $\mathbb{R}^n$ satisfying the condition [H0] and the positive real-valued function $\varepsilon(x)$ defined for each $x \in K$, forms a cone field over K.

Let $\mathcal{C} = \{\mathcal{C}_x\} = \{C_{\varepsilon(x)}(E^s_x, E^u_x, \|\cdot\|_x)\}$ be a cone field over an f-invariant compact set $K \subset U$. The diffeomorphism f is said to be strongly (λ_s, λ_u)-dominating on $\mathcal{C}$ over K, where

$$\lambda_s^{-1}\lambda_u > 1,$$

if for each $x \in K$ we have $\dim E^{s,u}_x = \dim E^{s,u}_{f(x)}$ and there exists a norm $\|\cdot\|_x$ on $\mathbb{R}^n$ such that [H0], [H1] and

[SD2] $\|D_{f(x)}f^{-1}v\|_x \geq \lambda_s^{-1}\|v\|_{f(x)}$, $\|D_x f w\|_{f(x)} \geq \lambda_u \|w\|_x$ for $x \in K$, $v \in \mathbb{R}^n \setminus \mathcal{C}_{f(x)}$, $w \in \mathcal{C}_x$

hold.

If, additionally, $\lambda_s < 1$ and $\lambda_u > 1$ we say that f is λ_u-expanding and λ_s-co-expanding on $\mathcal{C}$ over K.

Now, Theorem 2.4 is a direct consequence of the following Lemma.

Lemma 2.5 ([14]). *Let $f \colon U \to f(U) \subset \mathbb{R}^n$ be a C^1 diffeomorphism and $K \subset U$ a compact invariant set.*

(i) *If K is $(\lambda_s, \lambda_u, \mu_s, \mu_u)$-semi-hyperbolic according to splittings $\mathbb{R}^n = E^s_x \oplus E^u_x$ and norms $\|\cdot\|_x$ on $\mathbb{R}^n$, then f is $(\lambda_u - \varepsilon\mu_u)$-expanding and $(\lambda_s + \varepsilon^{-1}\mu_s)$-co-expanding on the cone field*

$$\mathcal{C} = \{\mathcal{C}_x\} = \{C_{\varepsilon(x)}(E^s_x, E^u_x, \|\cdot\|_x)\}$$

over K, where

$$\varepsilon(x) = \varepsilon = \frac{c\mu_s}{1-\lambda_s} \quad \text{for all } x \in K, \tag{3}$$

and c is any constant satisfying

$$1 < c < \frac{(1-\lambda_s)(\lambda_u - 1)}{\mu_s \mu_u}.$$

(ii) *If f is λ_u-expanding and λ_s-co-expanding on a cone field $\mathcal{C} = \{\mathcal{C}_x\}$ over K, then K is (λ_s, λ_u)-hyperbolic with respect to some invariant splittings $\mathbb{R}^n = E^s_x \oplus E^u_x$ satisfying $E^s_x \subset \mathbb{R}^n \setminus \mathcal{C}_x$ and $E^u_x \subset \mathcal{C}_x$ for $x \in K$.*

Proof. (i) The conditions [H1] and $\dim E_x^{s,u} = \dim E_{f(x)}^{s,u}$ are obviously held. Since

$$\lambda_s + \tfrac{\mu_s}{\varepsilon} = \lambda_s + \tfrac{1-\lambda_s}{c} < \lambda_s + 1 - \lambda_s = 1,$$
$$\lambda_u - \varepsilon\mu_u = \lambda_u - \tfrac{c\mu_s\mu_u}{1-\lambda_s} > \lambda_u - \tfrac{(1-\lambda_s)(\lambda_u-1)}{1-\lambda_s} = 1,$$

it remains to show the existence of a family of norms $\|\cdot\|_x$ $(x \in K)$ on $\mathbb{R}^n$ satisfying [H0] and the following estimates for each $x \in K$, $v \in \mathbb{R}^n \setminus \mathcal{C}_{f(x)}$, $w \in \mathcal{C}_x$:

$$\|D_{f(x)}f^{-1}v\|_x \geq (\lambda_s + \frac{\mu_s}{\varepsilon})^{-1}\|v\|_{f(x)},\ \|D_x f w\|_{f(x)} \geq (\lambda_u - \varepsilon\mu_u)\|w\|_x.$$

Take $x \in K$ and put

$$\|v\|_x = \max\{\varepsilon^{-1}\|\!|P_x^s v|\!\|_x, \|\!|P_x^u v|\!\|_x\} \ \text{ for } \ v \in \mathbb{R}^n.$$

Then the norms $\|\cdot\|_x$ satisfy the condition [H0] and for all $v, w \in \mathbb{R}^n$ such that

$$\|\!|P_x^s w|\!\|_x \leq \varepsilon\|\!|P_x^u w|\!\|_x,\ \|\!|P_{f(x)}^s v|\!\|_{f(x)} > \varepsilon\|\!|P_{f(x)}^u v|\!\|_{f(x)},$$

we have:

$$\begin{aligned}
\|D_x f w\|_{f(x)} &\geq \|\!|P_{f(x)}^u D_x f w|\!\|_{f(x)} \geq \|\!|P_{f(x)}^u D_x f P_x^u w|\!\|_{f(x)} - \|\!|P_{f(x)}^u D_x f P_x^s w|\!\|_{f(x)} \\
&\geq \lambda_u \|\!|P_x^u w|\!\|_x - \mu_u \|\!|P_x^s w|\!\|_x \geq \lambda_u \|\!|P_x^u w|\!\|_x - \mu_u \varepsilon \|\!|P_x^u w|\!\|_x \\
&= (\lambda_u - \varepsilon\mu_u)\|w\|_x, \\
\varepsilon\|v\|_{f(x)} &= \|\!|P_{f(x)}^s v|\!\|_{f(x)} = \|\!|P_{f(x)}^s D_x f D_{f(x)} f^{-1} v|\!\|_{f(x)} \\
&\leq \|\!|P_{f(x)}^s D_x f P_x^s D_{f(x)} f^{-1} v|\!\|_{f(x)} + \|\!|P_{f(x)}^s D_x f P_x^u D_{f(x)} f^{-1} v|\!\|_{f(x)} \\
&\leq \lambda_s \|\!|P_x^s D_{f(x)} f^{-1} v|\!\|_x + \mu_s \|\!|P_x^u D_{f(x)} f^{-1} v|\!\|_x \\
&\leq (\varepsilon\lambda_s + \mu_s)\|D_{f(x)} f^{-1} v\|_x,
\end{aligned}$$

which completes the proof of (i).

(ii) Since f is strongly (λ_s, λ_u)-dominating on $\mathcal{C}$ over K, from Theorem 1.2 of [19] for all $x \in K$ we obtain a splitting $\mathbb{R}^n = E_x^s \oplus E_x^u$ such that [H1], $D_x f E_x^{s,u} = E_{f(x)}^{s,u}$, $E_x^s \subset \mathbb{R}^n \setminus \mathcal{C}_x$ and $E_x^u \subset \mathcal{C}_x$ hold. Hence, the rest of condition [H2] follows directly from [SD2], which finishes the proof of (ii). □

Remark 2.1. Let $K \subset \mathbb{R}^n$ be a compact invariant set for a diffeomorphism f. We assume additionally that K is semihyperbolic with the splitting $\mathbb{R}^n = E_x^s \oplus E_x^u$. As we know from the previous result, K is hyperbolic, and therefore there is a uniquely determined invariant hyperbolic splitting $\mathbb{R}^n = F_x^s \oplus F_x^u$.

However, we also have

$$\dim E_x^s = \dim F_x^s,\ \dim E_x^u = \dim F_x^u.$$

To observe this one needs to look at the proof of Lemma 4.2.12 from [10], where it is showed even in the case of semihyperbolic sequence of matrices (in our situation it is enough to apply it for the sequence $A_n = D_{f^n(x)}f$).

3. Set-valued approach

In this section we are going to generalize the notion of semihyperbolicity for set-valued maps.

Let E, F be a Banach spaces and let $\mathcal{L}(E, F)$ denote the space of all bounded linear operators from E to F. Each nonempty bounded (with respect to the usual operator norm) set $\mathbb{S} \subset \mathcal{L}(E, F)$ will be called a *set-operator*. In this context, we define some natural notions:

- $\mathbb{S}\mathbb{T} := \{ST \mid S \in \mathbb{S} \text{ and } T \in \mathbb{T}\}$ *(superposition)*;
- $\mathbb{S}$ is said to be invertible if every $S \in \mathbb{S}$ is invertible, and $\mathbb{S}^{-1} := \{S^{-1} \mid S \in \mathbb{S}\}$ is also a set operator *(inverse)*;
- if F is a subspace of E, then $\mathbb{S}|_F := \{S|_F \mid S \in \mathbb{S}\}$ *(restriction)*;
- $\|\mathbb{S}\| := \sup_{S\in\mathbb{S}} \|S\|$ *(norm)*.

Let us also note, that we can identify an operator $S \in \mathcal{L}(E)$ with the set-operator $\mathbb{S} := \{S\}$.

Definition 3.1. We say that a set-operator $\mathbb{S} \subset B(E, F)$ *is* $(\lambda_s, \lambda_u, \mu_s, \mu_u)$*-semihyperbolic* if there exist splittings $E = E^s \oplus E^u$, $F = F^s \oplus F^u$ such that each element of $\mathbb{S}$ is $(\lambda_s, \lambda_u, \mu_s, \mu_u)$-semihyperbolic with the above splitting.

For given sets G and H, by $F\colon G \rightrightarrows H$ we denote a set-valued map, that is a map from G to 2^H. We define

$$\mathrm{dom}(F) := \{x \in G : F(x) \neq \emptyset\}.$$

To formulate the main theorem of this paper we need the following definition.

Definition 3.2. Let G be a finite set and $F\colon G \rightrightarrows G$, $DF\colon G \rightrightarrows B(E)$ be given multivalued maps such that $DF(\sigma)$ is a set-operator on E for all $\sigma \in G$. Let $\lambda_s, \lambda_u, \mu_s, \mu_u \in \mathbb{R}_+$ satisfy (2).

We say that the pair (F, DF) is $(\lambda_s, \lambda_u, \mu_s, \mu_u)$*-semihyperbolic* if for each $\sigma \in G$ there exist a norm $\|\cdot\|_\sigma$ on E and a splitting $E = E^s_\sigma \oplus E^u_\sigma$ such that for every $\sigma \in G, \tau \in F(\sigma)$ the set-operator

$$DF(\sigma) : (E^s_\sigma \oplus E^u_\sigma, \|\cdot\|_\sigma) \to (E^s_\tau \oplus E^u_\tau, \|\cdot\|_\tau)$$

is $(\lambda_s, \lambda_u, \mu_s, \mu_u)$-semihyperbolic.

Remark 3.1. Let $\|\cdot\|$ be a given norm on $\mathbb{R}^n$. As an immediate consequence of the above definition we obtain the existence of such constants $c, h > 0$ that

$$c^{-1}\|v\| \le \|v\|_\sigma \le c\|v\|$$

and

$$\sup\{\|P^s_\sigma\|, \|P^u_\sigma\|\} \le h$$

for every $v \in E$, $\sigma \in G$.

Corollary 3.3. *Consider two finite sets G and H such that $G \subset H$. Let $F\colon H \rightrightarrows H$ and $DF\colon H \rightrightarrows \mathcal{L}(R^n)$ be multivalued maps such that (F, DF) is $(\lambda_s, \lambda_u, \mu_s, \mu_u)$-semihyperbolic. Then so is $(F|_G, DF|_G)$.*

Before proceeding further, we need to establish some notation.

Let G be a finite collection of compact subsets of $\mathbb{R}^n$. Let $F\colon G \rightrightarrows G$ and let $DF\colon G \rightrightarrows \mathcal{L}(\mathbb{R}^n)$ be a multivalued map with set-operator values.

Let $f : U \to \mathbb{R}^n$ be a fixed C^1 map and let K be a subset of U.

Definition 3.4. We say that f *inherits dynamics of* (F, DF) *on* $K \subset U$, which we write $f \lhd_K (F, DF)$ if

(1) $K \subset \langle \mathrm{dom}(F) \rangle$;
(2) $(x \in K \cap \sigma,\ f(x) \in K \cap \tau$ for some $\sigma, \tau \in G) \Rightarrow \tau \in F(\sigma)$;
(3) $(x, f(x) \in K,\ x \in \sigma$ for some $\sigma \in G) \Rightarrow D_x f \in DF(\sigma)$.

In order to justify Definition 3.4 we provide the following simple, but important theorem.

Theorem 3.5. *Let $f\colon \mathbb{R}^n \supset U \to f(U) \subset \mathbb{R}^n$ be a C^1 diffeomorphism and $K \subset U$ be a hyperbolic compact invariant set for f. Then there exists a semihyperbolic pair (F, DF) such that $f \lhd_K (F, DF)$.*

Proof. To construct the desired pair, fix $\varepsilon > 0$ and take any compact finite covering G of the set K such that

$$\sigma \cap K \neq \emptyset \quad \text{and} \quad \operatorname{diam} \sigma \leq \varepsilon \quad \text{for every } \sigma \in G. \tag{4}$$

We put $E := \mathbb{R}^n$ and define multivalued maps $F\colon G \rightrightarrows G$ and $DF\colon G \rightrightarrows B(E)$ in the following way:

$$F(\sigma) := \{\tau \in G \mid f(\sigma) \cap \tau \neq \emptyset\}$$

and

$$DF(\sigma) := \{D_x f \mid x \in \sigma\}$$

for $\sigma \in G$. (The first condition in (4) implies that $F(\sigma) \neq \emptyset$ for each $\sigma \in G$.) We show that if ε is sufficiently small, the pair (F, DF) has discrete semihyperbolic structure.

For any $\sigma \in G$ we choose an arbitrary point $x_\sigma \in \sigma$ and define

$$\|\cdot\|_\sigma := \|\cdot\|_{x_\sigma}, \quad P^s_\sigma := P^s_{x_\sigma}, \quad P^u_\sigma := P^u_{x_\sigma},$$

where $\|\cdot\|_x$, $P^s_x\colon E \to E^s_x$ and $P^u_x\colon E \to E^u_x$ denote a norm and projections related to the point $x \in K$ according to the definition of hyperbolicity condition (with $C = 1$). Taking $\sigma \in G$, $\tau \in F(\sigma)$, $x \in \sigma$, $v \in E^s_\sigma$ and $w \in E^u_\sigma$, we obtain the following estimates:

$$\begin{aligned}
\|P^s_\tau D_x f v\|_\tau &\leq \|P^s_{x_\tau} D_{x_\sigma} f v\|_\tau + \|P^s_{x_\tau}(D_x f - D_{x_\sigma} f)v\|_\tau \\
&\leq \|D_{x_\sigma} f v\|_\tau + \|(P^s_{x_\tau} - P^s_{f(x_\sigma)})D_{x_\sigma} f v\|_\tau + \|P^s_{x_\tau}(D_x f - D_{x_\sigma} f)v\|_\tau, \\
\|P^u_\tau D_x f v\|_\tau &\leq \|P^u_{x_\tau} D_{x_\sigma} f v\|_\tau + \|P^u_{x_\tau}(D_x f - D_{x_\sigma} f)v\|_\tau \\
&\leq \|(P^u_{x_\tau} - P^u_{f(x_\sigma)})D_{x_\sigma} f v\|_\tau + \|P^u_{x_\tau}(D_x f - D_{x_\sigma} f)v\|_\tau, \\
\|P^s_\tau D_x f w\|_\tau &\leq \|P^s_{x_\tau} D_{x_\sigma} f w\|_\tau + \|P^s_{x_\tau}(D_x f - D_{x_\sigma} f)v\|_\tau \\
&\leq \|(P^s_{x_\tau} - P^s_{f(x_\sigma)})D_{x_\sigma} f w\|_\tau + \|P^s_{x_\tau}(D_x f - D_{x_\sigma} f)w\|_\tau, \\
\|P^u_\tau D_x f w\|_\tau &\geq \|P^u_{x_\tau} D_{x_\sigma} f w\|_\tau - \|P^u_{x_\tau}(D_x f - D_{x_\sigma} f)w\|_\tau \\
&\geq \|D_{x_\sigma} f w\|_\tau - \|(P^u_{x_\tau} - P^u_{f(x_\sigma)})D_{x_\sigma} f w\|_\tau - \|P^u_{x_\tau}(D_x f - D_{x_\sigma} f)w\|_\tau.
\end{aligned}$$

Owing to uniform equivalence of $\|\cdot\|_\sigma$ for $\sigma \in G$, uniform continuity of P^s_x, P^u_x and $\|\cdot\|_x$ according to the point $x \in K$, as well as uniform boundedness of $D_x f$ on the set K, taking notice into the fact that

$$\sup_{\sigma \in G, x \in \sigma} \operatorname{dist}(x, x_\sigma) \to 0 \quad \text{and} \quad \sup_{\sigma \in G, \tau \in F(\sigma)} \operatorname{dist}(x_\tau, f(x_\sigma)) \to 0 \quad \text{as } \varepsilon \to 0,$$

one can easily check that when ε is sufficiently small, the pair (F, DF) admits the discrete semihyperbolic structure given by the projections P^s_σ, P^u_σ and the norms $\|\cdot\|_\sigma$ for $\sigma \in G$. □

The above result shows that we can work with the "discretized" version of hyperbolicity condition in the context of compact hyperbolic sets. However, what is crucial from our point of view is, in some sense, the inverse problem that we fully solve in Theorem 4.1.

4. Main results

Now we are ready to formulate the main theorem of the paper. We recall that $\mathrm{Inv}(A, f)$ denotes an invariant part of A, i.e,

$$\mathrm{Inv}(A, f) := \{x \mid f^n(x) \in A \text{ for all } n \in \mathbb{Z}\}.$$

Theorem 4.1. *Let a pair (F, DF) be $(\lambda_s, \lambda_u, \mu_s, \mu_u)$-semihyperbolic. Let K be a compact subset of $\mathbb{R}^n$ and let f be a diffeomorophism such that $f \lhd_K (F, DF)$. Then f is $(\lambda_s, \lambda_u, \mu_s, \mu_u)$-semihyperbolic on K.*

Consequently, the set $\mathrm{inv}(K, f)$ is hyperbolic.

Proof. For each $x \in K$ choose $\sigma_x \in G$ as an arbitrary set containing x. Then, to fill in all the semihyperbolicity conditions, define

$$\|\cdot\|_x := \|\cdot\|_{\sigma_x},\ P^s_x := P^s_{\sigma_x},\ P^u_x := P^u_{\sigma_x},$$

and

$$E^s_x = P^s_x(\mathbb{R}^n),\ E^u_x = P^u_x(\mathbb{R}^n).$$

Since $f(x) \in \sigma_{f(x)}$ we have $\sigma_{f(x)} \in F(\sigma_x)$ (see Definition 3.4) and consequently we obtain the following estimates:

$$\begin{aligned}
\|P^s_{f(x)} D_x f|_{E^s_x}\|^{f(x)}_x &\leq \|P^s_{\sigma_{f(x)}} DF(\sigma_x)|_{E^s_{\sigma_x}}\|^{\sigma_{f(x)}}_{\sigma_x} \leq \lambda_s,\\
\|P^s_{f(x)} D_x f|_{E^u_x}\|^{f(x)}_x &\leq \|P^s_{\sigma_{f(x)}} DF(\sigma_x)|_{E^u_{\sigma_x}}\|^{\sigma_{f(x)}}_{\sigma_x} \leq \mu_s,\\
\|P^u_{f(x)} D_x f|_{E^s_x}\|^{f(x)}_x &\leq \|P^u_{\sigma_{f(x)}} DF(\sigma_x)|_{E^s_{\sigma_x}}\|^{\sigma_{f(x)}}_{\sigma_x} \leq \mu_u,\\
\|(P^u_{f(x)} D_x f|_{E^u_x})^{-1}\|^x_{f(x)} &\leq \|(P^u_{\sigma_{f(x)}} DF(\sigma_x)|_{E^u_{\sigma_x}})^{-1}\|^{\sigma_x}_{\sigma_{f(x)}} \leq (\lambda_u)^{-1},
\end{aligned}$$

where $\lambda_s, \lambda_u, \mu_s, \mu_u$ are the constants arising in Definition 3.2. Hence, Remark 3.1 and Theorem 2.4 makes the proof complete. □

Now let us proceed to the two-dimensional case. We will need the following obvious result.

Observation 4.2. *Let K be a compact subset of $\mathbb{R}^2$ which is (λ_s, λ_u)-hyperbolic for the diffeomorphism f. Assume that at each point of K the hyperbolic splitting contains one dimensional subspaces. Then for every $x \in K$ we have two (forward) Liapunov exponents $\chi^s(x), \chi^u(x)$ which satisfy*

$$\chi^s(x) \leq \ln(\lambda_s),\ \chi^u(x) \geq \ln(\lambda_u).$$

In the case of $\mathbb{R}^2$ we can obtain an easy criterion for hyperbolicity.

Proposition 4.3. *Let $f \lhd_K (F, DF)$ for some C^1 map f, a pair (F, DF) and a set K. Let $\lambda_s \in [0, 1)$, $\lambda_u > 1$, $\mu_s \geq 0$, $\mu_u \geq 0$ satisfying (1) be given. Suppose that for each $\sigma \in G$ we are given a base e^s_σ, e^u_σ of $\mathbb{R}^2$ such that for every $\sigma \in G$, $\tau \in F(\sigma)$*

$$B_\tau \circ DF(\sigma) \circ (B_\sigma)^{-1} \subset \begin{bmatrix} [-\lambda_s, \lambda_s] & [-\mu_s, \mu_s] \\ [-\mu_u, \mu_u] & \mathbb{R} \setminus (-\lambda_u, \lambda_u) \end{bmatrix},$$

where $B_\sigma = [e^s_\sigma, e^u_\sigma]$.

Then

1. *(F, DF) is $(\lambda_s, \lambda_u, \mu_s, \mu_u)$-semihyperbolic;*
2. *the set K is semihyperbolic for f;*

(3) *if f is invertible then the set $S := \operatorname{inv}(K, f)$ is hyperbolic for f and at each point $x \in S$ the Liapunov exponents of f satisfy*

$$\chi^s(x) \le \ln\left(\lambda_s + \frac{\mu_s\mu_u}{\lambda_u - 1}\right), \quad \chi^u(x) \ge \ln\left(\lambda_u - \frac{\mu_s\mu_u}{1-\lambda_s}\right).$$

Proof. Ad 1. To prove that the pair (F, DF) is semihyperbolic we need to construct the splitting and norms. For each $\sigma \in G$ we put

$$E^s_\sigma = \mathbb{R} \cdot e^s_\sigma,\ E^u_\sigma = \mathbb{R} \cdot e^u_\sigma,$$

and define the appropriate norms by the formula

$$\|x\|_\sigma = \max(\alpha, \beta) \quad \text{for } x = \alpha e^s_\sigma + \beta e^u_\sigma \in \mathbb{R}^2. \tag{5}$$

One can now easily check that all semihyperbolicity conditions are satisfied.

Ad 2. This is a direct consequence of point 1. and Theorem 4.1.

Ad 3. This is a corollary of 2 and Theorem 2.4.

Ad 4. By the Remark 2.1 we obtain that the dimensions of the subspaces in the hyperbolic splitting coincide with that of semihyperbolic one. Consequently, the hyperbolic splitting in each point has one stable and one unstable direction. Now we apply jointly Theorem 2.4 with Observation 4.2. □

Note that in Proposition 4.3 the norm which we need for semihyperbolicity is in fact „hidden" in the choice of the bases e^s_σ, e^u_σ, see (5).

5. Hyperbolicity of the invariant set in $\mathbb{R}^2$

In this section we explain the method we use to verify semihyperbolicity of the neighborhood of the Hénon attractor with some parameter values (in fact it can be used, at least theoretically, to any two-dimensional discrete diffeomorphism).

As we have seen, in the definition of semihyperbolicity we can take arbitrary norm in the tangent space in x, which is uniformly (with respect to x) equivalent to the Euclidean one. Since this choice is crucial for the numerical verification of the semihyperbolicity condition, below we describe the procedure of construction of such a norm in details.

For a sequence of vector spaces E_n and invertible operators $A_n \colon E_n \to E_{n+1}$ we define the operator

$$A(k, n) \colon E_k \to E_n$$

by the formula

$$A(k,n) := \begin{cases} A_{n-1} \circ \cdots \circ A_k & \text{if } n > k, \\ \mathrm{Id}_{E_k} & \text{if } n = k, \\ A(n,k)^{-1} & \text{if } n < k. \end{cases}$$

One of the crucial steps in our reasoning is described in the following observation.

Observation 5.1. *Let $\{A_n\}_{n\in\mathbb{Z}}$ be a sequence of invertible operators $A_n : E_n \to E_{n+1}$, where E_n are given Banach spaces. We assume that the operator sequences (A_n) and (A_n^{-1}) are uniformly bounded. Then the forward and backward expansion rates are finite, that is*

$$\rho_f := \limsup_{k\to\infty} \left(\sup_{n\in\mathbb{Z}} \|A(n, n+k)\|^{\frac{1}{k}}\right) < \infty,$$

$$\rho_b := \limsup_{k\to\infty} \left(\sup_{n\in\mathbb{Z}} \|A(n, n-k)\|^{\frac{1}{k}}\right) < \infty,$$

Moreover, for any $\varepsilon > 0$ we can uniformly modify the norms in the spaces E_n in such a way that

$$|||A_n||| \leq (1+\varepsilon)\rho_f, \; |||A_n^{-1}||| \leq (1+\varepsilon)\rho_b.$$

Sketch of the Proof. Let $K \geq 1$ be arbitrarily fixed. We define a norm in E_n by the formula

$$|||x||| := \max\left(\sup_{k\in\{0,\dots,K\}} \frac{\|A(n,n+k)x\|}{[(1+\frac{\varepsilon}{2})\rho_f]^k}, \sup_{k\in\{0,\dots,K\}} \frac{\|A(n,n-k)x\|}{[(1+\frac{\varepsilon}{2})\rho_b]^k} \right),$$

for $x \in E_n$.

One can easily notice that (for sufficiently large K) the induced norms on the spaces $\mathcal{L}(E_n, E_{n+1})$ satisfy the desired conditions. □

The above procedure of changing the norm is called *renormalization*. Below we show a typical example of a simple hyperbolic system which does not satisfy the hyperbolicity estimations before renormalization.

Example 5.1. For every $n \in \mathbb{Z}$ let E_n be a one dimensional space with the base e_n and the norm satisfying $\|e_n\| = 1$. Consider the sequence of linear maps $A_n\colon E_n \to E_{n+1}$ defined on the base by the formula

$$A_n(e_n) = \frac{1}{2}e_{n+1} \text{ if } n \neq 0 \text{ and } A_0(e_0) = 5e_1.$$

Clearly, such a sequence of matrices is hyperbolic, see [20] (the expansion rate is, of course, equal to $\frac{1}{2}$). However, the operator A_0 is clearly not shrinking!

Choose $\varepsilon > 0$, $K \geq 1$ and define a new norm in every E_n (on the base) as in the previous observation, which in our case simplifies to

$$\|e_n\|_n := \max_{k\in\{-K,\dots,K\}} \frac{\|A(n,n+k)e_n\|}{(1+\frac{\varepsilon}{2})^{|k|}\cdot(\frac{1}{2})^k}.$$

Now, the normalized base in $(E_n, \|\cdot\|_n)$ is given by

$$b_n := \frac{e_n}{\|e_n\|_n}.$$

In this case we have

$$A_n(b_n) = \beta_n b_{n+1},$$

where

$$\beta_n \in \frac{1}{2}\cdot((1+\varepsilon)^{-1}, 1+\varepsilon)$$

(for K chosen sufficiently large).

Now we are going to present the numerical method we use to prove hyperbolicity of the Hénon attractor for some parameter values. For the convenience of the reader we divide the description into a few steps (the standard ones we describe only briefly).

STEP 1. Cubical representation of the neighborhood of the invariant set. One needs to obtain theoretical bounds for the position of the invariant set. In our case we obtain the bounds by applying a result from [8]. Then we use interval arithmetic and set-valued mappings with cubical values to obtain a sufficiently small subdivision. We start with discrete representation of the function (see Figure 1).

Next we reduce the representation to invariant part only and divide each cube into four smaller ones (note that all cubes are subdivided). Then we go back to find

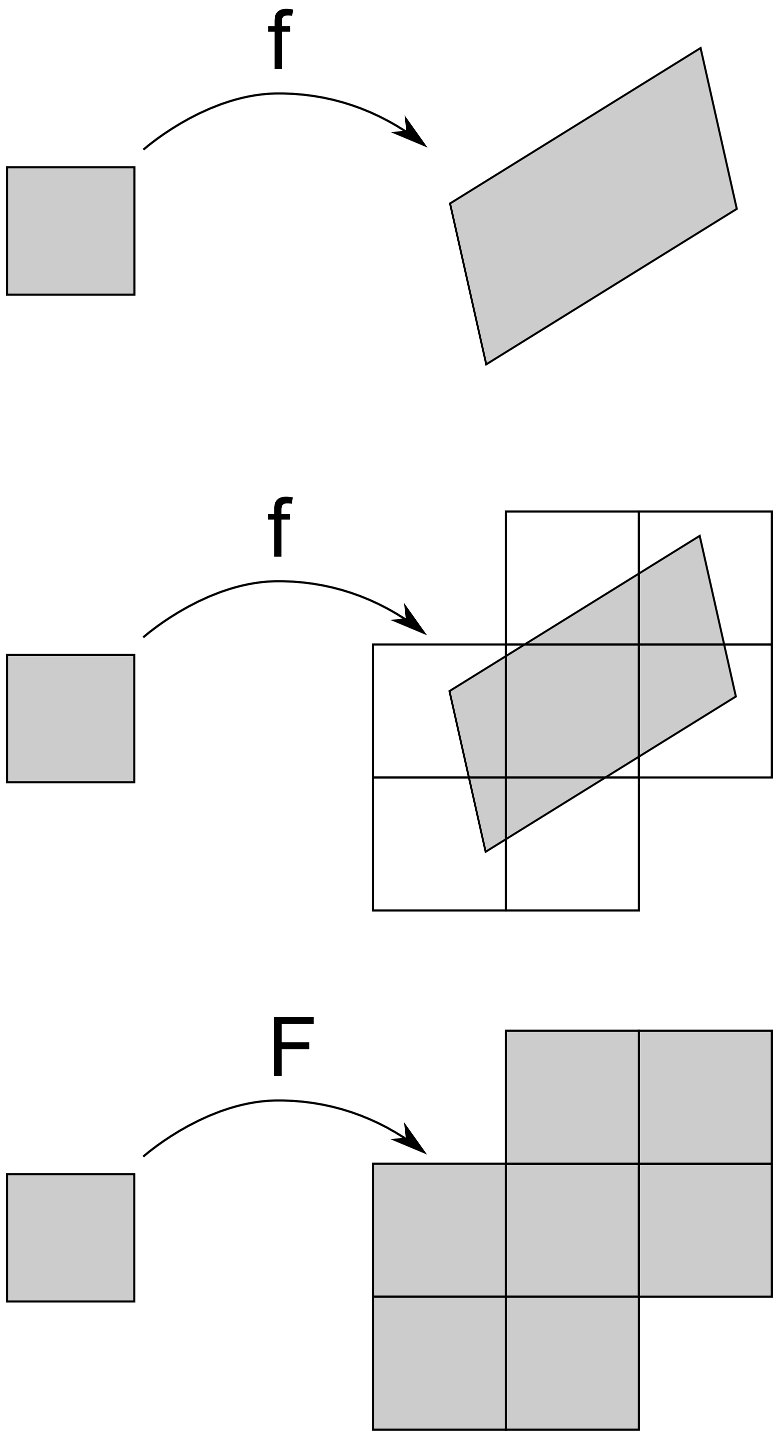

FIGURE 1. Cubic representation of the function.

new cubical representation and repeat the whole procedure until the desired level of subdivision is reached (see Figure 2). We call this set of cubes G (observe that G contains only the cubes of the same size).

STEP 2. Stable and unstable directions of the base. We present the construction of the base for the cube $\sigma \in G$. Fix an arbitrary point $x \in \sigma$ and $K \in \mathbb{N}$ (in our program we take $K = 5$). As before we use the convention:

$$A_x(k, n) := D_{f^{n-1}x} f \circ \cdots \circ D_{f^k(x)} f.$$

To construct the unstable direction, as e_u^x we take the (normalized) column of $A_x(0, K)$ which has the greatest norm. It is similar to finding the most expanding direction with the power method. To construct the stable direction, as e_s^x we take the (normalized) column of $A_x(-K, 0)^{-1}$ which has the greatest norm.

Such defined directions e_s^x, e_s^x are nearly invariant (that is $D_x f e_s^x$ lies near to the space $\mathbb{R} e_s^{f(x)}$, the analogue holds for the unstable direction).

However, it can happen that the Euclidean norm of $D_x f e_s^x$ may be greater then one and the norm of $D_x f e_u^x$ may be smaller then one (opposite as one should have). That is why in the next steps we apply the method described in Observation 5.1 to renormalize this bases. Note that Example 5.1 shows that without renormalization the conditions guaranteeing hyperbolicity may not be satisfied.

STEP 3. Renormalization. In this step (which is to some extent the most important one) we renormalize the stable and unstable directions to obtain the desired semihyperbolic behavior .

We describe the procedure only for the unstable case (the stable one is analogous). Fix $x \in \mathrm{inv}(\langle G \rangle, f)$. As $E_n^{u,s}$, for $n \in \mathbb{Z}$, we take $\mathbb{R} \cdot e_{f^n(x)}^{u,s}$ and define the operators

$$A_n \colon E_n^u \ni y \mapsto P_n^u D_{f^n(x)} f y \in E_{n+1}^u,$$

where P_n^u is the projection onto E_n^u with respect to the splitting $\mathbb{R}^2 = E_n^u \oplus E_n^s$. Then we renormalize the basis as in Observation 5.1 and Example 5.1. The same we apply for the stable direction.

STEP 4. Verification. Now it is enough to apply Proposition 4.3 to obtain the desired theoretical result proving the hyperbolicity of the invariant set.

One can find the program that verifies semihyperbolicity of the invariant set for a Hénon map with a short description and program documentation on the webpage of our project [17]. It is a short, object-oriented package written in C++ language. It widely uses Standard Template Library (STL) which makes it easy to utilize for people familiar with STL. It also employs interval arithmetic from Boost package [4]. The algorithms developed inside the package work on abstract objects like cubes, sets, etc. making it better to understand. The program itself is divided into a few independent parts. This allows easy and fast modifications and further development of particular fragments like finding cubic representation of the invariant set or approximating expanding and retracting directions.

We are able to prove hyperbolicity of the Hénon attractor for $a = 5.4$ and $b = -1$ in less then 10 seconds on 2GHz machine.

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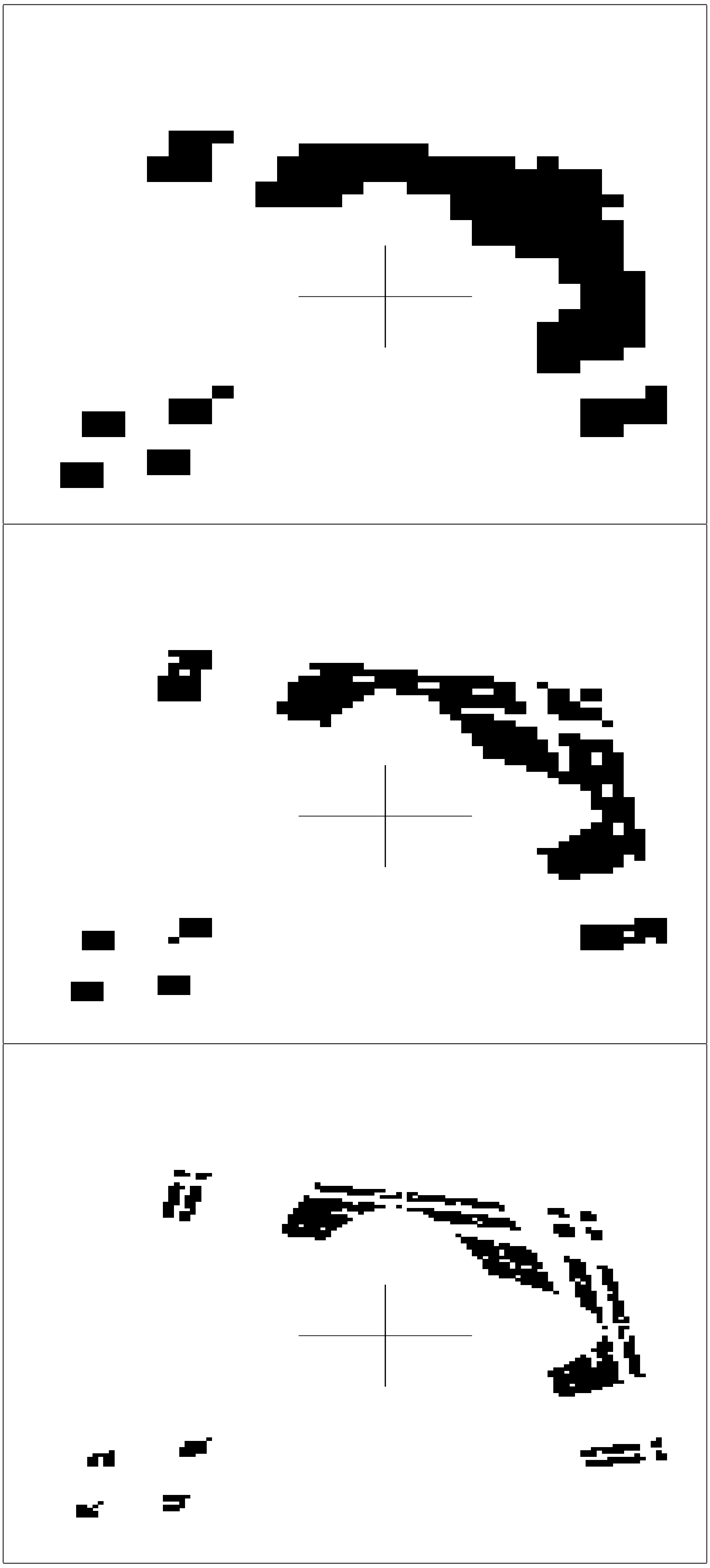

FIGURE 2. Consecutive approximations of the invariant set.

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E-mail address, Marcin Mazur: `Marcin.Mazur@im.uj.edu.pl`

E-mail address, Jacek Tabor: `tabor@ii.uj.edu.pl`