SEMI-HYPERBOLICITY AND HYPERBOLICITY

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Abstract. We prove that for $\mathcal{C}^1$-diffeomorfisms semi-hyperbolicity of an invariant set implies its hyperbolicity. Moreover, we provide some exact estimations of hyperbolicity constants by semi-hyperbolicity ones, which can be useful in strict numerical computations.

1. Introduction

As is well-known [7] hyperbolicity is one of the most important notions in the modern theory of dynamical systems. It follows from the fact that, roughly speaking, a given dynamical system possesses a highly stable behaviour in a neighbourhood of a hyperbolic invariant set.

It is thus not surprising that investigation of hyperbolic systems and sets occupies so many mathematical attention. However, the problem with hyperbolicity condition lies in the fact that it is very hard to verify by strict numerical computation (the reason behind it is that the invariance of the splitting, which arises in the definition, defies numerical verification).

Let us illustrate the above considerations with the example of the Hénon map (one of the simplest and oldest mapping showing chaotic behaviour)

$$H_{a,b}\colon \mathbb{R}^2 \ni (x,y) \mapsto (a - x^2 + by, x) \in \mathbb{R}^2.$$

The first proof of hyperbolicity of the Hénon map was obtained in 1979 by Devaney and Nitecki [4], who showed that for any fixed b and sufficiently large a the non-wandering set is hyperbolic and chaotic. However, up till now the Hénon map rejected the efforts to verify its hyperbolicity with the use of computer assisted proof. Recently, Arai [3] showed that for a large number of parameters a and b the Hénon map is quasi-hyperbolic (it is a weaker version of hyperbolicity which guarantees "real" hyperbolicity only on the non-wandering set).

Our aim is to provide some possible tools which (we hope) will enable to deal with the hyperbolicity of an invariant set. Instead of the notion of quasi-hyperbolicity, we investigate the notion of semi-hyperbolicity (see the next section for details), which is also well-adapted to numerical verifiction. However, the greatest advantage of semi-hyperbolicity over quasi-hyperbolicity is that it guarantees classical hyperbolicity condition.

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2. Hyperbolic and semi-hyperbolic mappings

For the convenience of the reader and to establish notation we recall some known definitions and theorems. The concept of semi-hyperbolicity for linear operators was introduced in [1] and for maps (even for only locally Lipschitz maps) in [5]. However, we would like to stress that we slightly modify notation to be suitable for our setting.

Let $(E, \|\cdot\|)$ be a Banach space.

Definition 2.1. We say that a linear operator $A\colon E \to E$ is (λ_s, λ_u)-*hyperbolic* (for $\lambda_s < 1 < \lambda_u$) if

$$\sigma(A) \cap \{\lambda \in \mathbb{C} \mid \lambda_s < |\lambda| < \lambda_u\} = \emptyset.$$

Obviously, A is hyperbolic if it is (λ_s, λ_u)-hyperbolic with some $\lambda_s < 1 < \lambda_u$.

It is well-known that if an operator is hyperblic then there exists a uniquely determined A-invariant splitting of E into $E^s \oplus E^u$ such that

$$\sigma(A^s) = \{\lambda \in \sigma(A) \mid |\lambda| < 1\},\ \sigma(A^u) = \{\lambda \in \sigma(A) \mid |\lambda| > 1\},$$

where $A^s = A|_{E^s}$ and $A^u = A|_{E^u}$.

We say that a hyperbolic operator A is $(\lambda_s, \lambda_u; C)$-*hyperbolic* if

$$\|(A^s)^k\| \leq C\lambda_s^k,\ \|(A^u)^{-k}\| \leq C\lambda_u^{-k} \quad \text{for } k \in \mathbb{N}.$$

It is well-known that A is (λ_s, λ_u)-hyperbolic if and only if for every sufficiently small $\varepsilon > 0$ there exist $C \geq 1$ such that A is $(\lambda_s + \varepsilon, \lambda_u - \varepsilon; C)$-hyperbolic.

Let us notice that, contrary to the hyperbolicity of the operator which depends only on the spectrum of A, the constant C from the above definition is dependent on chose of the particular norm on E (for example one can always change the norm on E into an equivalent one so that A is $(\lambda_s + \varepsilon, \lambda_u - \varepsilon; 1)$-hyperbolic in this new norm).

Now we proceed to the analogues of the above definition for C^1-diffeomorphisms. Let M be a compact Riemannian manifold (with Riemanian norm $\|\cdot\|$ on TM) and let $f\colon M \to M$ be a C^1-diffeomorphism. An f-invariant subset K of M is said to be (λ_s, λ_u)-*hyperbolic* ($\lambda_s < 1 < \lambda_u$) if there exists a Riemannian norm $\|\!|\cdot|\!\|$ on TM such that for each $x \in K$ there exists a splitting $T_xM = E_x^s \oplus E_x^u$ with corresponding projections P_x^s and $P_x^u = I - P_x^s$ satisfying the following properties

- H0. $P^s\colon K \ni x \to P_x^s$ is continuous (continuity of the splitting);
- H1. $D_xfE_x^{s,u} = E_{f(x)}^{s,u}$ for $x \in K$ (the invariance of the splitting);
- H2. $|\!|\!|D_xf|_{E_x^s}|\!|\!| \leq \lambda_s$ and $|\!|\!|(D_xf|_{E_x^u})^{-1}|\!|\!| \leq \lambda_u^{-1}$ for $x \in K$.

Obviously, K is hyperbolic if there exist constants λ_s, λ_u such that K is (λ_s, λ_u)-hyperbolic. As is known [9], condition H0 in the above definition is implied by boundedness of P^s and P^u.

One can also prove that K is hyperbolic if and only if conditions H0, H1 hold and there exists $C > 0$ such that

- H2'. $\|D_{f^{k-1}(x)} \circ \cdots \circ D_xf|_{E_x^s}\| \leq C\lambda_s^k$ and $\|(D_{f^{k-1}(x)} \circ \cdots \circ D_xf|_{E_x^u})^{-1}\| \leq C\lambda_u^{-k}$ for $x \in K$.

(Here we use the original Riemannian norm on TM.) In this case we say that K is $(\lambda_s, \lambda_u; C)$-*hyperbolic*.

Roughly speaking, an operator is semi-hyperbolic if it is nearly hyperbolic in some equivalent norm – in fact, we do not assume the exact invariance of the splitting. We proceed to formal definitions.

Definition 2.2. We say that a linear operator $A\colon E \to E$ is $(\lambda_s, \lambda_u, \mu_s, \mu_u; h)$-*semi-hyperbolic* if there exists a splitting $E = E^s \oplus E^u$ with corresponding projections P^s and $P^u = I - P^s$ satisfying the following properties

SH0. $\lambda_s < 1 < \lambda_u$ and $(1-\lambda_s)(\lambda_u - 1) > \mu_s\mu_u$;
SH1. $\max\{\|P^s\|, \|P^u\|\} \le h$;
SH2. $P^u A|_{E^u}$ is invertible and $\|P^s A|_{E^s}\| \le \lambda_s$, $\|(P^u A|_{E^u})^{-1}\| \le (\lambda_u)^{-1}$, $\|P^s A|_{E^u}\| \le \mu_s$ and $\|P^u A|_{E^s}\| \le \mu_u$.

We say that A is just $(\lambda_s, \lambda_u, \mu_s, \mu_u)$-*semi-hyperbolic*, if it is $(\lambda_s, \lambda_u, \mu_s, \mu_u; h)$-semi-hyperbolic with some h.

In the above notation an f-invariant subset K is said to be $(\lambda_s, \lambda_u, \mu_s, \mu_u; h)$-*semi-hyperbolic* if SH0 holds and for all $x \in K$ there exists a splitting $T_xM = E^s_x \oplus E^u_x$ with corresponding projections P^s_x and $P^u_x = I - P^s_x$ satisfying the following properties

S1. $\sup_{x\in K}\{\|P^s_x\|, \|P^u_x\|\} \le h$;
S2. $P^u_{f(x)} D_x f|_{E^u_x}$ is invertible and $\|P^s_{f(x)} D_x f|_{E^s_x}\| \le \lambda_s$, $\|(P^u_{f(x)} D_x f|_{E^u_x})^{-1}\| \le (\lambda_u)^{-1}$, $\|P^s_{f(x)} D_x f|_{E^u_x}\| \le \mu_s$ and $\|P^u_{f(x)} D_x f|_{E^s_x}\| \le \mu_u$ for all $x \in K$.

Note that neither the continuity nor the invariance of the splitting are not assumed in the definition of semi-hyperbolicity condition.

The following theorem is a direct consequence of the main result of this paper (Theorem 5.1).

Theorem. *Let M be a Riemannian manifold, let $f\colon M \to M$ be a C^1-diffeomorphism and let K be an invariant subset of M.*

Assume that K is $(\lambda_s, \lambda_u, \mu_s, \mu_u; h)$-semi-hyperbolic. Let γ^*_s, γ^*_u be arbitrary reals such that

$$\tfrac{\lambda_s+\lambda_u}{2} - \tfrac{\sqrt{(\lambda_u-\lambda_s)^2-4\mu_s\mu_u}}{2} < \gamma^*_s < 1 < \gamma^*_u < \tfrac{\lambda_s+\lambda_u}{2} + \tfrac{\sqrt{(\lambda_u-\lambda_s)^2-4\mu_s\mu_u}}{2}.$$

Let

$$C^* = \max\{\tfrac{(\lambda_u-\lambda_s+\mu_s+\mu_u)h}{(\gamma^*_s-\lambda_s)(\lambda_u-\gamma^*_s)-\mu_s\mu_u}, \tfrac{(\lambda_u-\lambda_s+\mu_s+\mu_u)h}{(\gamma^*_u-\lambda_s)(\lambda_u-\gamma^*_u)-\mu_s\mu_u}\}.$$

Then the set K is $(\gamma^*_s, \gamma^*_u; C^*)$-hyperbolic.

Before proceeding further, we would like to comment that the fact that semi-hyperbolicity implies hyperbolicity is essentially known. However, it is not easy to give a printed reference and up to our knowledge this result is a "mathematical folklore" (for the idea of the general proof see [6], for the linear case see [8]).

What is crucial from our point of view is strictly numerical character of the main result (the calculation of constants $\lambda^*_s, \lambda^*_u, C^*$). We hope that this estimations will help us in our future work on strict numerical verification of the hyperbolicity condition via the semi-hyperbolicity approach.

3. Linear case

Let E be a Banach space and let $A\colon E \to E$ be a bounded linear operator. The aim of this section is to generalize some results showing that semi-hyperbolicity implies hyperbolicity [2, 8]. Let us quote the following

Theorem [2, Th. 2]**.** *If A is a $(\lambda_s, \lambda_u, \mu_s, \mu_u)$-semi-hyperbolic operator with the splitting $E = E^s \oplus E^u$, then A is $(1-\gamma, 1+\gamma)$-hyperbolic with*

$$\gamma = \min\left\{1, \frac{1}{2}(\lambda_u - \lambda_s - \sqrt{(\lambda_u-\lambda_s)^2 - 4(1-\lambda_s)(\lambda_u-1) + 4\mu_s\mu_u})\right\}.$$

To obtain our results we will need the following proposition

Proposition 3.1. *If A is $(\lambda_s, \lambda_u, \mu_s, \mu_u)$-semi-hyperbolic operator then A is $(\lambda_s^*, \lambda_u^*)$-hyperbolic, with*

$$\begin{aligned} \lambda_s^* &= \frac{\lambda_s + \lambda_u}{2} - \frac{\sqrt{(\lambda_u - \lambda_s)^2 - 4\mu_s\mu_u}}{2}, \\ \lambda_u^* &= \frac{\lambda_s + \lambda_u}{2} + \frac{\sqrt{(\lambda_u - \lambda_s)^2 - 4\mu_s\mu_u}}{2}. \end{aligned} \tag{1}$$

Before proceeding to the proof let us remark, that Proposition 3.1 shows some improvement over the previous theorem, as it may be directly verified that $\lambda_s^* < 1-\gamma$ and $1+\gamma < \lambda_u^*$.

Proof of Proposition 3.1. Take a semi-hyperbolic splitting $E = E^s \oplus E^u$, corresponding projections P^s and P^u. Put

$$A_\lambda = \frac{1}{\lambda} A \text{ for any } \lambda \in \mathbb{C} \setminus \{0\}.$$

Obviously, if $\lambda \in \sigma(A)$ then A_λ is not hyperbolic. Since

$$\|P^s A_\lambda|_{E^s}\| = \frac{\|P^s A|_{E^s}\|}{|\lambda|} \le \frac{\lambda_s}{|\lambda|}$$

and, analogously,

$$\|P^s A_\lambda|_{E^u}\| \le \frac{\mu_s}{|\lambda|}, \quad \|P^u A_\lambda|_{E^s}\| \le \frac{\mu_u}{|\lambda|} \text{ and } \|(P^u A_\lambda|_{E^u})^{-1}\| \le \frac{|\lambda|}{\lambda_u},$$

we conclude that the operator A_λ is semi-hyperbolic if the following inequalities hold

$$\lambda_s < |\lambda| < \lambda_u \quad \text{and} \quad \mu_s\mu_u < (|\lambda| - \lambda_s)(\lambda_u - |\lambda|).$$

Solving the above system we obtain

$$|\lambda| \in (\lambda_s^*, \lambda_u^*).$$

The proof is completed by the following sequence of implications

$$|\lambda| \in (\lambda_s^*, \lambda_u^*) \Rightarrow A_\lambda \text{ is semi-hyperbolic} \Rightarrow A_\lambda \text{ is hyperbolic} \Rightarrow \lambda \notin \sigma(A).$$

□

At the end of this section we would like to fix our attention on the inverse problem. Thus we would like to ask the following question

Problem 3.1. *Suppose that the operator A is $(\lambda_s^*, \lambda_u^*)$-hyperbolic.*

Does there exist an equivalent norm on E such that for arbitrary $\lambda_s \in [0, \lambda_s^*]$ the operator A is $(\lambda_s, \lambda_u, \mu_s, \mu_u)$-semi-hyperbolic, for some λ_u, μ_s, μ_u for which (1) *holds?*

Roughly speaking we ask if we can "decrease" the value of λ_s^* by "putting" it into the not invariant splitting (according to the semi-hyperbolicity condition). Of course, one can also ask the dual question for $\lambda_u \in [\lambda_u^*, \infty)$ but since this is an analogue, we consider only the case of λ_s.

Let us first show that under no additional assumptions the answer for the above question is negative.

Example 3.1. Consider the mapping

$$A\colon \mathbb{R} \ni x \to x/2 \in \mathbb{R}.$$

Then A is $(1/2,\infty)$-hyperbolic (the space E_u is the zero space). However, one can easily notice that there does not exist $\lambda_s < 1/2$ and λ_u, μ_s, μ_u such that (1) holds and that A is $(\lambda_s, \lambda_u, \mu_s, \mu_u)$-semi-hyperbolic.

We show that in some cases, the inverse holds.

Example 3.2. Let $\lambda_s^* \in [0,1), \lambda_u^* \in (1,\infty)$. Consider the mapping $A\colon \mathbb{R}^2 \to \mathbb{R}^2$ given in the matrix form by

$$A := \begin{bmatrix} \lambda_s^* & 0 \\ 0 & \lambda_u^* \end{bmatrix}.$$

Then A is $(\lambda_s^*, \lambda_u^*)$-hyperbolic with the splitting $\mathbb{R}^2 = (\mathbb{R}\times\{0\}) \oplus (\{0\}\times\mathbb{R})$.

To show that in this case the answer to Problem 3.1 is positive, let us fix $b_1, b_2 \in \mathbb{R}$ such that

$$b_1 \cdot b_2 \in [0, \lambda_s^*/\lambda_u^*].$$

Consider the operator $B\colon \mathbb{R}^2 \to \mathbb{R}^2$, defined, in the matrix form, by

$$B = \begin{bmatrix} 0 & b_1 \\ b_2 & 0 \end{bmatrix}.$$

Put

$$E^s = (\mathrm{Id} - B)(\mathbb{R}\times\{0\}) \quad \text{and} \quad E^u = (\mathrm{Id} - B)(\{0\}\times\mathbb{R}),$$

where Id denotes the identity operator. It is easy to see that the projections corresponding to the splitting $E^s \oplus E^u$ are given by

$$P^s = \frac{1}{1-b_1b_2}\begin{bmatrix} 1 & b_1 \\ -b_2 & -b_1b_2 \end{bmatrix} \quad \text{and} \quad P^u = \frac{1}{1-b_1b_2}\begin{bmatrix} -b_1b_2 & -b_1 \\ b_2 & 1 \end{bmatrix}.$$

Then, choosing a norm $\|\cdot\|$ on $\mathbb{R}^2$ such that $\|e^s\| = \|e^u\| = 1$, where $e^s = [1,-b_2]^T \in E^s$ and $e^u = [-b_1, 1]^T \in E^u$, we obtain

$$\|P^sAe^s\| = \left|\frac{\lambda_s^* - \lambda_u^* b_1b_2}{1-b_1b_2}\right| \|e^s\| = \frac{\lambda_s^* - \lambda_u^* b_1b_2}{1-b_1b_2} = \frac{1-\frac{\lambda_u^*}{\lambda_s^*}b_1b_2}{1-b_1b_2}\lambda_s^* \le \lambda_s^* < 1,$$

$$\|P^uAe^s\| = \left|\frac{\lambda_s^* b_2 - \lambda_u^* b_2}{1-b_1b_2}\right| \|e^u\| = \frac{\lambda_u^* - \lambda_s^*}{1-b_1b_2}|b_2|,$$

$$\|P^sAe^u\| = \left|\frac{\lambda_u^* b_1 - \lambda_s^* b_1}{1-b_1b_2}\right| \|e^s\| = \frac{\lambda_u^* - \lambda_s^*}{1-b_1b_2}|b_1|,$$

$$\|P^uAe^u\| = \left|\frac{\lambda_u^* - \lambda_s^* b_1b_2}{1-b_1b_2}\right| \|e^u\| = \frac{\lambda_u^* - \lambda_s^* b_1b_2}{1-b_1b_2} = \frac{1-\frac{\lambda_s^*}{\lambda_u^*}b_1b_2}{1-b_1b_2}\lambda_u^* \ge \lambda_u^* > 1.$$

Thus, putting

$$\lambda_s = \frac{\lambda_s^* - \lambda_u^* b_1b_2}{1-b_1b_2}, \quad \lambda_u = \frac{\lambda_u^* - \lambda_s^* b_1b_2}{1-b_1b_2}, \quad \mu_s = \frac{\lambda_u^* - \lambda_s^*}{1-b_1b_2}|b_1| \ \text{and} \ \mu_u = \frac{\lambda_u^* - \lambda_s^*}{1-b_1b_2}|b_2|,$$

one can easily check, by direct computations, that (1) holds and that

$$\lambda_s \in [0, \lambda_s^*], \quad \mu_s\mu_u < (1-\lambda_s)(\lambda_u - 1).$$

We have obtained even more then we wanted – by modifying the values of b_1 and b_2 we can not only realize any value $\lambda_s \in [0, \lambda_s^*]$, but we control also one of the constants μ_s, μ_u.

The above example leads us to the following

Conjecture 3.1. *Let A be a hyperbolic operator such that the spaces E^s and E^u are isomorphic. Then the answer to Problem 3.1 is positive.*

4. Functional calculus and its consequences

Since we obtain our results with the use of functional calculus, we recall here some of its consequences which are important for us. We also establish some notation which is valid from now on.

By $S_r = \{\lambda \in \mathbb{C} \mid |\lambda| = r\}$ we denote the positively oriented circle. Let A be $(\lambda_s^*, \lambda_u^*)$-hyperbolic operator. We put

$$P_*^s = \frac{1}{2\pi i}\int_{S_r} (\lambda I - A)^{-1}\, d\lambda \ \text{ and } \ P_*^u = I - P_*^s \quad \text{for } r \in (\lambda_s^*, \lambda_u^*). \tag{2}$$

Then P_*^s and P_*^u does not depend on the choice of $r \in (\lambda_s^*, \lambda_u^*)$. Moreover, P_*^s and P_*^u are the projections corresponding to the unique splitting from the definition of hyperbolicity (we have $E = E_*^s \oplus E_*^u$ for $E_*^{s,u} = P_*^{s,u}(E)$).

Lemma 4.1. *Let A be $(\lambda_s, \lambda_u, \mu_s, \mu_u; h)$-semi-hyperbolic operator (hence hyperbolic). Then*

$$\|(\lambda \mathrm{Id} - A)^{-1}\| \le \frac{\lambda_u - \lambda_s + \mu_s + \mu_u}{(|\lambda| - \lambda_s)(\lambda_u - |\lambda|) - \mu_s\mu_u} h \quad \text{for } \lambda \in \mathbb{C}, |\lambda| \in (\lambda_s^*, \lambda_u^*),$$

where λ_s^* and λ_u^* are given by (1).

Proof. Take $x, y \in E$, $\lambda \in \mathbb{C}$ such that

$$|\lambda| \in (\lambda_s^*, \lambda_u^*) \quad \text{and} \quad (\lambda \mathrm{Id} - A)x = y.$$

Then $\lambda x = y + Ax$ and hence

$$\lambda P^s x = P^s y + P^s A P^s x + P^s A P^u x,$$
$$\lambda P^u x = P^u y + P^u A P^s x + P^u A P^u x.$$

Since

$$U = P^u A|_{E^u} : E^u \to E^u$$

is an invertible operator with $\|U^{-1}\| \le (\lambda_u)^{-1}$, we obtain

$$P^u x = U^{-1}(\lambda P^u x - P^u y - P^u A P^s x).$$

It follows that

$$|\lambda| \|P^s x\| \le h\|y\| + \lambda_s \|P^s x\| + \mu_s \|P^u x\|,$$
$$\lambda_u \|P^u x\| \le h\|y\| + |\lambda| \|P^u x\| + \mu_u \|P^s x\|$$

and, in consequence,

$$(|\lambda| - \lambda_s)\|P^s x\| - \mu_s \|P^u x\| \le h\|y\|,$$
$$(\lambda_u - |\lambda|)\|P^u x\| - \mu_u \|P^s x\| \le h\|y\|.$$

Then

$$\mu_u(|\lambda| - \lambda_s)\|P^s x\| - \mu_s\mu_u \|P^u x\| \le \mu_u h \|y\|,$$
$$(|\lambda| - \lambda_s)(\lambda_u - |\lambda|)\|P^u x\| - \mu_u(|\lambda| - \lambda_s)\|P^s x\| \le (|\lambda| - \lambda_s) h \|y\|,$$

and hence

$$\|P^u x\| \le \frac{\mu_u + |\lambda| - \lambda_s}{(|\lambda| - \lambda_s)(\lambda_u - |\lambda|) - \mu_s\mu_u} h \|y\|.$$

Analogously,

$$\|P^s x\| \leq \frac{\mu_s + \lambda_u - |\lambda|}{(|\lambda| - \lambda_s)(\lambda_u - |\lambda|) - \mu_s\mu_u} h\|y\|.$$

Let us note that the above calculations are based on the following estimates

$$\lambda_s < |\lambda| < \lambda_u \quad \text{and} \quad \mu_s\mu_u < (|\lambda| - \lambda_s)(\lambda_u - |\lambda|),$$

which can be easily verified as in the proof of Proposition 3.1. Thus we have

$$\|(\lambda \mathrm{Id} - A)^{-1}y\| = \|x\| \leq \|P^s x\| + \|P^u x\| \leq \frac{\lambda_u - \lambda_s + \mu_s + \mu_u}{(|\lambda| - \lambda_s)(\lambda_u - |\lambda|) - \mu_s\mu_u} h\|y\|.$$

□

As a direct consequence of Lemma 4.1 and (2) we obtain the following

Corollary 4.2. *Let A be $(\lambda_s, \lambda_u, \mu_s, \mu_u; h)$-semi-hyperbolic operator and let*

$$L := \frac{\lambda_u - \lambda_s + \mu_s + \mu_u}{(1 - \lambda_s)(\lambda_u - 1) - \mu_s\mu_u} h.$$

Then

$$\|P^s_*\| \leq L,\ \|P^u_*\| \leq L + 1.$$

Let, as before, A be $(\lambda^*_s, \lambda^*_u)$-hyperbolic operator and let $r \in (\lambda^*_s, \lambda^*_u)$. Our next theorem deals with the estimation of iterates of A on spaces E^s_* and E^u_*. We will use the following consequences of functional calculus

$$A^k x_s = \int_{S_r} \lambda^k (\lambda \mathrm{Id} - A)^{-1} x_s d\lambda \quad \text{for } x_s \in E^s_*, k \in \mathbb{N},$$

$$(A|_{E^u_*})^{-k}(x_u) = -\int_{S_r} \lambda^{-k} (\lambda \mathrm{Id} - A)^{-1} x_u d\lambda \quad \text{for } x_u \in E^s_*, k \in \mathbb{N}, k \geq 1.$$

The main result of this section, which gives exact estimations of iterates of A on subspaces E^s_*, E^u_*, is a trivial corollary of the above equalities and Lemma 4.1.

Theorem 4.3. *Let A be $(\lambda_s, \lambda_u, \mu_s, \mu_u; h)$-semi-hyperbolic operator, let λ^*_s, λ^*_u be given by* (1) *and let $r \in (\lambda^*_s, \lambda^*_u)$ be arbitrary.*

We put

$$C_r := \frac{\lambda_u - \lambda_s + \mu_s + \mu_u}{(r - \lambda_s)(\lambda_u - r) - \mu_s\mu_u} h$$

Then

$$\|(A|_{E^s_*})^k\| \leq C_r \cdot r^k, \ \ \|(A|_{E_u})^{-k}\| \leq C_r \cdot r^{-k} \quad \textit{for } k \in \mathbb{N}, k \geq 1.$$

Clearly, if we are interested in the estimation of $\|(A|_{E^s_*})^k\|$ we should take $r \in (\lambda^*_s, 1)$, and if want to estimate $\|(A|_{E_u})^{-k}\|$ we should take $r \in (1, \lambda^*_u)$.

5. General case

Let M be a compact Riemannian manifold and let $f\colon M \to M$ be a C^1-diffeomorphism. For $x \in M$ consider the Banach space

$$\mathcal{E}_x = \{\mathrm{v} = (v_n) \in \prod_{n\in\mathbb{Z}} T_{f^n(x)}M \mid \sup_{n\in\mathbb{Z}} \|v_n\|_{x,n} < \infty\},$$

endowed with the supremum norm

$$\|\mathrm{v}\|^\infty_x = \sup_{n\in\mathbb{Z}} \|v_n\|_{x,n}.$$

Here and subsequently $\|\cdot\|_{x,n}$ denotes, for each $n \in \mathbb{Z}$, an underlying Riemannian norm on $T_{f^n(x)}M$.

We define the bounded operator $\mathcal{A}_x$ on $\mathcal{E}_x$ by the following formula

$$(\mathcal{A}_x \mathrm{v})_{n+1} = D_{f^n(x)} f v_n \quad \text{for } \mathrm{v} \in \mathcal{E}_x.$$

From the above definition it directly follows that we can identify the operator $\mathcal{A}_x$ with $\mathcal{A}_y$ in the case of x and y lying on the same trajectory of f.

Now we are ready to present the main result of the paper.

Theorem 5.1. *Assume that K is a compact invariant subset of M.*

(i) *If the set K is $(\lambda_s, \lambda_u, \mu_s, \mu_u; h)$-semi-hyperbolic for f then for every $x \in K$ the operator $\mathcal{A}_x$ is $(\lambda_s, \lambda_u, \mu_s, \mu_u, h)$-semi-hyperbolic according to the norm $\|\cdot\|_x^\infty$.*

(ii) *Assume that for every $x \in K$ the operator $\mathcal{A}_x$ is $(\lambda_s, \lambda_u, \mu_s, \mu_u; h)$-semi-hyperbolic according to the norm $\|\cdot\|_x^\infty$. Let λ_s^*, λ_u^* be given by (1) and let γ_s^*, γ_u^* be arbitrary reals such that*

$$\lambda_s^* < \gamma_s^* < 1 < \gamma_u^* < \lambda_u^*.$$

Let

$$C^* = \max\{\frac{(\lambda_u - \lambda_s + \mu_s + \mu_u)h}{(\gamma_s^* - \lambda_s)(\lambda_u - \gamma_s^*) - \mu_s\mu_u}, \frac{(\lambda_u - \lambda_s + \mu_s + \mu_u)h}{(\gamma_u^* - \lambda_s)(\lambda_u - \gamma_u^*) - \mu_s\mu_u}\}.$$

Then the set K is $(\gamma_s^*, \gamma_u^*; C^*)$-hyperbolic.

Proof. (i) Take $x \in K$ and, for each $n \in \mathbb{Z}$, projections $P^s_{x,n}$ and $P^u_{x,n}$ corresponding to a given semi-hyperbolic splitting $T_{f^n(x)}M = E^s_{x,n} \oplus E^u_{x,n}$. Then it is easy to see that the operators $\mathcal{P}^s_x$ and $\mathcal{P}^u_x$, defined by

$$\mathcal{P}^s_x = \prod_{n\in\mathbb{Z}} P^s_{x,n} \quad \text{and} \quad \mathcal{P}^u_x = \prod_{n\in\mathbb{Z}} P^u_{x,n},$$

are bounded projections inducing a $(\lambda_s, \lambda_u, \mu_s, \mu_u; h)$-semi-hyperbolic splitting of the space $\mathcal{E}_x$ for the operator $\mathcal{A}_x$.

(ii) Fix $\gamma_s^* \in (\lambda_s^*, 1)$, $\gamma_u^* \in (1, \lambda_u^*)$ and take $x \in K$. Then we obtain a hyperbolic splitting $\mathcal{E}_x = \mathcal{E}^s_x \oplus \mathcal{E}^u_x$, corresponding projections $\mathcal{P}^s_x$ and $\mathcal{P}^u_x$. By Corollary 4.2 and Theorem 4.3 we obtain that

$$\|(\mathcal{A}_x|_{\mathcal{E}^s_x})^k\|_x^\infty \le C^*(\gamma_s^*)^k, \;\; \|(\mathcal{A}_x|_{\mathcal{E}^u_x})^{-k}\|_x^\infty \le C^*(\gamma_u^*)^{-k} \text{ for all } k > 0,$$
$$\max\{\|\mathcal{P}^s_x\|_x^\infty, \|\mathcal{P}^u_x\|_x^\infty\} \le L + 1.$$

where

$$L = \frac{\lambda_u - \lambda_s + \mu_s + \mu_u}{(1-\lambda_s)(\lambda_u - 1) - \mu_s\mu_u} h,$$

$$C^* = \max\{\frac{(\lambda_u - \lambda_s + \mu_s + \mu_u)h}{(\gamma_s^* - \lambda_s)(\lambda_u - \gamma_s^*) - \mu_s\mu_u}, \frac{(\lambda_u - \lambda_s + \mu_s + \mu_u)h}{(\gamma_u^* - \lambda_s)(\lambda_u - \gamma_u^*) - \mu_s\mu_u}\}.$$

For each $n \in \mathbb{Z}$ put

$$E_{x,n} = T_{f^n(x)}M, \;\; A_{x,n} = D_{f^n(x)}f$$

and consider the spaces $E^s_{x,n}, E^u_{x,n} \subset E_{x,n}$ defined as follows

$$E^s_{x,n} = \{v \in E_{x,n} \mid \sup_{k\ge 0} \|A_{x,n+k-1}\cdots A_{x,n}v\|_{x,n+k} < \infty\},$$

$$E^u_{x,n} = \{v \in E_{x,n} \mid \sup_{k\ge 0} \|A^{-1}_{x,n-k}\cdots A^{-1}_{x,n-1}v\|_{x,n-k} < \infty\}.$$

Obviously, since $A_{x,n}$ is invertible, $A_{x,n}(E^s_{x,n}) = E^s_{x,n+1}$ and $A_{x,n}(E^u_{x,n}) = E^u_{x,n+1}$.

Firstly note that $P_x^n(\mathcal{E}_x^s) \subset E_{x,n}^s$ and $P_x^n(\mathcal{E}_x^u) \subset E_{x,n}^u$, where P_x^n denotes the canonical projection on the n-th coordinate in $\prod_{n\in\mathbb{Z}} E_{x,n}$. Indeed, taking sequences $\mathrm{v}^s \in \mathcal{E}_x^s$ and $\mathrm{v}^u \in \mathcal{E}_x^u$ it is easily seen that for $k \in \mathbb{N}$, $k \geq 1$

$$\|A_{x,n+k-1}\cdots A_{x,n}v_n^s\|_{x,n+k} \leq \|\mathcal{A}_x^k \mathrm{v}^s\|_x^\infty \leq C^*(\gamma_s^*)^k \|\mathrm{v}^s\|_x^\infty \leq C^*\|\mathrm{v}^s\|_x^\infty,$$
$$\|A_{x,n-k}^{-1}\cdots A_{x,n-1}^{-1}v_n^u\|_{x,n-k} \leq \|\mathcal{A}_x^{-k}\mathrm{v}^u\|_x^\infty \leq C^*(\gamma_u^*)^{-k}\|\mathrm{v}^u\|_x^\infty \leq C^*\|\mathrm{v}^u\|_x^\infty.$$

Now we show that, for each $n \in \mathbb{Z}$, the spaces $E_{x,n}^s$ and $E_{x,n}^u$ form splitting of the space E_n.

Indeed, if $v \in E_{x,n}^s \cap E_{x,n}^u$ then

$$\mathrm{v} = (\ldots, A_{x,n-2}^{-1}A_{x,n-1}^{-1}v, A_{x,n-1}^{-1}v, v_n = v, A_{x,n}v, A_{x,n+1}A_{x,n}v, \ldots) \in \mathcal{E}_x$$

and since $\mathcal{A}_x$ is hyperbolic and $\mathcal{A}_x\mathrm{v} = \mathrm{v}$ we obtain $\mathrm{v} = 0$. The equality $E_{x,n} = E_{x,n}^s + E_{x,n}^u$ follows immediately from the following observation

$$v = v_n = (\mathcal{P}_x^s\mathrm{v} + \mathcal{P}_x^u\mathrm{v})_n = (\mathcal{P}_x^s\mathrm{v})_n + (\mathcal{P}_x^u\mathrm{v})_n \in P_x^n(\mathcal{E}_x^s) + P_x^n(\mathcal{E}_x^u) \subset E_{x,n}^s + E_{x,n}^u$$

for any $\mathrm{v} \in \mathcal{E}_x$ with $v_n = v \in E_{x,n}$, e.g., $\mathrm{v} = (\ldots, 0, v_n = v, 0, \ldots)$.

It finishes the proof that $E_{x,n} = E_{x,n}^s \oplus E_{x,n}^u$.

Finally, we also obtain that

$$\mathcal{P}_x^s = \prod_{n\in\mathbb{Z}} P_{x,n}^s, \quad \text{and} \quad \mathcal{P}_x^u = \prod_{n\in\mathbb{Z}} P_{x,n}^u,$$

as the consequence of the earlier remarks and the following sequence of implications that hold for each $n \in \mathbb{Z}$

$$\begin{aligned} \mathrm{v} \in \mathcal{E}_x \quad &\Rightarrow \quad v_n = (\mathcal{P}_x^s\mathrm{v})_n + (\mathcal{P}_x^u\mathrm{v})_n \in E_{x,n}^s + E_{x,n}^u \\ &\Rightarrow \quad P_{n,x}^s v_n = (\mathcal{P}_x^s\mathrm{v})_n \text{ and } P_{n,x}^u v_n = (\mathcal{P}_x^u\mathrm{v})_n. \end{aligned}$$

In particular, if $\mathrm{v} = (\ldots, 0, v_n = v, 0, \ldots)$ then

$$\mathcal{P}_x^s\mathrm{v} = (\ldots, 0, P_{x,n}^s v, 0, \ldots) \quad \text{and} \quad \mathcal{P}_x^u\mathrm{v} = (\ldots, 0, P_{x,n}^u v, 0, \ldots).$$

Now we prove the (γ_s^*, γ_u^*)-hyperbolicity of the splitting $E_{x,n} = E_{x,n}^s \oplus E_{x,n}^u$.

To do this take $v^s \in E_{x,n}^s, v^u \in E_{x,n}^u$, corresponding sequences

$$\mathrm{v}^s = (\ldots, 0, v_n^s = v^s, 0, \ldots) = (\ldots, 0, P_{x,n}^s v^s, 0, \ldots) = \mathcal{P}_x^s\mathrm{v}^s \in \mathcal{E}_x^s,$$
$$\mathrm{v}^u = (\ldots, 0, v_n^u = v^u, 0, \ldots) = (\ldots, 0, P_{x,n}^u v^u, 0, \ldots) = \mathcal{P}_x^u\mathrm{v}^u \in \mathcal{E}_x^u,$$

and note that the following estimates hold for all $k > 0$

$$\|A_{x,n+k-1}\cdots A_{x,n}v^s\|_{x,n+k} \leq \|\mathcal{A}_x^k\mathrm{v}^s\|_x^\infty \leq C^*(\gamma_s^*)^k\|\mathrm{v}^s\|_x^\infty = C^*(\gamma_s^*)^k\|v^s\|_{x,n},$$
$$\|A_{x,n-k}^{-1}\cdots A_{x,n-1}^{-1}v^u\|_{x,n-k} \leq \|\mathcal{A}_x^{-k}\mathrm{v}^u\|_x^\infty \leq C^*(\gamma_u^*)^{-k}\|\mathrm{v}^u\|_x^\infty = C^*(\gamma_u^*)^{-k}\|v^u\|_{x,n}.$$

Since the uniform boundedness of projections guarantees continuity [9], it is enough to verify that the projections $P_{x,n}^s$ and $P_{x,n}^u$, corresponding to the splitting $E_{x,n} = E_{x,n}^s \oplus E_{x,n}^u$, are uniformly bounded, i.e.

$$\sup_{x\in K, n\in\mathbb{Z}} \max\{\|P_{x,n}^s\|_{x,n}, \|P_{x,n}^u\|_{x,n}\} < \infty.$$

Indeed, for $v \in E_{x,n}$ and $\mathrm{v} = (\ldots, 0, v_n = v, 0, \ldots)$ we have

$$\mathcal{P}_x^s\mathrm{v} = (\ldots, 0, P_{x,n}^s v, 0, \ldots) \quad \text{and} \quad \mathcal{P}_x^u\mathrm{v} = (\ldots, 0, P_{x,n}^u v, 0, \ldots)$$

and then we obtain the following estimates

$$\|P_{n,x}^s v\|_{x,n} = \|\mathcal{P}_x^s\mathrm{v}\|_x^\infty \leq \|\mathcal{P}_x^s\|_x^\infty\|\mathrm{v}\|_x^\infty = \|\mathcal{P}_x^s\|_x^\infty\|v\|_{x,n} \leq (L+1)\|v\|_{x,n},$$

$$\|P^u_{n,x}v\|_{x,n} = \|\mathcal{P}^u_x \mathrm{v}\|^\infty_x \leq \|\mathcal{P}^u_x\|^\infty_x \|\mathrm{v}\|^\infty_x = \|\mathcal{P}^u_x\|^\infty_x \|v\|_{x,n} \leq (L+1)\|v\|_{x,n}.$$

But L depends neither on $x \in K$ nor on $n \in \mathbb{Z}$, which makes the proof complete. $\square$

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