On semi-hyperbolicity

Jacek Tabor
Jagiellonian University

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Motivation

M a C^1-manifold, $f : M \to M$ a diffeomorphism. We are given an invariant subset S of M [that is $f(S) = S$].
Interested in: the behaviour of f in (the neighbourhood) of S.

A lot of information comes from the fact that f is hyperbolic on S.

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- important: guarantees stability, shadowing, etc.;
- hard to verify;

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Definition of hyperbolicity

By $\mathcal{T}_x M$ we denote the tangent space to x at the point M.

$$\mathcal{T}M = \bigcup_{x \in M} \mathcal{T}_x M, \; \mathcal{T}_S M = \bigcup_{x \in S} \mathcal{T}_x M.$$

We consider the linear mapping

$$Df : \mathcal{T}M \to \mathcal{T}M$$

restricted to $\mathcal{T}_S M$. Roughly speaking, we say that f is hyperbolic on an invariant set S if the linearization Df is hyperbolic on $\mathcal{T}_S M$.

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Definition of hyperbolic operator

We therefore go to the linear case. X is a Banach space and $A : X \to X$ is a linear operator.

Definition

We say that A is hyperbolic if there exists a splitting $X = X_s \oplus X_u$ and an equivalent norm on X such that

- X_s, X_u are A-invariant;
- there exist $\lambda_s < 1$, $\lambda_u > 1$ such that

$$\|Ax_s\| \leq \lambda_s \|x_s\|, \; \|Ax_u\| \geq \lambda_u \|x_u\| \quad \text{for } x_s \in X_s, x_u \in X_u.$$

One can easily see that if A is hyperbolic then A has no nontrivial bounded orbit. Small modification of a hyperbolic operator is still a hyperbolic one, but the invariant subspaces change.

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Reformulation:

Definition

We say that A is hyperbolic if there exists a splitting $X = X_s \oplus X_u$ and an equivalent norm on X such that

- A has the matrix form
$$\begin{bmatrix} A_s & 0 \\ 0 & A_u \end{bmatrix};$$
- there exist $\lambda_s < 1$, $\lambda_u > 1$ such that
$$\|A_s\| \leq \lambda_s,\ \|A_u^{-1}\| \leq \frac{1}{\lambda_u};$$

Henon

Open problem:
hyperbolicity of the Henon attractor for the classical parameter values.

Partial Answer: Z. Arai, *On Hyperbolic Plateaus of the Hénon Maps* [to appear in Experimental Mathematics].

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The use of the quasi-hyperbolicity.

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Definition

A linear mapping A is called quasi-hyperbolic if it does not have a nontrivial bounded orbit.

Zin Arai proved that the Henon attractor is quasi-hyperbolic for the large set of parameter values.

Theorem (Churchill et al. 77, Sacker and Sell 74)

Assume that $f|_\Lambda$ is chain recurrent. Then f is hyperbolic on Λ if and only if f is quasi-hyperbolic on it.

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How to deal with hyperbolicity?

One of possible solutions is semi-hyperbolicity [introduced by P. Diamond, P. E. Kloeden, V. S. Kozyakin, A. V. Pokrovskii]:

- easier to check;
- allows lipschitzian perturbations;

A.A. Al-Nayef, P.E. Kloeden, A.V. Pokrovskii, *Semi-hyperbolic mappings, condensing operators, and neutral delay equations,* J. Differential Equations **137** (1997), 320–339.

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$$\begin{bmatrix} \lambda_s & \mu_s \\ \mu_u & \lambda_u \end{bmatrix}$$

$$\lambda_s < 1 < \lambda_u \text{ and } (1-\lambda_s)(\lambda_u - 1) > \mu_s \mu_u$$

Definition of hyperbolicity

X a Banach space, $X = X_s \oplus X_u$. $A : X \to X$ a linear operator, and assume that X_s and X_u are A-invariant. Then in the matrix form we have

$$A = \begin{bmatrix} A_s & 0 \\ 0 & A_u \end{bmatrix}$$

Definition

We say that a linear operator A is hyperbolic if

- $\lambda_s < 1 < \lambda_u$;
- $\|A_s\| \leq \lambda_s$, $\|A_u^{-1}\| \geq \frac{1}{\lambda_u}$;
- $\|P_s\| \leq \mu_s$, $\|P_u\| \leq \mu_u$;

Definition of semi-hyperbolicity

X a Banach space, $X = X_s \oplus X_u$. $A : X \to X$ a linear operator, but we do not assume that X_s and X_u are A-invariant. Then in the matrix form we have

$$A = \begin{bmatrix} A_s & P_s \\ P_u & A_u \end{bmatrix}$$

Definition

We say that a linear operator A is semi-hyperbolic if

- $\lambda_s < 1 < \lambda_u$ and $(1 - \lambda_s)(\lambda_u - 1) > \mu_s \mu_u$;
- $\|A_s\| \leq \lambda_s$, $\|A_u^{-1}\| \geq \frac{1}{\lambda_u}$;
- $\|P_s\| \leq \mu_s$, $\|P_u\| \leq \mu_u$;

Basic results

We are ready to proceed to our investigation. Our aim is to present an approach to investigation of hyperbolic operators via semi-hyperbolicity. We obtain some numerical estimations of hyperbolicity constants (in the nonlinear case) with the use of semi-hyperbolicity.

M. Mazur, J. Tabor, P. Kocielniak, *Semi-hyperbolicity and hyperbolicity,* to appear in Discrete Contin. Dynam. Syst (IM UJ preprint 2007/04, http://www.im.uj.edu.pl/badania/preprinty/).

Let us begin with the following result

Theorem

If the operator A is semi-hyperbolic with respect $(\lambda_s, \lambda_u, \mu_s, \mu_u)$, then A is hyperbolic.

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Estimation of the spectrum

Theorem

We assume that A is semi-hyperbolic with respect to a split $(\lambda_s, \lambda_u, \mu_s, \mu_u)$. Then the spectrum of A does not intersect the ring

$$\mathcal{R}(\lambda_s^*, \lambda_u^*) := \{\lambda \in \mathbb{C} : \lambda_s^* < |\lambda| < \lambda_u^*\},$$

where

$$\begin{aligned} \lambda_s^* &= \frac{\lambda_s + \lambda_u}{2} - \frac{\sqrt{(\lambda_u - \lambda_s)^2 - 4\mu_s\mu_u}}{2} < 1, \\ \lambda_u^* &= \frac{\lambda_s + \lambda_u}{2} + \frac{\sqrt{(\lambda_u - \lambda_s)^2 - 4\mu_s\mu_u}}{2} > 1. \end{aligned} \quad (1)$$

Let us mention that the estimation obtained in (1) are sharp. To observe this, consider the linear operator $A : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$A = \begin{bmatrix} \lambda_s & \mu_s \\ \mu_u & \lambda_u \end{bmatrix}.$$

Obviously, A is semi-hyperbolic with respect to the the split from the above theorem, but the eigenvalues of A are exactly λ_s^* and λ_u^*.

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Projection constant

Let A be a semi-hyperbolic operator with the splitting $X = X_s \oplus X_u$. By P_s (P_u) we denote the projection onto X_s (X_u).
Then by the previous result we know that A is hyperbolic, and therefore there exists an A-invariant splitting $X = X_s^* \oplus X_u^*$. By P_s^* (P_u^*) we denote the projection onto X_s^* (X_u^*).

Theorem

Let A be a semi-hyperbolic operator with respect to $(\lambda_s, \lambda_u, \mu_s, \mu_u)$. Then

$$\|P_s^*\| \leq L, \; \|P_u^*\| \leq L + 1,$$

where

$$L = \frac{\lambda_u - \lambda_s + \mu_s + \mu_u}{(1-\lambda_s)(\lambda_u - 1) - \mu_s \mu_u} h,$$

and $h := \max\{\|P_s\|, \|P_u\|\}$.

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Main technical result

Theorem

Let S be a compact invariant subset of M.
Assume that for every $x \in S$ the operator $Df_{\mathrm{orb}(x)}$ is $(\lambda_s, \lambda_u, \mu_s, \mu_u; h)$-semi-hyperbolic according to the norm $\| \cdot \|_\infty$. Let λ_s^*, λ_u^* be given by (1) and let γ_s^*, γ_u^* be arbitrary reals such that

$$\lambda_s^* < \gamma_s^* < 1 < \gamma_u^* < \lambda_u^*.$$

Let

$$h^* = \max\{\frac{(\lambda_u - \lambda_s + \mu_s + \mu_u)h}{(\gamma_s^* - \lambda_s)(\lambda_u - \gamma_s^*) - \mu_s\mu_u}, \frac{(\lambda_u - \lambda_s + \mu_s + \mu_u)h}{((\gamma_s^*)^{-1} - \lambda_s)(\lambda_u - (\gamma_s^*)^{-1}) - \mu_s\mu_u}\}.$$

Then then the set S is $(\gamma_s^*, \gamma_u^*; h^*)$-hyperbolic.

Hyperbolicity of the Henon attractor

Now we are ready to formulate the main result of the paper. We consider the Henon mapping

$$H_{a,b}(x,y) = (-x^2 + by + a, x)$$

The Henon attractor is the set of all points which have bounded orbits.

Theorem

If $a = 5.4$ and $b = -1$ then the Henon attractor is hyperbolic.