On various concepts of computational hyperbolicity

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Joint work with
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Motivation

Let M be a C^1-manifold and $f\colon M \to M$ a diffeomorphism.

We are given an invariant subset K of M (i.e. $f(K) = K$).

Field of interest:

The behaviour of f in (the neighbourhood of) K.

A lot of information follows from the fact that K is hyperbolic for f.

Hyperbolicity:

- guarantees stability, shadowing, expansivity, symbolic dynamics, Markov partition, topological entropy, etc.;
- hard to verify.

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Idea of hyperbolicity

By $\mathcal{T}M$ we denote the tangent boundle and by $\mathcal{T}f$ the tangent map on $\mathcal{T}M$.

Let $\mathcal{T}K = \mathcal{T}M|_K$.

Definition

We say that K is hyperbolic for f if $\mathcal{T}K$ splits into a direct sum $\mathcal{T}K = E^s \oplus E^u$ of two $\mathcal{T}f$-invariant subbundles such that $\mathcal{T}f|_{E^s}$ is contracting (co-expanding) and $\mathcal{T}f|_{E^u}$ is expanding with respect to some Riemannian norm on $\mathcal{T}M$.

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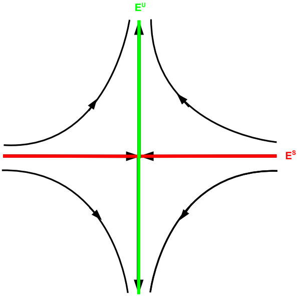
Trivial example

Standard situation in $\mathbb{R}^2$ – a saddle fixed point:

$$A = \begin{bmatrix} \lambda_s & 0 \\ 0 & \lambda_u \end{bmatrix}$$

$$\lambda_s < 1 < \lambda_u$$



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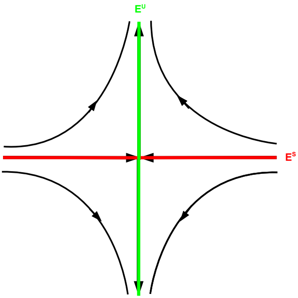

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How to deal with hyperbolicity?

Main problem: invariance of the splitting.

There are three independent concepts of computational hyperbolicity:

1. via quasi-hyperbolicity (Arai)

   Z. Arai, *On Hyperbolic Plateaus of the Hénon Maps*, Experiment. Math. 16 (2007), 181–188.

2. via cone field condition (Hruska)

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Quasi-hyperbolicity

Definition

We say that f is quasi-hyperbolic on K if $\mathcal{T}f$ has no nontrivial bounded orbit.

Proposition (Z. Arai)

Assume that there exists an isolating neighborhood $N \subset \mathcal{T}K$ containing K. Then the set K is quasi-hyperbolic.

$$\text{hyperbolicity} \quad \Rightarrow \quad \text{quasi-hyperbolicity}$$

Theorem (Churchill, Franke, Selgrade and Sacker, Sell)

Assume that $f|_K$ is chain recurrent. Then the set K is hyperbolic if and only if f is quasi-hyperbolic on K.

Algorithm: find a graph representation of f and $\mathcal{T}f$ according to a given cubical grid, construct an isolating nbd N containing $K = \mathcal{CR}(f)$ (subdivide cubes if necessary).

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Cone field condition

Definition

Let $\varepsilon > 0$ and $\mathbb{R}^n = E^s \oplus E^u$. The set $C_\varepsilon(E^s, E^u) = \{v \in \mathbb{R}^n \mid \|P^s v\| \leqslant \varepsilon \|P^u v\|\}$ is called a cone.

Theorem (Newhouse, Palis)

The set K is hyperbolic if and only if there exists a cone field

$$\{\mathcal{C}_x\}_{x \in K} = \{C_{\varepsilon(x)}(E^s_x, E^u_x)\}_{x \in K},$$

such that $\mathcal{T}_x f$ ($\mathcal{T}_x f^{-1}$) uniformly expands vectors in $\mathcal{C}_x$ ($\mathcal{T}_x M \setminus \mathcal{C}_x$) with respect to some continuous norm on $\mathcal{T}M$.

Hruska developed a discrete condition on a graph representation (box chain model) of f on $K = J$, called box hyperbolicity, and proved that it implies hyperbolicity of K.

Algorithm: find a box chain model of f on J, show that it satisfies box hyperbolicity condition.

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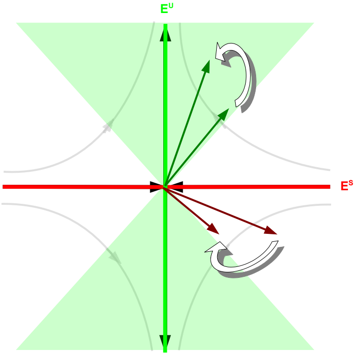
Trivial example

$$A = \begin{bmatrix} \lambda_s & 0 \\ 0 & \lambda_u \end{bmatrix} : E^s \oplus E^u \to E^s \oplus E^u$$

$$\lambda_s < 1 < \lambda_u$$



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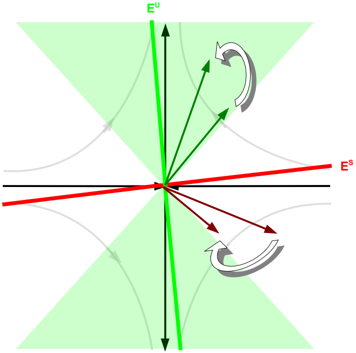
$$A = \begin{bmatrix} \lambda_s^* & \mu_s^* \\ \mu_u^* & \lambda_u^* \end{bmatrix} : E^s \oplus E^u \to E^s \oplus E^u$$

$$\lambda_s^* < 1 < \lambda_u^*, \quad \mu_s^*, \mu_u^* - \text{ small enough}$$



Semi-hyperbolicity – linear case

Let E be a Banach space, $E = E^s \oplus E^u$.

We consider a linear operator $A \in B(E)$ and assume that E^s and E^u are A-invariant. Then in the matrix form we have

$$A = \begin{bmatrix} A_{ss} & 0 \\ 0 & A_{uu} \end{bmatrix}.$$

Definition

We say that A is hyperbolic if

- $\|A_{ss}\| \leqslant \lambda_s$, $\|A_{uu}^{-1}\| \leqslant \lambda_u^{-1}$;
- $\lambda_s < 1 < \lambda_u$;

for some equivalent norm $\|\cdot\|$ on E.

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Let E be a Banach space, $E = E^s \oplus E^u$.

We consider a linear operator $A \in B(E)$ and do not assume that E^s and E^u are A-invariant. Then in the matrix form we have

$$A = \begin{bmatrix} A_{ss} & A_{su} \\ A_{us} & A_{uu} \end{bmatrix}.$$

Definition

We say that A is semi-hyperbolic if

- $\|A_{ss}\| \leqslant \lambda_s$, $\|A_{uu}^{-1}\| \leqslant \lambda_u^{-1}$, $\|A_{su}\| \leqslant \mu_s$, $\|A_{us}\| \leqslant \mu_u$;
- $\lambda_s < 1 < \lambda_u$, $(1 - \lambda_s)(\lambda_u - 1) > \mu_s \mu_u$;

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$$A = \begin{bmatrix} \lambda_s & \mu_s \\ \mu_u & \lambda_u \end{bmatrix} : E^s \oplus E^u \to E^s \oplus E^u$$

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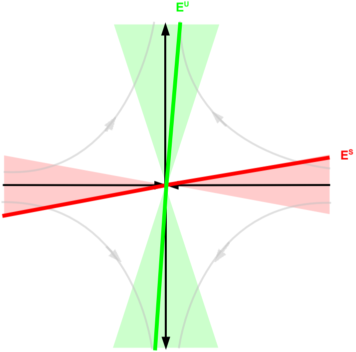

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We say that K is semi-hyperbolic for f if $\mathcal{T}K$ splits into a direct sum $\mathcal{T}K = E^s \oplus E^u$ of two subbundles such that the related projections P^s_x, P^u_x are uniformly bounded and each operator

$$\mathcal{T}_x f\colon E^s_x \oplus E^u_x \to E^s_x \oplus E^u_x$$

is semi-hyperbolic with respect the same constants and "uniformly equivalent" norm on $\mathcal{T}M$.

Theorem (MM, Tabor, Kościelniak and Diamond et al.)

The set K is hyperbolic if and only if K is semi-hyperbolic.

Note that the concepts of cone field condition and semi-hyperbolicity are very similar – see the proceedings.

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Semi-hyperbolicity – computational aspect (in $\mathbb{R}^n$)

A set $\mathbb{S} \subset B(\mathbb{R}^n)$ we call a set-operator on $\mathbb{R}^n$.

We say that a set operator $\mathbb{S}$ is $(\lambda_s, \lambda_u, \mu_s, \mu_u)$-semi-hyperbolic if each of its elements is $(\lambda_s, \lambda_u, \mu_s, \mu_u)$-semi-hyperbolic with respect to the same splitting and norm.

Definition

Let G be a finite set and $F\colon G \rightrightarrows G$, $DF\colon G \rightrightarrows B(\mathbb{R}^n)$ be given multivalued maps such that $DF(\sigma)$ is a set-operator on E for all $\sigma \in G$.

We say that the pair (F, DF) is $(\lambda_s, \lambda_u, \mu_s, \mu_u)$-semi-hyperbolic if for each $\sigma \in G$ there exist a norm $\|\cdot\|_\sigma$ on E and a splitting $E = E^s_\sigma \oplus E^u_\sigma$ such that for every $\sigma \in G, \tau \in F(\sigma)$ the set-operator

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- $K \subset \bigcup_{\sigma \in \mathrm{dom}(F)} \sigma$;
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Algorithm: find a cubical representation of (a neighborhood of) K, construct a pair (F, DF) such that $f \lhd_K (F, DF)$, prove that (F, DF) satisfies semi-hyperbolicity condition with respect to some collection of norms $\|\cdot\|_\sigma$, $\sigma \in G$ (it is necessary to compute approximate stable and unstable directions for each cube).

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Nontrivial examples – Hénon map

Problem:

Hyperbolicity of a nontrivial invariant set K of a real or complex Hénon map

$$H_{a,b} = (a - x^2 + by, x)$$

for various values of the parameters $a, b \in \mathbb{R}$.

Clasical result:

Theorem (Devaney, Nitecki)

If K is equal to the set of all bounded bounded orbits of a real Hénon map with

$$a \geqslant (5 + 2\sqrt{5})(1 + |b|)^2/4,$$

then K is hyperbolic.

R. Devaney, Z. Nitecki, *Shift automorphisms in the Hénon mapping*, Commun. Math. Phys. 67 (1979), 137–146.

S. Newhouse, *Cone-fields, domination, and hyperbolicity*, in: Modern dynamical systems and applications, 419–432, Cambridge Univ. Press, Cambridge, 2004.

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$$H_{a,b} = (a - x^2 + by, x)$$

for various values of the parameters $a, b \in \mathbb{R}$.

Clasical result:

Theorem (Devaney, Nitecki)

If K is equal to the set of all bounded bounded orbits of a real Hénon map with

$$a \geqslant (5 + 2\sqrt{5})(1 + |b|)^2/4,$$

then K is hyperbolic.

R. Devaney, Z. Nitecki, *Shift automorphisms in the Hénon mapping*, Commun. Math. Phys. 67 (1979), 137–146.

S. Newhouse, *Cone-fields, domination, and hyperbolicity*, in: Modern dynamical systems and applications, 419–432, Cambridge Univ. Press, Cambridge, 2004.

Nontrivial examples – Hénon map

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Computer assisted results

Theorem (Arai – quasi-hyperbolicity)

If K is equal to the chain recurrent set of a real Hénon map with $(a, b) \in P$, where P is union of finitely many rectangles, then K is hyperbolic.

Theorem (Hruska – cone field condition)

If K is equal to the Julia set of a complex Hénon map with

$$(a, b) = (0.3, -0.1), (0, 0.22), (3, 0.25) \text{ or } (-1.5, 0.5),$$

then K is hyperbolic.

Theorem (MM, Tabor, Kułaga – semi-hyperbolicity)

If K is equal to the invariant part of the rectangle $R = [-\frac{K}{2}, \frac{K}{2}]^2$, where $K = 1 + |b| + \sqrt{(1 + |b|)^2 + 4a}$, for a real Hénon map with

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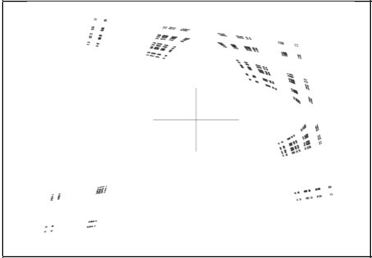
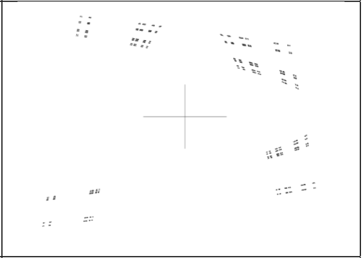
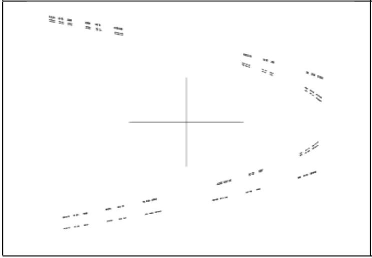
Some figures

$a = 5.4,\ b = -1$	$a = 5.5,\ b = -0.8$
	
$a = 3,\ b = 0.25$	
	

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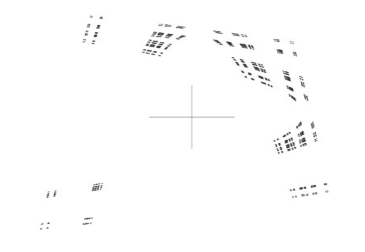
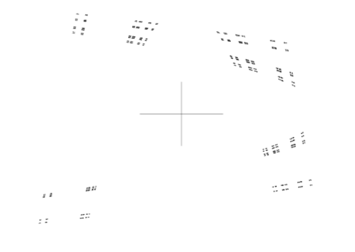
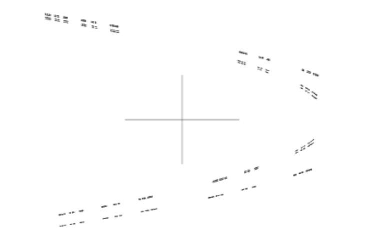
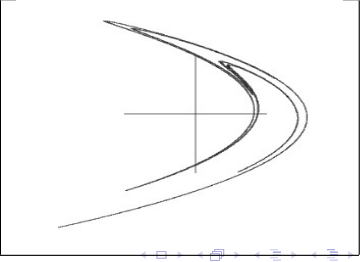
	
$a = 3,\ b = 0.25$	$a = 1.17,\ b = -0.3$
	

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http://www.im.uj.edu.pl/MarcinMazur/comphyp/