Semi-hyperbolicity and hyperbolicity

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Joint work with
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Motivation

Let M be a C^1-manifold and $f\colon M \to M$ a diffeomorphism. We are given an invariant subset K of M (i.e. $f(K) = K$).

Field of interest:

The behaviour of f in (the neighbourhood of) K.

A lot of information follows from the fact that K is hyperbolic for f.

Hyperbolicity:

- guarantees stability, shadowing, expansivity, symbolic dynamics, etc.;
- hard to verify.

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Idea of hyperbolicity

By $\mathcal{T}_x M$ we denote the tangent space at the point x.
We consider a compact invariant set $K \subset M$ and the collection of linear tangent maps

$$D_x f \colon \mathcal{T}_x M \to \mathcal{T}_{f(x)} M$$

for $x \in K$.

Roughly speaking, we say that K is hyperbolic if the linear operators $D_{orb(x)} f$ $(x \in K)$ are all hyperbolic "with the same hyperbolicity constants".

$$\begin{array}{ccccccc}
L^\infty(orb(x)) & \ni & (v_n) & \mapsto & D_{orb(x)} f\,(v_n) & \in & L^\infty(orb(x)) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathcal{T}_{f(x)} M & \ni & v_1 & \mapsto & D_{f(x)} f\, v_1 & \in & \mathcal{T}_{f^2(x)} M \\
\mathcal{T}_x M & \ni & v_0 & \mapsto & D_x f\, v_0 & \in & \mathcal{T}_{f(x)} M \\
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We therefore go to the linear case $\longrightarrow$ not so obvious!

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Definition of hyperbolicity – linear case

Let X be a Banach space and $A\colon X \to X$ a linear operator.

Definition

We say that A is hyperbolic if there exist a splitting $X = X_s \oplus X_u$ and an equivalent norm $\|\cdot\|$ on X such that

- X_s, X_u are A-invariant;
- there exist $\lambda_s < 1$, $\lambda_u > 1$ such that

$$\|Ax_s\| \leqslant \lambda_s \|x_s\|, \ \|Ax_u\| \geqslant \lambda_u \|x_u\| \quad \text{for } x_s \in X_s, x_u \in X_u.$$

Standard situation in $\mathbb{R}^2$ – saddle fixed point:

$$A = \begin{bmatrix} \lambda_s & 0 \\ 0 & \lambda_u \end{bmatrix}$$

$$\lambda_s < 1 < \lambda_u$$

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Hénon map

Problem:

Hyperbolicity of the Hénon attractor for various parameter values.

Partial Answer:

Theorem (Z. Arai)

Hénon attractor is hyperbolic on its chain recurrent part for a large set of parameter values.

Z. Arai, *On Hyperbolic Plateaus of the Hénon Maps*, to appear in Experimental Mathematics.

A computer assisted proof $\longrightarrow$ the use of quasi-hyperbolicity.

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How to deal with hyperbolicity?

Standard situation in $\mathbb{R}^2$:

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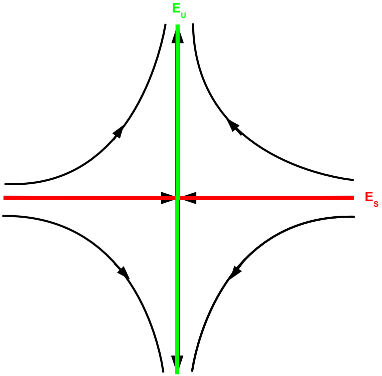

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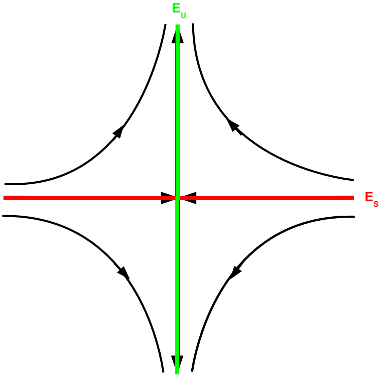

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$$A = \begin{bmatrix} \lambda_s & \mu_s \\ \mu_u & \lambda_u \end{bmatrix}$$

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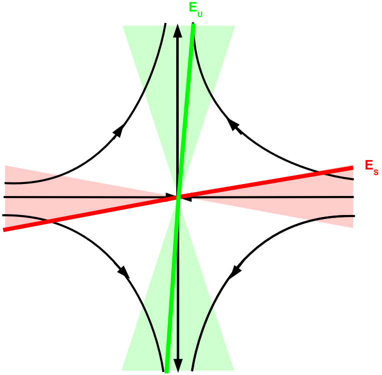

How to deal with hyperbolicity?

One of possible solutions $\longrightarrow$ semi-hyperbolicity:

- easier to check;
- allows lipschitzian perturbations.

A.A. Al-Nayef, P.E. Kloeden, A.V. Pokrovskii, *Semi-hyperbolic mappings, condensing operators, and neutral delay equations*, J. Differential Equations **137** (1997), 320–339.

P. Diamond, P.E. Kloeden, V. S. Kozyakin, A.V. Pokrovskii, *Semi-hyperbolic mappings*, in preparation.

Definition of hyperbolicity – linear case (reformulation)

Let X be a Banach space, $X = X_s \oplus X_u$.
We consider a linear operator $A\colon X \to X$ and assume that X_s and X_u are A-invariant. Then in the matrix form we have

$$A = \begin{bmatrix} A_{ss} & 0 \\ 0 & A_{uu} \end{bmatrix}.$$

Definition

We say that A is hyperbolic if

- $\lambda_s < 1 < \lambda_u$;
- $\|A_s\| \leqslant \lambda_s$, $\|A_u^{-1}\| \leqslant \frac{1}{\lambda_u}$;
- $\|A_{su}\| \leqslant \mu_s$, $\|A_{us}\| \leqslant \mu_u$;

for some equivalent norm $\|\cdot\|$ in X.

Definition of semi-hyperbolicity – linear case

Let X be a Banach space, $X = X_s \oplus X_u$.
We consider a linear operator $A\colon X \to X$ and do not assume that X_s and X_u are A-invariant. Then in the matrix form we have

$$A = \begin{bmatrix} A_{ss} & A_{su} \\ A_{us} & A_{uu} \end{bmatrix}.$$

Definition

We say that A is semi-hyperbolic if

- $\lambda_s < 1 < \lambda_u$ and $(1-\lambda_s)(\lambda_u - 1) > \mu_s \mu_u$;
- $\|A_s\| \leqslant \lambda_s$, $\|A_u^{-1}\| \leqslant \frac{1}{\lambda_u}$;
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Results

Theorem

If the operator A is semi-hyperbolic, then A is hyperbolic.

M. Mazur, J. Tabor, K. Stolot, *Semi-hyperbolicity implies hyperbolicity in the linear case,* Proceedings of the Conference "Topological Methods in Differential Equations and Dynamical Systems" (Krakow-Przegorzaly, 1996), Univ. Iagell. Acta Math. **36** (1998), 121–126.

Theorem

Let K be a compact invariant semi-hyperbolic subset of M. Then K is hyperbolic.

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Main results

Theorem (MM, J. Tabor, P. Kościelniak)

Let M be a Riemannian manifold, $f : M \to M$ be a C^1-diffeomorphism and K be an invariant subset of M. Assume that K is $(\lambda_s, \lambda_u, \mu_s, \mu_u; h)$-semi-hyperbolic. Let γ_s^*, γ_u^* be arbitrary reals such that

$$\frac{\lambda_s+\lambda_u}{2} - \frac{\sqrt{(\lambda_u-\lambda_s)^2-4\mu_s\mu_u}}{2} < \gamma_s^* < 1 < \gamma_u^* < \frac{\lambda_s+\lambda_u}{2} + \frac{\sqrt{(\lambda_u-\lambda_s)^2-4\mu_s\mu_u}}{2}.$$

Let

$$C^* = \max\left\{\frac{(\lambda_u-\lambda_s+\mu_s+\mu_u)h}{(\gamma_s^*-\lambda_s)(\lambda_u-\gamma_s^*)-\mu_s\mu_u}, \frac{(\lambda_u-\lambda_s+\mu_s+\mu_u)h}{(\gamma_u^*-\lambda_s)(\lambda_u-\gamma_u^*)-\mu_s\mu_u}\right\}.$$

Then the set K is $(\gamma_s^*, \gamma_u^*; C^*)$-hyperbolic.

Theorem (MM, J. Tabor)

Hénon attractor is hyperbolic for some parameter values.

A computer assisted proof ⟶ the C++ program that verifies the semi-hyperbolicity conditions for the Hénon attractor.

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Work for (near) future

- improvement of the HENON-HYPER program: stricter estimations, interval parameters, etc.;
- development and implementation of the algorithm for systems in $\mathbb{R}^3$;
- looking for interesting examples.

http://www.im.uj.edu.pl/MarcinMazur/comphyp/

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