Swap rate a la stock: Bermudan swaptions made easy

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References

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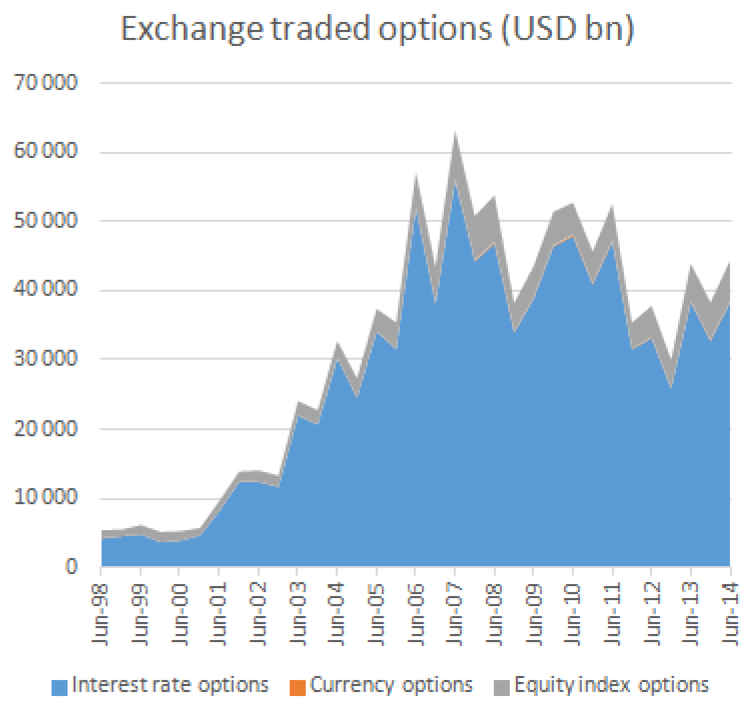
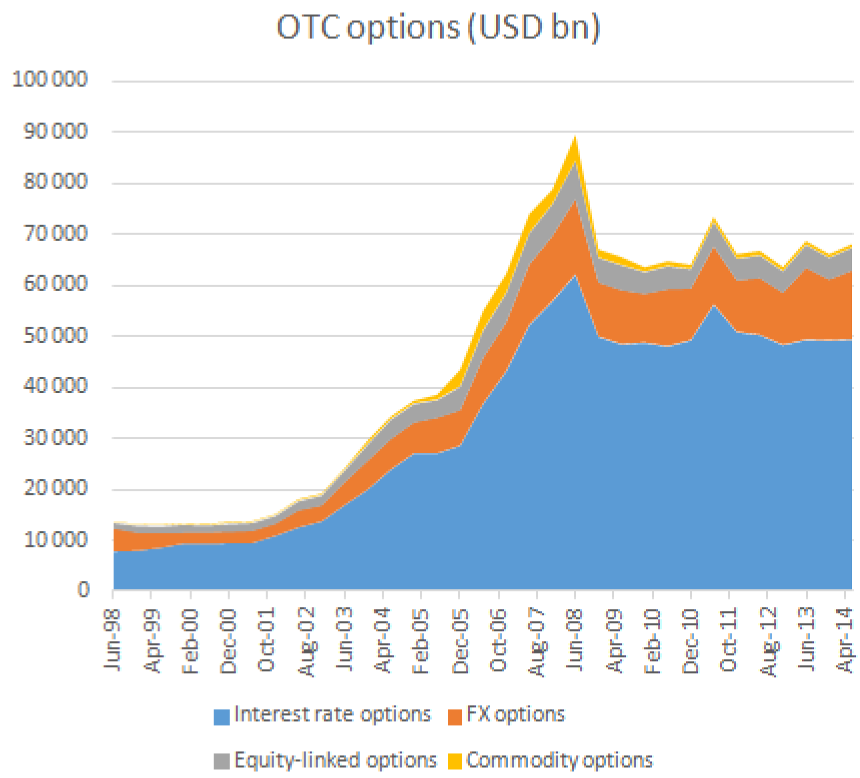
Stylized facts

- OTC interest rate derivatives market has grown remarkably over the past 20 years
- interest rates have become the dominant underlying for options contracts (by notional) in both OTC and organised exchanges





HJM framework

We start in the standard Heath-Jarrow-Morton setting, with time t price of a zero coupon bond maturing at time T, $B(t,T)$, given by

$$B(t,T) = \exp\left\{-\int_t^T f(t,s)ds\right\} = 1 - \int_t^T B(t,s)f(t,s)ds$$

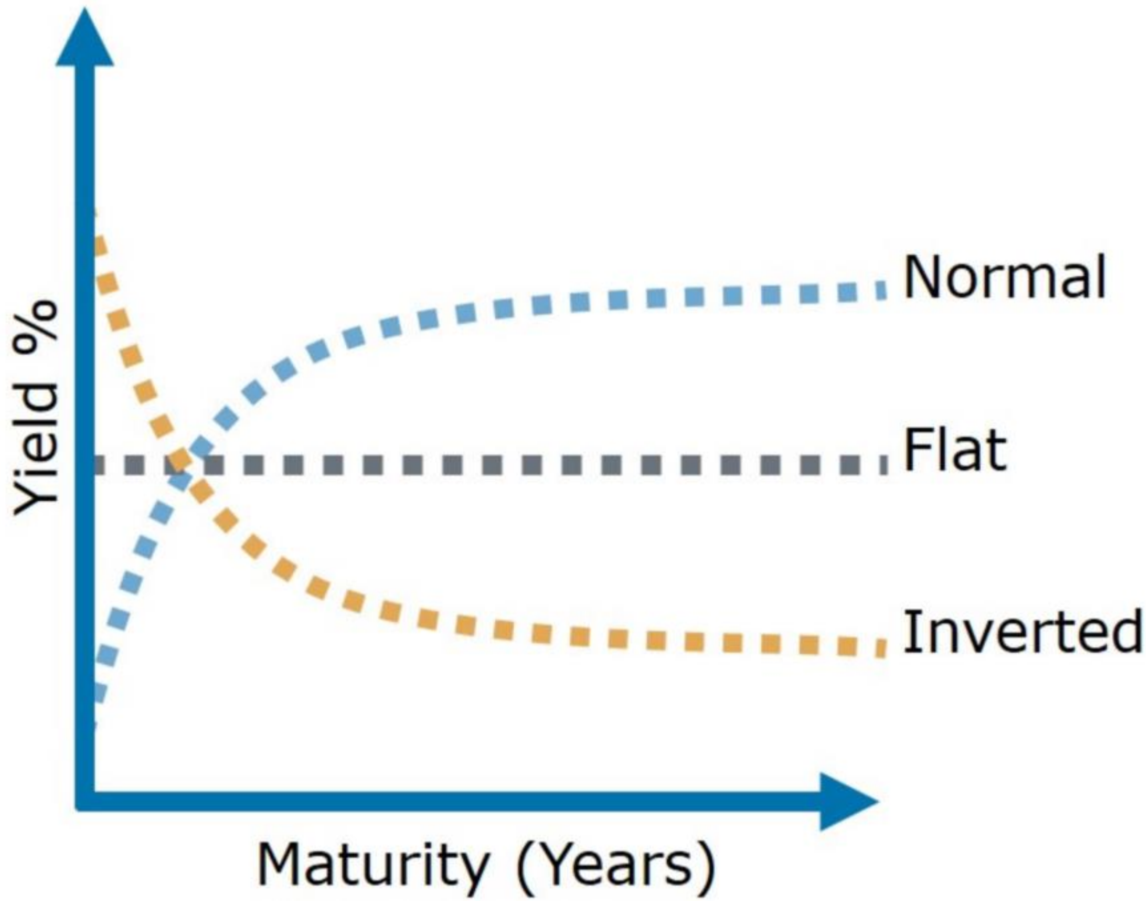
where $f(t,T)$ is the instantaneous forward rate. Define by $r(s) = f(s,s)$ the short rate. Under the risk-neutral measure

$$B(dt,T) = B(t,T)\{r(t) - \Sigma(t,T) \cdot dW(t)\},$$

$$df(t,T) = \sigma(t,T) \cdot \int_t^T \sigma(t,s)dsdt + \sigma(t,T) \cdot dW(t),$$

$$\sigma(t,T) = \frac{\partial \Sigma(t,T)}{\partial t}.$$

Yield curve



Swap options

Denote by $B(t,T)$ zero coupon bond price, $T > t$, $S(t,T)$ the swap rate

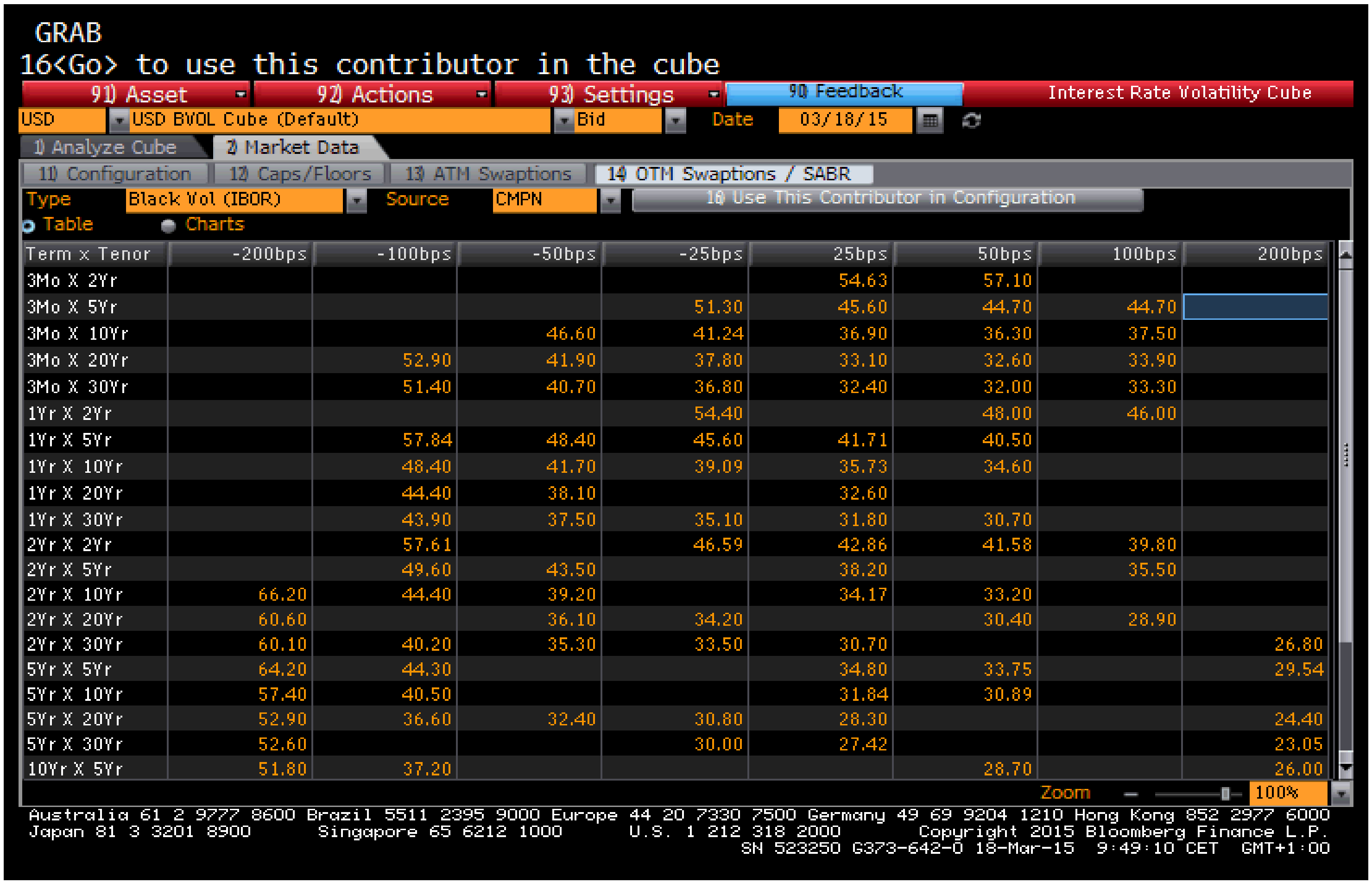
$$N(t,T) = \int_t^T B(t,s)ds\,,$$

$$S(t,T) = \frac{\int_t^T B(t,s)f(t,s)ds}{N(t,T)} = \frac{1 - B(t,T)}{N(t,T)},$$

$$R(t,T) = \exp\left\{-\int_t^T r(s)ds\right\},$$

$$C(K,t,T) = E\{R(0,t)N(t,T)(S(t,T) - K)^+\}.$$

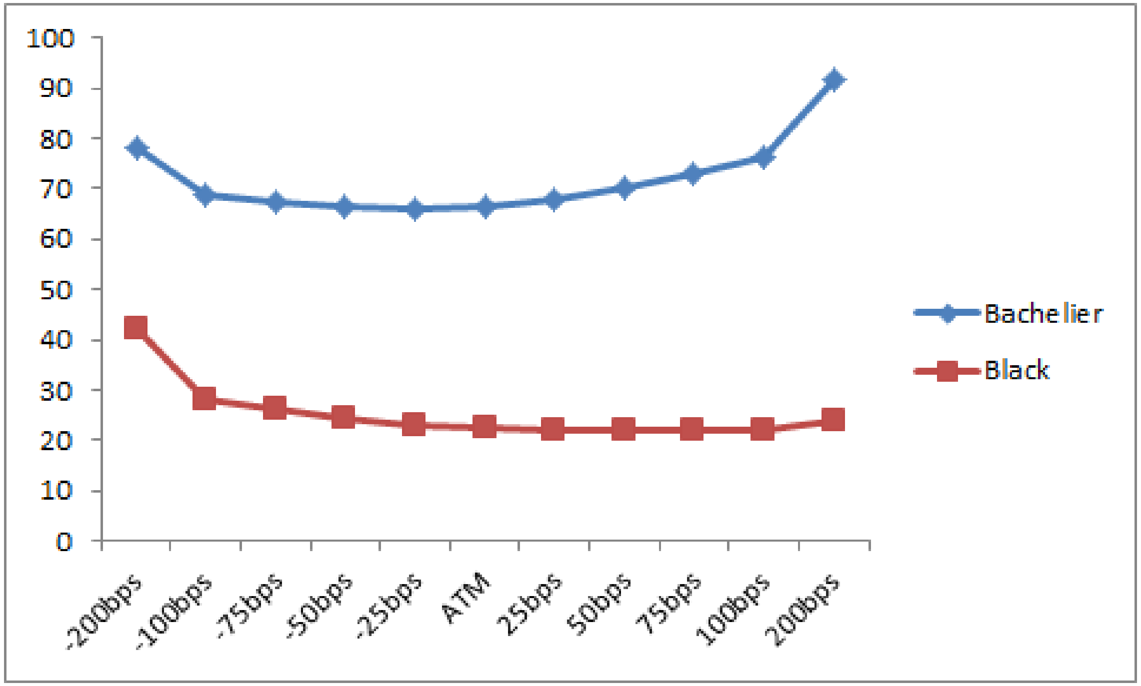
Swaption cube



Term x Tenor	-200bps	-100bps	-50bps	-25bps	25bps	50bps	100bps	200bps
3Mo X 2Yr					54.63	57.10		
3Mo X 5Yr				51.30	45.60	44.70	44.70	
3Mo X 10Yr			46.60	41.24	36.90	36.30	37.50	
3Mo X 20Yr		52.90	41.90	37.80	33.10	32.60	33.90	
3Mo X 30Yr		51.40	40.70	36.80	32.40	32.00	33.30	
1Yr X 2Yr				54.40		48.00	46.00	
1Yr X 5Yr		57.84	48.40	45.60	41.71	40.50		
1Yr X 10Yr		48.40	41.70	39.09	35.73	34.60		
1Yr X 20Yr		44.40	38.10		32.60			
1Yr X 30Yr		43.90	37.50	35.10	31.80	30.70		
2Yr X 2Yr		57.61		46.59	42.86	41.58	39.80	
2Yr X 5Yr		49.60	43.50		38.20		35.50	
2Yr X 10Yr	66.20	44.40	39.20		34.17	33.20		
2Yr X 20Yr	60.60		36.10	34.20		30.40	28.90	
2Yr X 30Yr	60.10	40.20	35.30	33.50	30.70			26.80
5Yr X 5Yr	64.20	44.30			34.80	33.75		29.54
5Yr X 10Yr	57.40	40.50			31.84	30.89		
5Yr X 20Yr	52.90	36.60	32.40	30.80	28.30			24.40
5Yr X 30Yr	52.60			30.00	27.42			23.05
10Yr X 5Yr	51.80	37.20				28.70		26.00

Quoting conventions

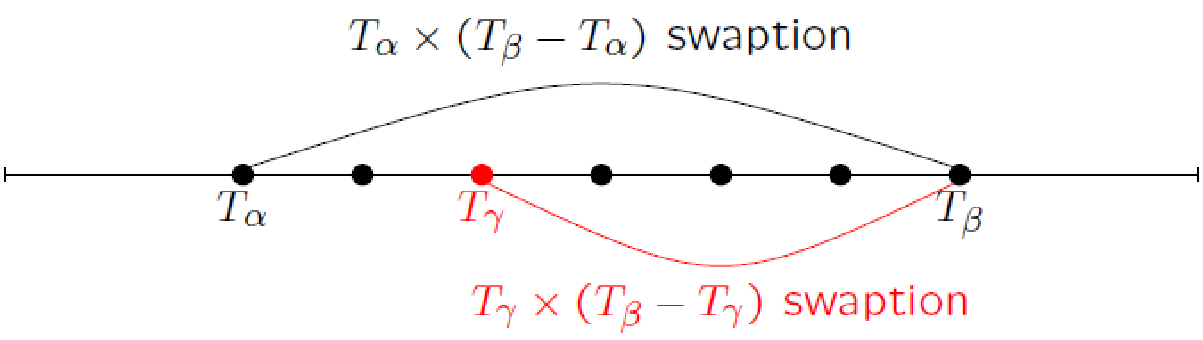
Black and Bachelier implied volatilities for 1x10 USD swaption (in bp; as of 27 April 2018)



Bermudan swaption challenge

Bermudan swaption is a best-of chooser option giving right to choose among several European options on swap rates with different fixing dates.

Consider a Bermudan to enter at $T_{ex} \in \{T_\alpha, T_{\alpha+1}, ..., T_\beta\}$ into fixed-for-floating swap terminating at T_β:



Bermudan swaption pricing

Price of a Bermudan option

$$A(K,T) = \max_{\tau \le T} E\{R(0,\tau)N(\tau,T)(S(\tau,T)-K)^+\}.$$

Markovian approach.

Let $f(t,x) = g(t,Y(t))(x)$, where Y is a Markov process on a finite or infinite dimensional state space. Then there exists a function G such that

$$G(Y(t),t) = N(t,T)(S(t,T)-K)^+.$$

If

$$V(y,T_i) = \max\left\{G(y,T_i); E\{R(T_i,T_{i+1})V(Y(T_{i+1}),T_{i+1}) | Y(T_i) = y\}\right\},$$
$$V(y,T_N) = G(y,T_N).$$

Then

$$A(K,T_N) = V(Y(0),0).$$

Free boundary problem

Stopping area $V(y, T_i) = G(y, T_i)$.
Continuation area

$$V(y, T_i) = E\{R(T_i, T_{i+1})V(Y(T_{i+1}), T_{i+1}) | Y(T_i) = y\}.$$

Interest rate models

1. Hull-White – one, two-dimensional.
2. Black-Karasiński – one, two-dimensional.
3. Cox-Ingersoll-Ross – one, two-dimensional.
4. Gaussian Heath-Jarrow-Morton – N-dimensional.
5. Libor market model – infinite dimensional.
6. Cheyette $- N \times (N + 1)$-dimensional.
7. Linear Gauss-Markov $- N$-dimensional.

$$\text{HW} \approx \text{HJM} \approx \text{LGM}$$

Calibration

Calibrate to:

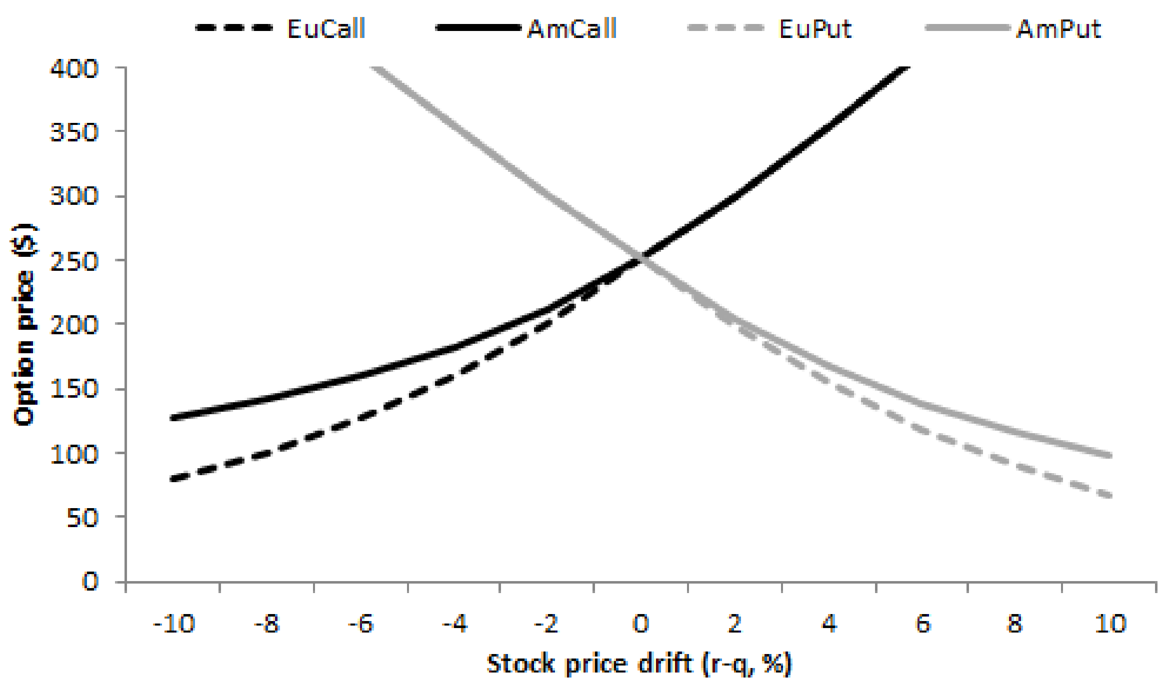
1. Full swaption cube.
2. Co-terminal swaptions with full strike structure.
3. All swaptions with given strike prices.
4. Co-terminal swaptions with given strike prices.

Scylla and Charibdis

1. Larger dimension means better calibration.
2. Bermudan pricing is accurate only in dimenion 1.

American-style options

- Number of factors has little impact on the American/Bermudan option price.
- Value of the early exercise right depends on the dividend yield and its relation to the short rate



Swap rate dynamics

Define

$D(t) = N(t,T)R(0,t)$ - discounted riskless asset,
$S(t) \; = \; S(t,T)$ – swaption rate,
$X(t) = (1 - B(t,T\,))R(0,t) = S(t)D(t)$ – discounted risky asset,
$C(K,t) = C(K,t,T)$.

Let

$$dS(t) = \varphi(t)dW(t) - (\dots)dt.$$

It is straightforward to calculate that

$$dD(t) = (\dots)dW(t) - R(0,t)dt,$$

$$dX(t) = (\dots)dW(t) - R(0,t)r(t)dt.$$

Option payoff dynamics

$$d(D(t)(S(t)-K)^+) = \frac{1}{2}\varphi^2(t)\delta(S(t)-K)D(t)dt$$
$$+I(S(t)>K)\{D(t)dS(t)+d\langle S,D\rangle_t\}+(S(t)-K)^+dD(t)$$
$$=\frac{1}{2}\varphi^2(t)\delta(S(t)-K)D(t)dt$$
$$+I(S(t)>K)\{D(t)dS(t)+d\langle S,D\rangle_t+(S(t)-K)dD(t)\}$$
$$=\frac{1}{2}\delta(S(t)-K)D(t)\varphi^2(t)dt+I(S(t)>K)\{dX(t)-KdD(t)\}.$$

Taking the expected value we get

$$\frac{\partial C(K,t)}{\partial t}-\frac{1}{2}\frac{\partial^2 C(K,t)}{\partial K^2}\varphi(K,t)$$
$$=ER(0,t)I(S(t)>K)R(0,t)\{K-r(t)\}$$
$$=\int_K^{\infty}\frac{\partial^2 C(x,t)}{\partial x^2}(g(x,t)-Kh(x,t))dx\,,$$

Option dynamics

where

$$\varphi^2(x,t) = E^t\{\varphi^2(t)|S(t) = x\},$$
$$g(x,t) = -E^t\left\{\frac{r(t)}{N(t,T)}\,\middle|S(t) = x\right\},$$
$$h(x,t) = -E^t\left\{\frac{1}{N(t,T)}\,\middle|S(t) = x\right\}$$

and

$$E^t\xi = \frac{ED(t)\xi}{ED(t)}.$$

Differentiating twice we get

$$\frac{\partial^3 C(K,t)}{\partial t\partial K^2} - \frac{1}{2}\frac{\partial^2}{\partial K^2}\left\{\frac{\partial^2 C(K,t)}{\partial K^2}\varphi(K,t)\right\}$$
$$-\frac{\partial}{\partial K}\left\{\frac{\partial^2 C(K,t)}{\partial K^2}(g(K,t) - Kh(K,t))\right\} + \frac{\partial^2 C(K,t)}{\partial K^2}h(K,t).$$

Markovian projection

Then $\frac{\partial^2 C(K,T)}{\partial K^2}$ is the density function for a stochastic process Z, namely

$$dZ(t) = \varphi(Z(t),t)dW(t) + (g((Z(t),t) - Z(t)h(Z(t),t))dt,$$
$$dQ(t) = Q(t)h(Z(t),t)dt.$$

$$Q(t) = \exp\left\{\int_0^t h(Z(s),s)ds\right\}.$$

$$C(K,t) = ED(t)(S(t)-K)^+ = EQ(t)(Z(t)-K)^+.$$

$$\mathrm{Law}\big(S(t)\big) \neq \mathrm{Law}\big(Z(t)\big).$$

$$A(K,T) \approx \max_{\tau \leq T} EQ(\tau)(Z(\tau)) - K)^+$$

Useful approximations

$$\frac{r(t)}{N(t,T)} = \frac{r(t)S(t)}{1-B(t,T)} \approx S(t)\,\frac{f(0,t)B(0,t)}{B(0,t)-B(0,T)} = p(t)S(t),$$

$$\frac{1}{N(t,T)} \approx \frac{B(0,t)}{\int_t^T B(0,s)ds} = q(t).$$

Hence $g(x,t) = -p(t)x$ and $h(x,t) = -q(t)$,

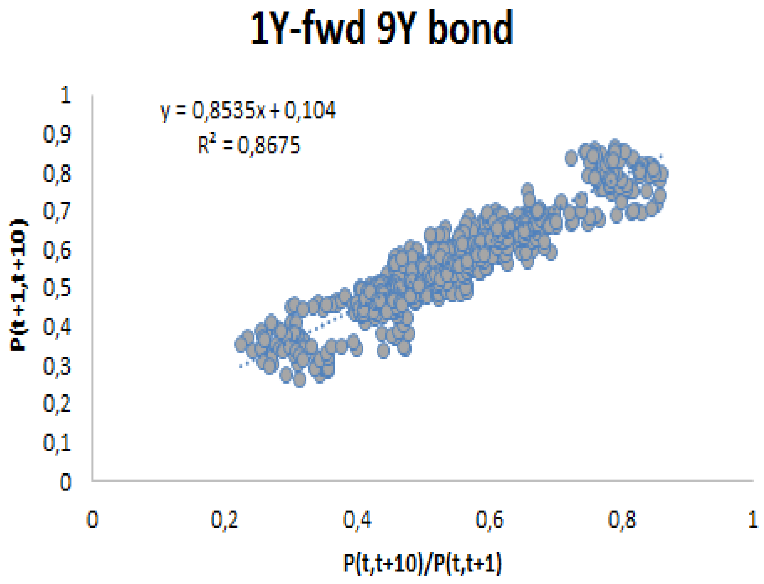
$$dZ(t) = \varphi(Z(t),t)dW(t) + Z(t)(q(t)-p(t))dt,$$
$$dQ(t) = -Q(t)q(t)dt.$$

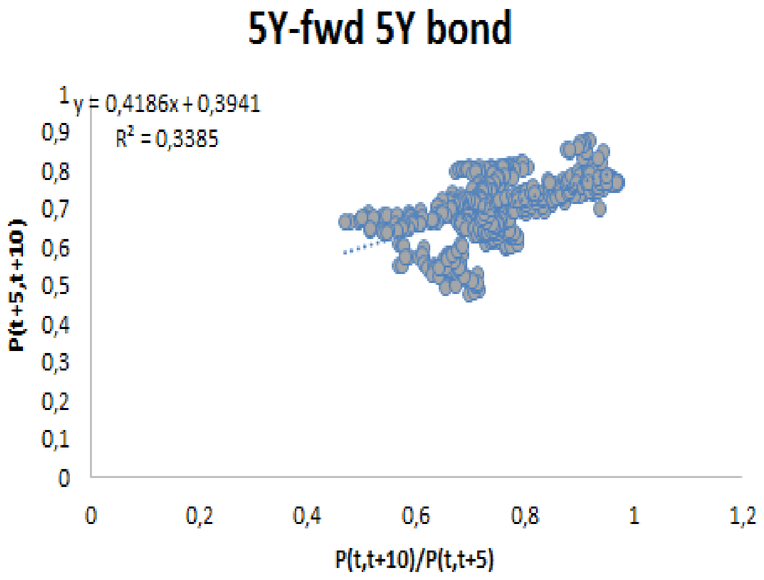
and

$$\frac{\partial C(K,t)}{\partial t} - \frac{1}{2}\frac{\partial^2 C(K,t)}{\partial K^2}\varphi(K,t) \approx K\,\frac{\partial C(K,t)}{\partial K}\big(p(t)-q(t)\big)$$
$$-p(t)C(K,t)$$

Quality of approximation

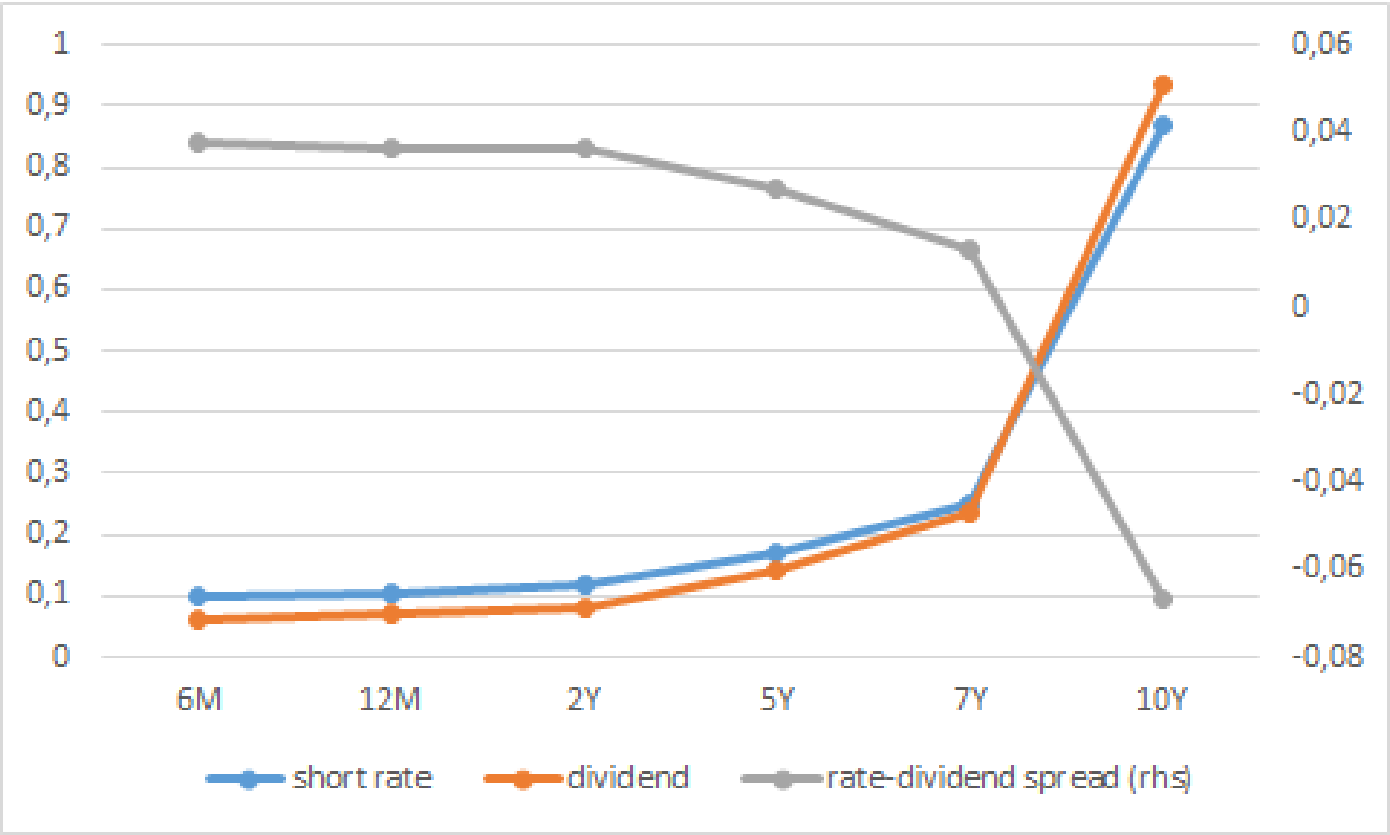
Regression fit of the 1-year forward 9-year zero coupon bond price and the 5-year forward 5-year zero coupon bond price against their time t „frozen" forecasts





Implied short rate and dividend

US yield curve data as of October 5, 2016



Cheyette model

Short rate dynamics $r(t) = f(0,t) + x(t)$

$$dx(t) = \big(y(t) - k(t)x(t)\big)dt + \nu(x(t), y(t), t)dW(t),$$
$$y'(t) = \nu(x(t), y(t), t)^2 - 2k(t)y(t),$$
$$x(0) = y(0) = 0,$$

where $k(t) = b'(t)/b(t)$ and b is a given deterministic function.
All zero coupon bonds are deterministic functions of $x(t)$ and $y(t)$:

$$B(t,T) = \frac{B(0,T)}{B(0,t)} \exp\left\{-G(t,T)x(t) - \frac{1}{2}G^2(t,T)y(t)\right\},$$

where

$$G(t,T) = \frac{1}{b(t)} \int_t^T b(s)ds\,.$$

Triangle approximation

Notice that

$$y(t) = \int_0^t \nu(x(s), y(t), t)^2 \exp\left(-2\int_s^t k(u)du\right) ds.$$

Apply triangle approximation for $y(t)$:

$$y(t, x(t)) \approx \int_0^t \nu\left(\frac{x(t)s}{t}, 0, s\right)^2 \exp\left(-2\int_s^t k(u)du\right) ds.$$

Now $x(t)$ follows a one-dimensional SDE:

$$dx(t) = \big(y(t, x(t)) - k(t)x(t)\big)dt + \tilde{\nu}(x(t), t)dW(t),$$
$$x(0) = 0.$$

Volatility

Finally, we link swaption volatility to our yield curve parameterization. Let

$$dS(t) = (\dots)dt + \varphi(S(t), t)dW(t).$$

Since $S(t) = S(x(t), y(t), t)$

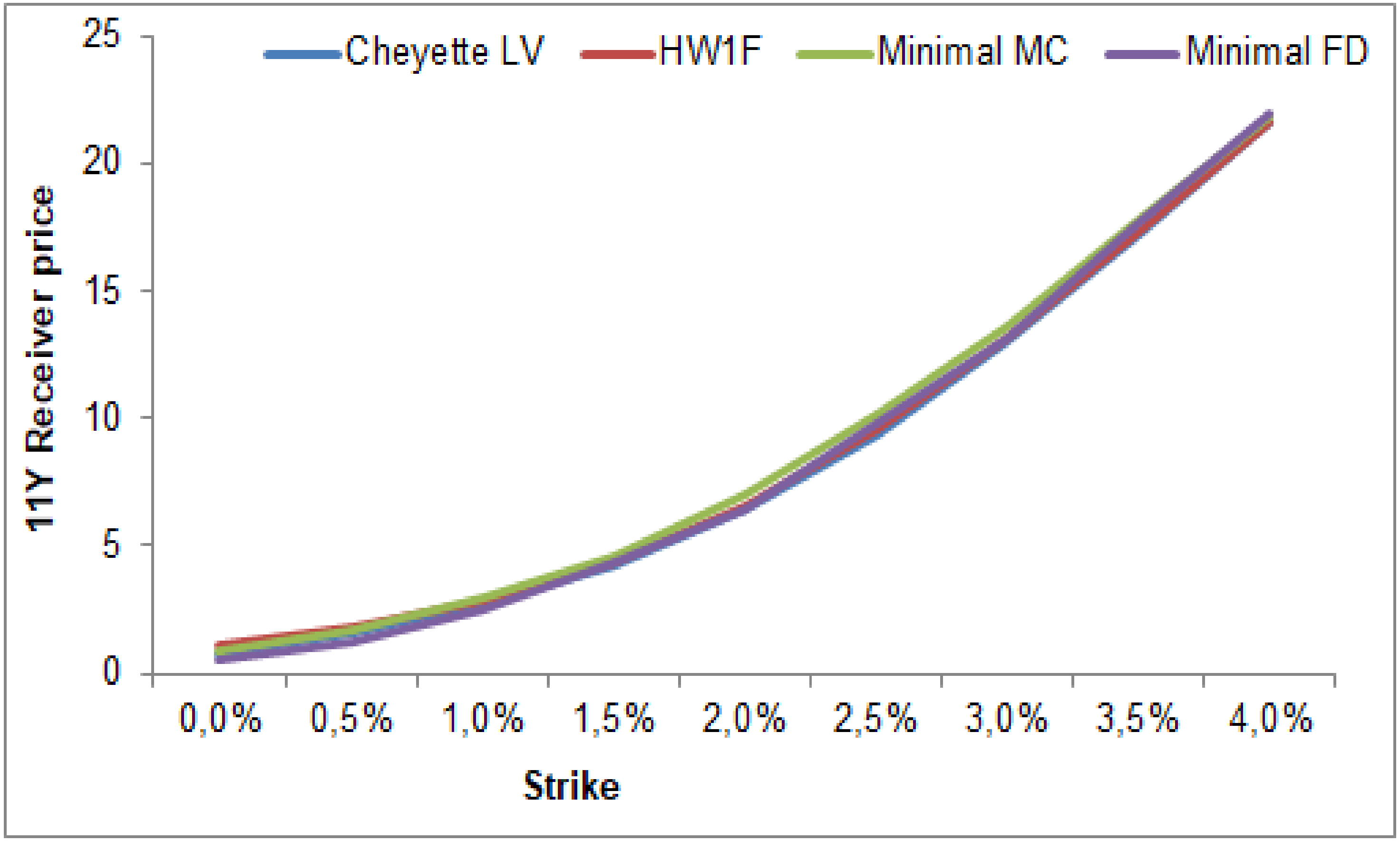
$$dS(x(t), y(t), t) = (\dots)dt + \frac{\partial S(x(t), y(t), t)}{\partial x(t)} \nu(x(t), y(t), t)dW(t).$$

Then

$$\nu(z, y, t) = \left(\frac{\partial S(z, y, t)}{\partial z}\right)^{-1} \varphi(S(z, y, t), t).$$

Bermudan swaption pricing

Prices of a Bermudan receiver swaption with 100 notional



Cheyette vs Markovian proxy

Cheyette model

- Complete arbitrage free model
- Accurately calibrated to European swaptions
- Able to price all products

Markovian proxy

- Short cut/dirty trick
- Swap rate evolves like a dividend paying stock
- Only swap rate derivatives may be prices
- Equity derivatives software may be used

Calibration + pricing is nothing more than an interpolation between prices!

Conclusions

We have presented a simple trick that allows us to price Bermudan swaptions essentially as if they were options on a dividend paying stock. The practical signficance of this contribution is that, while the approach is not strictly arbitrage free and unfortunately can be directly applied only to swap rate derivatives, it offers a potentially attractive solution especially in cases when the development of a fully-edged interest rate model is too time or resource consuming. Perhaps the most attractive feature of the model is that it makes it possible to transpose some of the well-established rules of thumb from the world of equities to interest rate space.