Contents

References

[Bou 1965] N. Bourbaki, *Topologie générale*, Hermann, Paris, 1965.

[Cal-Ros 1954] E. Calabi, M. Rosenlicht, *Complex analytic manifolds without countable base*, Proc. Amer. Math. Soc. 4 (1954), 335–340.

[Die 1944] J. Dieudonné, *Une généralisation des espaces compactes*, J. Math. Pure Appl. 23 (1944), 65–76.

[Gun-Nar 1967] R.C. Gunning, R. Narasimhan, *Immersion of open Riemann surfaces*, Math. Ann. 174 (1967), 103–108.

[Lej 1957] F. Leja, *Teoria funkcji analitycznych*, PWN, Warszawa, 1957.

[Lej 1971] F. Leja, *Funkcje zespolone*, PWN, Warszawa, 1971.

[Min-Sch 1983] D. Minda, G. Schober, *Another elementary approach to the theorems of Landau, Montel, Picard and Schottky*, Complex Analysis 2 (1983), 157–164.

[Rad 1925] T. Radó, *Über den Begriff der Riemannschen Fläche*, Acta Univ. Szeged 2, 1925.

CHAPTER 1

Basic facts

1.1. What we know so far

Standard notation: $\mathit{\Omega}, G, D \in \operatorname{top}\mathbb{C}$, D – a domain.

Definition 1.1.1. Let $f : \mathit{\Omega} \longrightarrow \mathbb{C}$. We say that f is *holomorphic in* $\mathit{\Omega}$ ($f \in \mathcal{O}(\mathit{\Omega})$), if for any point $a \in \mathit{\Omega}$ there exist a power series $\sum_{n=0}^{\infty} a_n(z-a)^n$ and $0 < r \le R$, where R is the *radius of convergence* of the series, such that $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$, $z \in B(a,r) \cap \mathit{\Omega}$. Recall that $R := \sup\{r > 0 : \text{the series } \sum_{n=0}^{\infty} a_n(z-a)^n \text{ is convergent uniformly in } B(a,r)\}$.

If $f \in \mathcal{O}(\mathbb{C})$, then we say that f is an *entire function*.

If $f : \mathit{\Omega} \longrightarrow G$ is a bijection, and $f \in \mathcal{O}(\mathit{\Omega})$, $f^{-1} \in \mathcal{O}(G)$, then we say that f is *biholomorphic* ($f \in \operatorname{Bih}(\mathit{\Omega}, G)$). Put $\operatorname{Aut}(\mathit{\Omega}) := \operatorname{Bih}(\mathit{\Omega}, \mathit{\Omega})$. A function $f \in \operatorname{Aut}(\mathit{\Omega})$ is called an *automorphism* of $\mathit{\Omega}$.

Let $\mathit{\Omega} \in \operatorname{top}\widehat{\mathbb{C}}$ be such that $\infty \in \mathit{\Omega}$ and let $R > 0$ be such that $\widehat{\mathbb{C}} \setminus \overline{B}(R) \subset \mathit{\Omega}$. We say that a function $f : \mathit{\Omega} \longrightarrow \mathbb{C}$ is *holomorphic* ($f \in \mathcal{O}(\mathit{\Omega})$), if:

- $f \in \mathcal{O}(\mathit{\Omega} \setminus \{\infty\})$ and
- the function $B(1/R) \ni z \longmapsto f(1/z) \in \mathbb{C}$ is holomorphic, where $1/0 := \infty$.

Remark 1.1.2. Let $f(z) := \sum_{n=0}^{\infty} a_n(z-a)^n$, $|z-a| < R$, where R is the radius of convergence. The following results are known:

[Remark 1.1.2$\longrightarrow$ Exer . . . .]

(a) For every $z \in B(a,R)$ the *complex derivative* $f'(z) := \lim_{\mathbb{C}\ni h\to 0} \frac{f(z+h)-f(z)}{h}$ exists and $f'(z) = \sum_{n=1}^{\infty} n a_n (z-a)^{n-1}$.

(b) The radius of convergence of the above series is equal to R.

(c) f has in $B(a,r)$ all complex derivatives $f^{(k)}(z)$ and $f^{(k)}(z) = \sum_{n=k}^{\infty} k!\binom{n}{k} a_n (z-a)^{n-k}$, $z \in B(a,R)$. In particular,

- f is real analytic as a function of two real variables, $f \in \mathcal{C}^{\omega}(B(a,R),\mathbb{C})$,
- $a_n = \frac{f^{(n)}(a)}{n!}$, $n \in \mathbb{Z}_+$,

- $f(z) = T_a f(z)$, $z \in B(a,R)$, where $T_a f(z) := \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n$ denotes the *Taylor* [1] *series of f at a.*

(d) (Identity principle) Let $D \subset \mathbb{C}$ be a domain, $f, g \in \mathcal{O}(D)$, $A := \{z \in D : f(z) = g(z)\}$. If A has an accumulation point in D, then $f \equiv g$. In particular, if $f \in \mathcal{O}(D)$, $f \not\equiv 0$, then points of the set $f^{-1}(0)$ are isolated.
(e) $\mathcal{O}(\Omega)$ is a $\mathbb{C}$-algebra.
(f) If $f, g \in \mathcal{O}(D)$, where D is a domain and $g \not\equiv 0$, then $f/g \in \mathcal{O}(D \setminus g^{-1}(0))$. In particular, every rational function $R = P/Q$, where P, $Q \in \mathcal{P}(\mathbb{C}, \mathbb{C})$, $Q \not\equiv 0$, is holomorphic in $\mathbb{C} \setminus Q^{-1}(0)$.
(g) The composition of holomorphic functions is holomorphic.
(h) If $f \in \mathrm{Bih}(D_1, D_2)$, then the mapping $\mathrm{Aut}(D_1) \ni \varphi \longmapsto f \circ \varphi \circ f^{-1} \in \mathrm{Aut}(D_2)$ is a group isomorphism.
(i) If $f \in \mathcal{O}(\Omega)$ and $a \in \Omega$ is such that $f'(a) \neq 0$, then there exists on open neighborhood $U \subset \Omega$ of a such that $V := f(U)$ is open and $f : U \longrightarrow V$ is biholomorphic.
(j) If $f \in \mathcal{O}(\Omega)$ and $f : \Omega \longrightarrow G$ is bijective, then $f \in \mathrm{Bih}(\Omega, G)$ if and only if $f'(z) \neq 0$, $z \in \Omega$ (cf. Theorem 5.2.1).

Theorem 1.1.3. *Let $I \subset \mathbb{R}$ be an open interval and $f \in \mathcal{C}^{\omega}(I, \mathbb{C})$. Then there exist a domain $D \subset \mathbb{C}$ and a function $\widetilde{f} \in \mathcal{O}(D)$ such that $D \cap \mathbb{R} = I$ and $\widetilde{f} = f$ on I.*

[Theorem 1.1.3⟶ Exer . . . .]

1.2. Elementary holomorphic functions

1.2.1. Homographies.

Definition 1.2.1. Let $a, b, c, d \in \mathbb{C}$ be such that $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq 0$. Then the mapping $h : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$, $h(z) := \frac{az+b}{cz+d}$ is called *a homography* ($h \in \mathcal{H}$) ($1/\infty : 0$).

Remark 1.2.2 (Basic properties)**.** [Remark 1.2.2⟶ Exer . . . .]

(a) Every homography is a homeomorphism $\widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$. The inverse of a homography is a homography. The set of all homographies is a group (with composition). $\mathcal{H}$ depends on 6 real parameters.
(b) Elementary homographies:

		Parameters	Number of real parameters	
translation	$z \longmapsto z + b$	$b \in \mathbb{C}$	2	subgroup
rotation	$z \longmapsto az$	$a \in \mathbb{T}$	1	subgroup
homothety	$z \longmapsto tz$	$t > 0$	1	subgroup
affine mapping	$z \longmapsto az + b$	$a \in \mathbb{C}_*$, $b \in \mathbb{C}$	4	subgroup
inversion	$z \longmapsto 1/z$		0	

[1] Brook Taylor (1717–1783).

(c) Every homography is a composition of elementary homographies. Every affine mapping is a composition of a rotation, homothety, and translation.
(d) Every homography h is a $\mathcal{C}^\infty$-diffeomorphism on $D := \mathbb{C} \cap h^{-1}(\mathbb{C})$.
(e) Every homography h is a conformal mapping on D, i.e. for every point $a \in D$ and for any $\mathcal{C}^1$-curves $\gamma_1,\ \gamma_2 : (-\varepsilon, \varepsilon) \longrightarrow D$ with $\gamma_1(0) = \gamma_2(0) = a$:
  - h preserves the angle measure: $\measuredangle(\gamma_1'(0), \gamma_2'(0)) = \measuredangle((h \circ \gamma_1)'(0), (h \circ \gamma_2)'(0))$;
  - h preserves the orientation: $O(\gamma_1'(0), \gamma_2'(0)) = O((h \circ \gamma_1)'(0), (h \circ \gamma_2)'(0))$.

(f)
$$S := \Big\{z \in \mathbb{C} : \Big|\frac{z-p}{z-q}\Big| = \lambda\Big\} = \begin{cases} \text{straight line } \{|z-p| = |z-q|\}, & \text{if } p \neq q,\ \lambda = 1 \\ \text{circle } C(\frac{p-\lambda^2 q}{1-\lambda^2},\ \frac{\lambda|p-q|}{|1-\lambda^2|}), & \text{if } p \neq q,\ 0 < \lambda \neq 1 \end{cases}.$$

The points p and q are symmetric with respect to S. In the case of a circle $C(z_0, r)$ this means that the points p, q are on the same half-line starting at z_0 and $|p-z_0||q-z_0| = r^2$. We assume that z_0 and ∞ are symmetric by definition. Moreover, for a straight line L we say that $L \cup \{\infty\}$ is an *improper circle*.
(g) Conversely, every circle or straight line may be represented as a set S. In the case of the circle $C(z_0, r)$ we take arbitrary $p \in \mathbb{C}\setminus(\{z_0\} \cup C(z_0, r))$ and set $q := z_0 + \frac{r^2}{\overline{p-z_0}}$, $\lambda := \frac{|p-z_0|}{r}$.
(h) Homographies map circles onto circles. The set S is mapped onto

$$\Big\{w \in \mathbb{C} : \Big|\frac{w-h(p)}{w-h(q)}\Big| = \lambda\Big|\frac{qc+d}{pc+d}\Big|\Big\}.$$

Symmetric points are mapped onto symmetric points.
(i) If h is an affine mapping, then h maps every proper circle (resp. a straight line) onto a proper circle (resp. a straight line).
(j) If h is an inversion, then the image of S is the set $\big\{w \in \mathbb{C} : \big|\frac{w-1/p}{w-1/q}\big| = \lambda\big|\frac{q}{p}\big|\big\}$. It implies that:
  - the image of a straight line is either a straight line (if $|p| = |q|$) or a circle (if $|p| \neq |q|$);
  - the image of a circle is either a circle (if $\lambda|q| \neq |p|$) or a straight line (if $\lambda|q| = |p|$).

(k) Let $\mathbb{H}^+ := \{x + iy \in \mathbb{C} : y > 0\}$. For any $a \in \mathbb{H}^+$ the homography $h(z) := \frac{z-a}{z-\bar{a}}$ maps $\mathbb{H}^+$ onto the unit disc $\mathbb{D}$.
(l) For any $a \in \mathbb{D}$, $\zeta \in \mathbb{T}$, the homography $h(z) := \zeta h_a(z)$, where $h_a(z) := \frac{z-a}{1-\bar{a}z}$, maps $\mathbb{D}$ onto $\mathbb{D}$.
(m) Let $\operatorname{Aut}_{\mathcal{H}}(\mathbb{D}) := \{h \in \mathcal{H} : h(\mathbb{D}) = \mathbb{D}\}$. Then $\operatorname{Aut}_{\mathcal{H}}(\mathbb{D}) = \{h \in \mathcal{H} : h \text{ is of the form (l)}\}$. In particular, $\operatorname{Aut}_{\mathcal{H}}(\mathbb{D})$ depends on 3 real parameters. Moreover, $\operatorname{Aut}_{\mathcal{H}}(\mathbb{D})$ acts transitively on $\mathbb{D}$, i.e. for any $a, b \in \mathbb{D}$ there exists an $h \in \operatorname{Aut}_{\mathcal{H}}(\mathbb{D})$ such that $h(a) = b$.

1.2.2. Special elementary mappings.

Remark 1.2.3. [Remark 1.2.3⟶ Exer . . . .]

(a) (n-th root) Let $f(z) := e^{\frac{1}{n}\operatorname{Log} z}$, where $\operatorname{Log} : \mathbb{C} \setminus \mathbb{R}_-$ is the principal branch of logarithm. Then f maps bijectively $\mathbb{C} \setminus \mathbb{R}_-$ onto $\{z \in \mathbb{C} \setminus \mathbb{R}_- : |\operatorname{Arg} z| < \pi/n\}$.

(b) (Zhukovsky function ([2])) $Z(z) := \frac{1}{2}(z+1/z)$, $z \in \mathbb{C}_*$. Let $f(z) = f(re^{it}) = u+iv$. Then $u = \frac{1}{2}(r+1/r)\cos t$, $v = \frac{1}{2}(r-1/r)\sin t$. We have:
- $Z(z) = Z(1/z)$, $z \in \mathbb{C}_*$;
- Z is injective on $\mathbb{D}_*$ and on $\mathbb{C} \setminus \overline{\mathbb{D}}$ and maps homeomorphically each of these domains onto $\mathbb{C} \setminus [-1,1]$;
- the inverse mapping has the form $\mathbb{C} \setminus [-1,1] \ni w \longmapsto w \pm \sqrt{w^2-1}$.
- for $r > 0$, $r \neq 1$, Z maps $C(r)$ onto the ellipse $\mathcal{E}(r)$ with foci ± 1 and half axes $\frac{1}{2}(r \pm 1/r)$.
- if $r \longrightarrow 0$, then $\mathcal{E}(r) \longrightarrow \infty$;
- if $r \longrightarrow 1$, then $\mathcal{E}(r) \longrightarrow [-1,1]$, which is twice covered by $Z(\mathbb{T})$.

(c) (exp) Let $u+iv = e^z = e^{x+iy}$, i.e. $u = e^x \cos y$, $v = e^x \sin y$.
- For any $y_0 \in \mathbb{R}$ the horizontal strip $\{x+iy : x \in \mathbb{R},\ y_0 - \pi < y \le y_0 + \pi\}$ is mapped bijectively (but not homeomorphically) by exp onto $\mathbb{C}_*$.
- The horizontal line $y = y_0$ goes to the ray $\{(e^x \cos y_0, e^x \sin y_0) : x \in \mathbb{R}\}$.
- What is the image of the open strip $\{x+iy : x \in \mathbb{R},\ y_0 - \pi < y < y_0 + \pi\}$?
- For any $p_0 \in \mathbb{R}_*$, $q_0 \in \mathbb{R}$, the strip $\{(x, p_0x+q) : x \in \mathbb{R},\ q_0 - \pi < q \le q_0 + \pi\}$ is mapped bijectively onto $\mathbb{C}_*$.
- The line $y = p_0x + q_0$ goes to the spiral curve $\{(e^x \cos(p_0x+q_0), e^x \sin(p_0x+q_0) : x \in \mathbb{R}\}$.

(d) (sin) *sin* maps homeomorphically the strip $\{x+iy : -\pi/2 < x < \pi/2,\ y \in \mathbb{R}\}$ onto $\mathbb{C} \setminus ((-\infty,1] \cup [1,+\infty))$.

The vertical line $x = 0$ is mapped onto $u = 0$. Every vertical line $x = c \neq 0$ is mapped bijectively onto one branch of the hyperbola $\frac{u^2}{\sin^2 c} - \frac{v^2}{\cos^2 c} = 1$.

1.2.3. Formal derivatives.

Definition 1.2.4. Let $\Omega \in \operatorname{top}\mathbb{C}$ and let $f : \Omega \longrightarrow \mathbb{C} \simeq \mathbb{R}^2$, $f = u+iv$, be Fréchet differentiable (in the real sense) at a point $a \in \Omega$. Let $f'_{\mathbb{R}}(a)$ denote the real Fréchet derivative of f at a. Then for $Z = X+iY \in \mathbb{C} \simeq \mathbb{R}^2$ we get

$$\begin{aligned} f'_{\mathbb{R}}(a)(Z) &= \frac{\partial f}{\partial x}(a)X + \frac{\partial f}{\partial y}(a)Y = \frac{\partial f}{\partial x}(a)\frac{Z+\overline{Z}}{2} + \frac{\partial f}{\partial y}(a)\frac{Z-\overline{Z}}{2i} \\ &= \frac{1}{2}\Big(\frac{\partial f}{\partial x}(a) - i\frac{\partial f}{\partial y}(a)\Big)Z + \frac{1}{2}\Big(\frac{\partial f}{\partial x}(a) + i\frac{\partial f}{\partial y}(a)\Big)\overline{Z} = \frac{\partial f}{\partial z}(a)Z + \frac{\partial f}{\partial \overline{z}}(a)\overline{Z}, \end{aligned}$$

where

$$\frac{\partial f}{\partial z}(a) := \frac{1}{2}\Big(\frac{\partial f}{\partial x}(a) - i\frac{\partial f}{\partial y}(a)\Big), \quad \frac{\partial f}{\partial \overline{z}}(a) := \frac{1}{2}\Big(\frac{\partial f}{\partial x}(a) + i\frac{\partial f}{\partial y}(a)\Big)$$

denote the *formal derivatives* of f at a. Of course, to define the above formal derivatives it suffices that the partial derivatives $\frac{\partial f}{\partial x}(a)$ and $\frac{\partial f}{\partial y}(a)$ exist.

Remark 1.2.5. [Remark 1.2.5⟶ Exer . . . .] The following conditions are equivalent:

(i) $f'(a)$ exists;

(ii) $f'_{\mathbb{R}}(a)$ exists and is $\mathbb{C}$-linear ($f'_{\mathbb{R}}(a)(Z) = f'(a)Z$);

([2]) Nikolai Zhukovsky (1847–1921).

(iii) $f'_{\mathbb{R}}(a)$ exists and $\frac{\partial f}{\partial \bar{z}}(a) = 0$, i.e. $\frac{\partial u}{\partial x}(a) = \frac{\partial v}{\partial y}(a)$, $\frac{\partial u}{\partial y}(a) = -\frac{\partial v}{\partial x}(a)$ – the Cauchy-Riemann ([3]) ([4]) equations.

We have $f'(a) = \frac{\partial f}{\partial x}(a) = -i\frac{\partial f}{\partial y}(a) = \frac{\partial f}{\partial z}(a)$.

Exercise 1.2.6. [Exercise 1.2.6⟶ Exer . . . .]

(a) Let $f(x+iy) := \sqrt{|xy|}$, $z = x + iy \in \mathbb{C}$. Then $\frac{\partial f}{\partial x}(0) = \frac{\partial f}{\partial y}(0) = 0$, but $f'(0)$ does not exist.
(b) If $f'(a)$ exists, then $\det f'_{\mathbb{R}}(a) = |f'(a)|^2$.
(c) Let $D \subset \mathbb{C}$ be a domain, $f = u + iv \in \mathcal{O}(D)$. If $|f| = \text{const}$, then $f = \text{const}$.

([3]) Augustin Cauchy (1789–1857).
([4]) Bernhard Riemann (1826–1866).

CHAPTER 2

Basic properties of holomorphic functions

2.1. Basic theorems

Definition 2.1.1. Let $\gamma : [\alpha, \beta] \longrightarrow \mathbb{C}$ be a *path*, i.e. a piecewise $\mathcal{C}^1$ curve, and let $f = u + iv : \gamma^* \longrightarrow \mathbb{C}$ be continuous. Define

$$\int_\gamma f dz := \int_\gamma (u+iv)d(x+iy) = \int_\gamma udx - vdy + i\int_\gamma vdx + udy = \int_\alpha^\beta f(\gamma(t))\gamma'(t)dt.$$

Remark 2.1.2. Observe that $\big|\int_\gamma f(z)dz\big| \le \boldsymbol{\ell}(\gamma)\|f\|_{\gamma^*}$, where $\boldsymbol{\ell}(\gamma) = \int_\alpha^\beta |\gamma'(t)|dt$.

[Remark 2.1.2⟶ Exer . . . .]

Lemma 2.1.3 (Lemma on production of holomorphic functions)**.** *Let $\gamma : [0,1] \longrightarrow \mathbb{C}$ be a path and let $g : \gamma^* \longrightarrow \mathbb{C}$ be continuous. Set*

$$f(z) := \frac{1}{2\pi i}\int_\gamma \frac{g(\zeta)}{\zeta - z}d\zeta, \quad z \in \mathbb{C}\setminus\gamma^*.$$

Then $f \in \mathcal{O}(\mathbb{C}\setminus\gamma^*)$,

$$f^{(k)}(z) = \frac{k!}{2\pi i}\int_\gamma \frac{g(\zeta)}{(\zeta - z)^{k+1}}d\zeta, \quad z \in \mathbb{C}\setminus\gamma^*,\ k \in \mathbb{N},\ \textit{and}$$

$$f(z) = \sum_{n=0}^\infty \frac{f^{(n)}(a)}{n!}(z-a)^n, \quad a \in \mathbb{C}\setminus\gamma^*,\ |z-a| < \operatorname{dist}(a,\gamma^*).$$

In particular, $d(T_a f) \ge \operatorname{dist}(a,\gamma^*)$, $a \in \mathbb{C}\setminus\gamma^*$.

Proof. Fix an $a \in \mathbb{C}\setminus\gamma^*$ and let $r := \operatorname{dist}(a,\gamma^*)$, $0 < \vartheta$. Then for $z \in B(a,\vartheta r)$ and $\zeta \in \gamma^*$ we get

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - a}\cdot\frac{1}{1 - \frac{z-a}{\zeta-a}} = \sum_{n=0}^\infty \frac{(z-a)^n}{(\zeta-a)^{n+1}}.$$

The series is uniformly convergent because $|\frac{z-a}{\zeta-a}| \le \vartheta$. Hence

$$f(z) = \sum_{n=0}^\infty \Big(\frac{1}{2\pi i}\int_\gamma \frac{g(\zeta)}{(\zeta-a)^{n+1}}d\zeta\Big)(z-a)^n, \quad z \in B(a,r). \qquad \square$$

Definition 2.1.4. We say that a bounded domain is *regular* if $D = D_0 \setminus (\overline{D}_1 \cup \dots \cup \overline{D}_p)$, where $D_0, \dots, D_p$ are Jordan domains, $\overline{D}_j \subset D_0$, $j = 1 \dots, p$, $\overline{D}_j \subset \operatorname{ext} D_k$, $j \neq k$, $j, k = 1, \dots, p$, and ∂D_j is a Jordan path which has the positive orientation with respect to D (like in the classical Green ([1]) theorem).

Theorem 2.1.5 (Cauchy-Green formula)**.** *Let $D \subset \mathbb{C}$ be a regular domain. Let $f \in \mathcal{C}^1(\overline{D})$, i.e. $f \in \mathcal{C}^1(\Omega)$, where $\Omega \in \operatorname{top} \mathbb{C}$ and $\overline{D} \subset \Omega$. Then*

$$f(z) = \frac{1}{2\pi i}\Big(\int\limits_{\partial D} \frac{f(\zeta)}{\zeta - z} d\zeta + \int\limits_{D} \frac{\frac{\partial f}{\partial \overline{\zeta}}(\zeta)}{\zeta - z} d\zeta \wedge d\overline{\zeta}\Big), \quad z \in D.$$

In particular, if additionally $f'(z)$ exists for all $z \in D$ (e.g. $f \in \mathcal{O}(D)$), then we get the Cauchy formula

$$f(z) = \frac{1}{2\pi i}\int\limits_{\partial D} \frac{f(\zeta)}{\zeta - z} d\zeta, \quad z \in D.$$

Proof. Fix an $a \in D$. Applying the Green formula to the domain $D_\varepsilon := D \setminus \overline{B}(a, \varepsilon)$, $0 < \varepsilon \ll 1$, we get:

$$\int\limits_{\partial D} \frac{f(\zeta)}{\zeta - a} d\zeta - \int\limits_{C(a,\varepsilon)} \frac{f(\zeta)}{\zeta - a} d\zeta = \int\limits_{\partial D_\varepsilon} \frac{f(\zeta)}{\zeta - a} d\zeta = \int\limits_{D_\varepsilon} d\Big(\frac{f(\zeta)}{\zeta - a} d\zeta\Big)$$

$$= -\int\limits_{D_\varepsilon} \frac{\frac{\partial f}{\partial \overline{\zeta}}(\zeta)}{\zeta - a} d\zeta \wedge d\overline{\zeta} \underset{\varepsilon \longrightarrow 0+}{\longrightarrow} -\int\limits_{D} \frac{\frac{\partial f}{\partial \overline{\zeta}}(\zeta)}{\zeta - a} d\zeta \wedge d\overline{\zeta}.$$

On the other hand $\Big|\frac{1}{2\pi i}\int\limits_{C(a,\varepsilon)} \frac{f(\zeta)}{\zeta - a} d\zeta - f(a)\Big| \le \max\{|f(\zeta) - f(a)| : \zeta \in C(a, \varepsilon)\} \underset{\varepsilon \to 0+}{\longrightarrow} 0.$ $\square$

Corollary 2.1.6. *If $f \in \mathcal{O}(\Omega)$, then $f(z) = \frac{1}{2\pi i}\int\limits_{C(a,r)} \frac{f(\zeta)}{z - \zeta} d\zeta$, $z \in B(a, r) \subset\subset \Omega$.*

Consequently, by Lemma 2.1.3, $d(T_a f \ge d_\Omega(a)$, $a \in \Omega$.
In particular, if $f \in \mathcal{O}(\mathbb{C})$, then $d(T_a f) = +\infty$, $a \in \mathbb{C}$.

Theorem 2.1.7 (Weierstrass theorem ([2]))**.** *Let $(f_k)_{k=1}^\infty \subset \mathcal{O}(\Omega)$ and suppose that $f_k \longrightarrow f_0$ locally uniformly in Ω. Then $f_0 \in \mathcal{O}(\Omega)$.*

Proof. Obviously, $f_0 \in \mathcal{C}(\Omega, \mathbb{C})$ and for each disc $B(a, r) \subset\subset \Omega$ we have

$$f_k(z) = \frac{1}{2\pi i}\int\limits_{C(a,r)} \frac{f_k(\zeta)}{z - \zeta} d\zeta, \quad z \in B(a, r),\ k \in \mathbb{N}.$$

Since $f_k \longrightarrow f_0$ uniformly on $C(a, r)$, we get $f_0(z) = \frac{1}{2\pi i}\int\limits_{C(a,r)} \frac{f_0(\zeta)}{z - \zeta} d\zeta$, $z \in B(a, r)$. It remains to apply the production lemma. $\square$

([1]) George Green (1793–1841).
([2]) Karl Weierstrass (1815–1897).

Theorem 2.1.8 (Maximum principle). *Let $D \subset \mathbb{C}$ be a domain and $f \in \mathcal{O}(D)$, $f \not\equiv \text{const}$. Then:*

(a) $|f|$ *does not have local maxima in* D.
(b) $|f|$ *does not have a local minimum at a point* $a \in D$ *with* $f(a) \neq 0$.
(c) *If* D *is bounded, then* $|f(z)| < \sup\{\limsup\limits_{w \to \zeta} |f(w)| : \zeta \in \partial D\}$, $z \in D$.
(d) *If* D *is bounded and* $|f|$ *extends to an upper semicontinuous function on* $\overline{D}$, *then* $|f(z)| < \max_{\overline{D}} |f|$, $z \in D$.

Proof. (a) Suppose that $|f(z)| \leq |f(a)|$, $z \in B(a,r) \subset\subset D$. By the Cauchy formula we get $|f(a)| \leq \frac{1}{\pi r^2} \int\limits_{B(a,r)} |f| d\mathcal{L}^2 \leq |f(a)|$. Thus $|f| = |f(a)|$ a.e. on $B(a,r)$, which implies that $|f| = |f(a)|$ on $B(a,r)$. By Exercise 1.2.6(c) $f = \text{const}$ on $B(a,r)$ and finally, by the identity principle, $f \equiv \text{const}$ on D — a contradiction.

(b) We apply (a) to $1/f$.

(c) Fix a $z_0 \in D$ and let $(D_k)_{k=1}^\infty$ be a sequence of domains such that $z_0 \in D_1 \subset D_k \subset D_{k+1} \subset\subset D$, $D = \bigcup\limits_{k=1}^\infty D_k$. For each k there exists a $w_k \in \overline{D}_k$ such that $|f(w_k)| = \max_{\overline{D}_k} |f|$. By (a) we get $|f(z_0)| < |f(w_k)| \leq |f(w_{k+1})|$. We may assume that $w_k \longrightarrow \zeta \in \partial D$. Then $|f(z_0)| < \limsup\limits_{k \to +\infty} |f(w_k)| \leq \limsup\limits_{w \to \zeta} |f(w)|$.

(d) follows from (c). $\square$

Theorem 2.1.9 (Cauchy inequalities). (a) *Let* $f \in \mathcal{O}(B(a,r))$, $|f| \leq C$. *Then* $|f^{(n)}(a)| \leq \frac{n!}{r^n} C$, $n \in \mathbb{N}$.
(b) *Let* $f \in \mathcal{O}(\Omega)$. *Then for any compact set* $K \subset\subset \Omega$ *and* $0 < r < d_\Omega(K)$ *we get* $\|f^{(n)}\|_K \leq \frac{n!}{r^n} \|f\|_{K^{(r)}}$, $n \in \mathbb{N}$.

Proof. (a) For every $0 < s < r$ we get

$$|f^{(n)}(a)| = \Big| \frac{n!}{2\pi i} \int\limits_{C(a,s)} \frac{f(\zeta)}{(\zeta - a)^{n+1}} d\zeta \Big| \leq \frac{n!}{2\pi} \int\limits_0^{2\pi} \frac{|f(a + s e^{i\vartheta})|}{s^n} d\vartheta \leq \frac{n!}{s^n} C, \quad n \in \mathbb{N}.$$

(b) follows from (a). $\square$

Corollary 2.1.10 (Weierstrass theorem II). *Let* $(f_k)_{k=1}^\infty \subset \mathcal{O}(\Omega)$ *and assume that* $f_k \longrightarrow f_0$ *locally uniformly in* Ω. *Then* $f_0 \in \mathcal{O}(\Omega)$ *and* $f_k^{(n)} \longrightarrow f_0^{(n)}$ *locally uniformly in* Ω *for every* $n \in \mathbb{N}$.

Definition 2.1.11. For $\Omega \in \operatorname{top} \mathbb{C}$ let $L_h^p(\Omega) := L^p(\Omega) \cap \mathcal{O}(\Omega)$, $1 \leq p \leq +\infty$.

- $\mathcal{H}^\infty(\Omega) := L_h^\infty(\Omega)$ is the space of all bounded holomorphic functions on Ω.
- $L_h^2(\Omega)$ is a unitary space with scalar product $L_h^2(\Omega) \times L_h^2(\Omega) \ni (f,g) \longmapsto \int\limits_\Omega f\overline{g} d\mathcal{L}^2$.

Theorem 2.1.12. (a) $\|f\|_K \leq \frac{1}{\pi r^2} \int\limits_{K^{(r)}} |f| d\mathcal{L}^2$, $f \in \mathcal{O}(\Omega)$, $0 < r < d_\Omega(K)$, $K \subset\subset \Omega$.

(b) $\|f\|_K \le \frac{1}{\pi r^2}(\mathcal{L}(K^{(r)}))^{1/q}\Big(\int\limits_{K^{(r)}} |f|^p d\mathcal{L}^2\Big)^{1/p}, \quad f \in \mathcal{O}(\Omega),\ 0 < r < d_\Omega(K),\ 1 < p < +\infty,$ *where* $1/p + 1/q = 1$.

(c) $L^p_h(\Omega)$ *is a Banach* ([3]) *space*, $1 \le p \le +\infty$.

(d) $L^2_h(\Omega)$ *is a Hilbert* ([4]) *space.*

Theorem 2.1.13 (Liouville theorem ([5])). *Let* $f \in \mathcal{O}(\mathbb{C})$. *Then* $f \in \mathcal{P}_d(\mathbb{C})$ *if and only if for some* $R, C > 0$ *we have* $|f(z)| \le C|z|^d$, $|z| \ge R$, *or equivalently,* $|f(z)| \le M(1 + |z|)^d$, $z \in \mathbb{C}$, *for an* $M > 0$.

Proof. It is clear that every polynomial satisfies the inequality (Exercise). Conversely, suppose that the inequality is fulfilled. We know that $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$, $z \in \mathbb{C}$ (cf. Corollary 2.1.6). Using the Cauchy inequalities, for $r \ge R$ and $n > d$ we have

$$|a_n| = \Big|\frac{f^{(n)}(0)}{n!}\Big| \le \frac{Cr^d}{r^n} = Cr^{d-n} \underset{r\to+\infty}{\longrightarrow} 0. \qquad \square$$

Theorem 2.1.14 (Schwarz lemma ([6])). (a) *Let* $f \in \mathcal{O}(B(r))$, $|f| \le C$, *and* $f(0) = 0$. *Then* $|f(z)| \le C|z|/r$, $z \in \mathbb{D}$, $|f'(0)| \le C/r$. *Moreover, if* $|f(z_0)| = C|z_0|/r$ *for a* $z_0 \in B_*(r)$ *or* $|f'(0)| = C/r$, *then* $f(z) = Ce^{i\vartheta_0}z/r$, $z \in B(r)$, *for a* $\vartheta_0 \in \mathbb{R}$.

(b) *Let* $f \in \mathcal{O}(B(r))$, $|f| \le C$, $f(0) = \dots = f^{(k-1)}(0) = 0$ $(k \in \mathbb{N})$. *Then* $|f(z)| \le C(|z|/r)^k$, $z \in \mathbb{D}$, $|f^{(k)}(0)| \le k!C/r^k$. *Moreover, if* $|f(z_0)| = C(|z_0|/r)^k$ *for a* $z_0 \in B_*(r)$ *or* $|f^{(k)}(0)| = k!C/r^k$, *then* $f(z) = Ce^{i\vartheta_0}(z/r)^k$, $z \in B(r)$, *for a* $\vartheta_0 \in \mathbb{R}$.

Proof. (a) follows from (b).

(b) Let $g(z) := \begin{cases} \frac{f(z)}{z^k}, & z \in B_*(r) \\ \frac{f^{(k)}(0)}{k!}, & z = 0 \end{cases}$, $z \in B(r)$. Obviously, $g \in \mathcal{O}(B(r))$ (Exercise). Moreover, by the maximum principle, we get $|g(z)| \le \sup_{\zeta \in C(r)} \limsup\limits_{w\to\zeta} |g(w)| \le C/r^k$, $z \in B(r)$, which implies the result. $\square$

Recall that $h_a(z) := \frac{z-a}{1-\overline{a}z}$, $z \in \mathbb{C} \setminus \{1/\overline{a}\}$. Observe that $(h_a)^{-1} = h_{-a}$,

$$h'_a(z) = \frac{1 - \overline{a}\ z - (z-a)(-\overline{a})}{(1-\overline{a}\ z)^2} = \frac{1-|a|^2}{(1-\overline{a}z)^2}.$$

In particular, $h'_a(a) = \frac{1}{1-|a|^2}$.

Theorem 2.1.15. $\operatorname{Aut}(\mathbb{D}) = \operatorname{Aut}_{\mathcal{H}}(\mathbb{D})$.

Proof. Fix a $g \in \operatorname{Aut}(\mathbb{D})$. Then $f := h_{g(0)} \circ g \in \operatorname{Aut}(\mathbb{D})$ and $f(0) = 0$. Thus it suffices to prove that the set $\operatorname{Aut}_0(\mathbb{D}) := \{f \in \operatorname{Aut}(\mathbb{D}) : f(0) = 0\}$ coincides with the group of rotations. By the Schwarz lemma (applied to f and f^{-1}) we conclude that $|f(z)| = |z|$, $z \in \mathbb{D}$. Hence f is a rotation. $\square$

([3]) Stefan Banach (1892–1945).
([4]) David Hilbert (1862–1943).
([5]) Joseph Liouville (1809–1882).

Definition 2.1.16. Set

$$\boldsymbol{m}(z',z''):=\Big|\frac{z'-z''}{1-z'\overline{z}''}\Big|=|h_{z''}(z')|,\quad z',z''\in\mathbb{D},\qquad \boldsymbol{\gamma}(z):=\frac{1}{1-|z|^2}=h_z'(z),\quad z\in\mathbb{D}.$$

The Schwarz lemma may be easily generalized to the following result.

Theorem 2.1.17 (Schwarz-Pick (7) lemma)**.** *Let $f\in\mathcal{O}(\mathbb{D},\mathbb{D})$. Then:*
[Theorem 2.1.17⟶ Exer . . . .]

(a) $\boldsymbol{m}(f(z'),f(z''))\le\boldsymbol{m}(z',z'')$, $z',z''\in\mathbb{D}$.
(b) $\boldsymbol{\gamma}(f(z))|f'(z)|\le\boldsymbol{\gamma}(z)$, $z\in\mathbb{D}$.
(c) *the following conditions are equivalent:*
  (i) $f\in\operatorname{Aut}(\mathbb{D})$;
  (ii) $\boldsymbol{m}(f(z'),f(z''))=\boldsymbol{m}(z',z'')$, $z',z''\in\mathbb{D}$;
  (iii) $\boldsymbol{m}(f(z_0'),f(z_0''))=\boldsymbol{m}(z_0',z_0'')$ *for some* z_0', $z_0''\in\mathbb{D}$, $z_0'\neq z_0''$;
  (iv) $\boldsymbol{\gamma}(f(z))|f'(z)|=\boldsymbol{\gamma}(z)$, $z\in\mathbb{D}$;
  (v) $\boldsymbol{\gamma}(f(z_0))|f'(z_0)|=\boldsymbol{\gamma}(z_0)$ *for a* $z_0\in\mathbb{D}$.

2.2. Normal families, Montel theorem, Vitali theorem

Definition 2.2.1. Let $D\subset\mathbb{C}$ be a domain. We say that a family $\mathcal{R}\subset\mathcal{O}(D)$ is *normal* in D, if every sequence $(f_n)_{n=1}^\infty\subset\mathcal{R}$ contains a subsequence $(f_{n_k})_{k=1}^\infty$ such that $f_{n_k}\longrightarrow f$ locally uniformly in D, where either $f:D\longrightarrow\mathbb{C}$ or $f\equiv\infty$. We say that $\mathcal{R}\subset\mathcal{O}(D)$ is *locally normal* if each point $a\in D$ has a connected neighborhood U such that $\mathcal{R}|_U$ is normal in U.

Lemma 2.2.2. *Every locally normal family is normal.*

Proof. For any $a\in D$ let $U_a\subset D$ be a disc centered at a such that $\mathcal{R}|_{U_a}$ is normal. By the Lindelöf theorem there exists a sequence $(a_k)_{k=1}^\infty\subset D$ such that $D=\bigcup_{k=1}^\infty U_{a_k}$. We fix an arbitrary sequence $(f_n)_{n=1}^\infty=(f_{0,n})_{n=1}^\infty\subset\mathcal{R}$. For $k\in\mathbb{N}$ let $(f_{k,n})_{n=1}^\infty$ be a subsequence of $(f_{k-1,n})_{n=1}^\infty$ such that $f_{k,n}\longrightarrow\widehat{f}_k$ locally uniformly on U_{a_k}. The diagonal method of selection gives a subsequence $(f_{n_\ell})_{\ell=1}^\infty$ such $f_{n_\ell}\longrightarrow\widehat{f}_k$ locally uniformly on U_{a_k} for every k. Since D is a domain, we easily exclude the situation where $\widehat{f}_{k'}(U_{a_{k'}})\subset\mathbb{C}$ but $\widehat{f}_{k''}\equiv\infty$ for some k',k'' (Exercise). □

Theorem 2.2.3 (Montel (8) theorem)**.** *Let $(f_k)_{k=1}^\infty\subset\mathcal{O}(\Omega)$ be locally bounded. Then there exists a locally uniformly convergent subsequence $(f_{k_n})_{n=1}^\infty$.*

Consequently, for every domain $D\subset\mathbb{C}$, every locally bounded family $\mathcal{R}\subset\mathcal{O}(D)$ is normal.

(7) Georg Alexander Pick (1859–1942).
(8) Paul Montel (1876–1975).

Proof. First observe that the sequence $(f_k)_{k=1}^\infty$ is equicontinuous. Indeed, if $B(a,2r) \subset\subset \Omega$ and $|f_k(\zeta)| \le C$, $\zeta \in C(a,2r)$, $k \in \mathbb{N}$, then for $z \in B(a,r)$ we have:

$$|f_k(z) - f_k(a)| = \Big|\frac{1}{2\pi i}\int\limits_{C(a,2r)} f_k(\zeta)\Big(\frac{1}{\zeta - z} - \frac{1}{\zeta - a}\Big)d\zeta\Big| = \Big|\frac{1}{2\pi i}\int\limits_{C(a,2r)} f_k(\zeta)\frac{z-a}{(\zeta - z)(\zeta - a)}d\zeta\Big|$$

$$\le \frac{1}{2\pi}\int_0^{2\pi} C\frac{|z-a|}{|a + 2re^{i\vartheta} - z|2r}2rd\vartheta \le \frac{C}{2\pi}|z-a|\int_0^{2\pi}\frac{1}{|a+2re^{i\vartheta} - z|}d\vartheta \le \frac{C}{r}|z-a|.$$

Now we can argue as the Arzela-Ascoli [9] [10] theorem. [11]

Let $A \subset \Omega$ be an arbitrary countable dense set. Using the diagonal method of selection we get a subsequence $(f_{k_n})_{n=1}^\infty$ that is pointwise convergent on A. Using the equicontinuity we conclude that this subsequence is locally uniformly convergent. Indeed, let $B(a,r) \subset\subset \Omega$ for an $a \in A$ and let $\varepsilon > 0$. Then there exists a $0 < \delta \le r$ such that $|f_{k_n}(z) - f_{k_n}(a)| \le \varepsilon$ for all $z \in B(a,\delta)$ and $n \in \mathbb{N}$. Moreover, there exists an n_0 such that for $n, m \ge n_0$ we obtain $|f_{k_n}(a) - f_{k_m}(a)| \le \varepsilon$. Then for $z \in B(a,\delta)$ and $n, m \ge n_0$ we get

$$|f_{k_n}(z) - f_{k_m}(z)| \le |f_{k_n}(z) - f_{k_n}(a)| + |f_{k_n}(a) - f_{k_m}(a)| + |f_{k_m}(a) - f_{k_m}(z)| \le 3\varepsilon. \qquad \square$$

The Montel theorem can be essentially strengthened.

Theorem* 2.2.5 (Montel theorem II). *For any domain $D \subset \mathbb{C}$, every family $\mathcal{R} \subset \mathcal{O}(D)$ such that there exist $w_1, w_2 \in \mathbb{C}$, $w_1 \ne w_2$, with $w_1, w_2 \notin f(D)$, $f \in \mathcal{R}$, is normal.*

Theorem 2.2.6 (Vitali [12] theorem). *Let $(f_k)_{k=1}^\infty \subset \mathcal{O}(D)$ be locally bounded and pointwise convergent on a set $A \subset D$ that has an accumulation point in D. Then $(f_k)_{k=1}^\infty$ converges locally uniformly in D.*

Proof. Suppose that for an $a \in D$ we have two subsequences $(f_{k_n})_{n=1}^\infty$ and $(f_{s_n})_{n=1}^\infty$ such that $\lim\limits_{n\to+\infty} f_{k_n}(a) \ne \lim\limits_{n\to+\infty} f_{s_n}(a)$. By the Montel theorem we may assume that $f_{k_n} \longrightarrow p$, $f_{s_n} \longrightarrow q$ locally uniformly D, where $p, q \in \mathcal{O}(D)$. We know that $p = q$ on A. Hence, by the identity principle, $p \equiv q$. In particular, $p(a) = q(a)$. Thus the sequence $(f_k)_{k=1}^\infty$ is pointwise convergent on D to a function f.

Suppose that $(f_k)_{k=1}^\infty$ is not locally uniformly convergent to f. Then there exist a compact $K \subset D$ and an $\varepsilon_0 > 0$ such that $\forall_{s\in\mathbb{N}}\ \exists_{n_s \ge s}$: $\|f_{n_s} - f\|_K \ge \varepsilon_0$. By the Montel theorem there exists a subsequence $(f_{n_{s_t}})_{t=1}^\infty$ such that $f_{n_{s_t}} \longrightarrow f$ locally uniformly. In particular, $\forall_{\varepsilon>0}\ \exists_{t_0\in\mathbb{N}} : \forall_{t\ge t_0} : \|f_{n_{s_t}} - f\|_K \le \varepsilon$ — a contradiction. $\square$

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(9) Cesare Arzelá (1847–1912).
(10) Giulio Ascoli (1843–1896).
(11)

Theorem 2.2.4 (Arzela-Ascoli theorem). *Let $(g_n)_{n=1}^\infty \subset \mathcal{C}(\Omega, \mathbb{C})$. Assume that the sequence $(g_n)_{n=1}^\infty$ is locally bounded and equicontinuous. Then there exists a subsequence $(g_{n_k})_{k=1}^\infty$ such that $(g_{n_k})_{k=1}^\infty$ converges locally uniformly in Ω.*

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(12) Giuseppe Vitali (1875–1932).

2.3. Complex derivatives vs. holomorphicity

Lemma 2.3.1. *Let $D \subset \mathbb{C}$ be a domain and let $f = u + iv : D \longrightarrow \mathbb{C}$ be continuous. Then the following conditions are equivalent:*

(i) *for any $a, b \in D$, the integral $\int_a^b f(z)dz := \int_\gamma f(z)dz$ is independent of the path γ joining a and b in D;*

(ii) *f has a* primitive function, *i.e. there exists a function $F : D \longrightarrow \mathbb{C}$ such that $F'(z) = f(z)$, $z \in D$.*

Proof. (ii) $\Longrightarrow$ (i): $\int_\gamma f(z)dz = \int_\alpha^\beta F'(\gamma(t))\gamma'(t)dt = \int_\alpha^\beta (F \circ \gamma)'(t)dt = F(\gamma(\beta)) - F(\gamma(\alpha))$.

(i) $\Longrightarrow$ (ii): The integral $\int_\gamma f(z)dz = \int_\gamma udx - vdy + i\int_\gamma vdx + udy$ is independent of the path if and only if each of the integrals $\int_\gamma udx - vdy$, $\int_\gamma vdx + udy$ is independent. Then there exist functions $\varphi, \psi \in \mathcal{C}^1(D, \mathbb{R})$ such that $\frac{\partial\varphi}{\partial x} = u$, $\frac{\partial\varphi}{\partial y} = -v$, $\frac{\partial\psi}{\partial x} = v$, $\frac{\partial\psi}{\partial y} = u$. Let $F := \varphi + i\psi$. Then F is $\mathcal{C}^1$ satisfies the Cauchy-Riemann equations and $F' = \varphi'_x + i\psi'_x = u + iv = f$. □

Theorem 2.3.2 (Characterization of holomorphic functions)**.** *Let $\Omega \in \operatorname{top}\mathbb{C}$ and $f : \Omega \longrightarrow \mathbb{C}$. Then the following conditions are equivalent:*

(i) *$f'(z)$ exists for each $z \in \Omega$;*

(ii) *$f'_{\mathbb{R}}(z)$ exists for each $z \in \Omega$ and $\frac{\partial f}{\partial \bar z}(z) = 0$, $z \in \Omega$;*

(iii) *$f \in \mathcal{C}(\Omega, \mathbb{C})$ and $\int_{\partial T} f(z)dz = 0$ for each triangle $T \subset\subset \Omega$ (the equivalence* (i) $\Longleftrightarrow$ (iii) *is called* Morera ([13]) theorem*);*

(iv) *$f \in \mathcal{C}(\Omega, \mathbb{C})$ and for each starlike domain $G \subset \Omega$ there exists an $F : G \longrightarrow \mathbb{C}$ such that $F' = f$ in G;*

(v) *$f \in \mathcal{C}(\Omega, \mathbb{C})$ and for each disc $B(a, r) \subset\subset \Omega$ we get*

$$f(z) = \frac{1}{2\pi i}\int_{C(a,r)} \frac{f(\zeta)}{z - \zeta}d\zeta, \quad z \in B(a, r);$$

(vi) *for each $a \in \Omega$ the function has all complex derivatives $f^{(n)}(a)$, $n \in \mathbb{N}$, and*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z - a)^n, \quad |z - a| < \operatorname{dist}(a, \partial\Omega);$$

(vii) *$f \in \mathcal{O}(\Omega)$.*

Proof. We need a few auxiliary results.

Theorem 2.3.3 (Cauchy- Goursat ([14]) theorem)**.** *Let If $f : \Omega \longrightarrow \mathbb{C}$ is such that $f'(z)$ exists for each $z \in \Omega$, then $\int_{\partial T} f(z)dz = 0$ for every triangle $T = \operatorname{conv}\{a, b, c\}$ ($\partial T := [a, b, c, a]$).*

([13]) Giacinto Morera (1856–1909).
([14]) Édouard Jean-Baptiste Goursat (1858–1936).

Proof. We may assume that $T_0 := T$ is non-degenerated. Using points $p := \frac{1}{2}(a+b)$, $q = \frac{1}{2}(b+c)$, and $r := \frac{1}{2}(c+a)$, we divide T_0 into four triangles $T_{0,1} = \operatorname{conv}\{a,p,r\}$, $T_{0,2} := \operatorname{conv}\{p,b,q\}$, $T_{0,3} := \operatorname{conv}\{q,c,r\}$, and $T_{0,4} := \operatorname{conv}\{p,q,r\}$. Then

$$\int\limits_{\partial T_0} f(z)dz = \sum_{j=1}^{4} \int\limits_{\partial T_{0,j}} f(z)dz.$$

Let $T_1 \in \{T_{0,1},\dots,T_{0,4}\}$ be such that $\Big|\int\limits_{\partial T_1} f(z)dz\Big| = \max\Big\{\Big|\int\limits_{\partial T_{0,j}} f(z)dz\Big| : j = 1,2,3,4\Big\}$. Obviously,

$$\Big|\int\limits_{\partial T_0} f(z)dz\Big| \le 4\Big|\int\limits_{\partial T_1} f(z)dz\Big|.$$

We repeat the above procedure and we get a sequence $(T_j)_{j=1}^{\infty}$ of triangles such that for all $j \in \mathbb{N}$:

- $T_{j+1} \subset T_j$,
- $\boldsymbol{\ell}(\partial T_j) = \frac{1}{2^j}\boldsymbol{\ell}(\partial T_0)$,
- $\Big|\int\limits_{\partial T_0} f(z)dz\Big| \le 4^j\Big|\int\limits_{\partial T_j} f(z)dz\Big|$.

Let $\{a\} := \bigcap\limits_{j=1}^{\infty} T_j$. We have $f(z) = f(a) + f'(a)(z-a) + \alpha(z)(z-a)$, where $\alpha(z) \longrightarrow 0$ when $z \longrightarrow a$. The function $z \longmapsto f(a) + f'(a)(z-a)$ has obviously a primitive. Thus, we finally get

$$\Big|\int\limits_{\partial T_0} f(z)dz\Big| \le 4^j\Big|\int\limits_{\partial T_j}(f(a)+f'(a)(z-a)+\alpha(z)(z-a))dz\Big| = 4^j\Big|\int\limits_{\partial T_j}\alpha(z)(z-a)dz\Big|$$
$$\le 4^j\boldsymbol{\ell}(\partial T_j)\max\{|\alpha(z)(z-a)| : z \in \partial T_j\} \le 4^j\boldsymbol{\ell}^2(\partial T_j)\|\alpha\|_{\partial T_j} = \boldsymbol{\ell}^2(\partial T_0)\|\alpha\|_{\partial T_j} \underset{j\to+\infty}{\longrightarrow} 0.$$

□

Theorem 2.3.4 (Cauchy integral formula). *Let $h : \varOmega \longrightarrow \mathbb{C}$ be such that $h'(z)$ exists for any $z \in \varOmega$ and let $B(c,r) \subset\subset \varOmega$. Then $h(a) = \frac{1}{2\pi i}\int\limits_{C(c,r)} \frac{h(z)}{z-a}dz$, $a \in B(c,r)$.*

Proof. Fix an a and let $g(z) := \begin{cases} \frac{h(z)-h(a)}{z-a}, & \text{if } z \in \varOmega\setminus\{a\} \\ h'(a), & \text{if } z = a\end{cases}$. It clear that g is continuous on $\varOmega$ and $g'(z)$ exists for $z \in \varOmega\setminus\{a\}$. By the Cauchy-Goursat theorem we get $\int\limits_{\partial T} g(z)dz = 0$ for any triangle $T \subset \varOmega\setminus\{a\}$. Since g is continuous, using an approximation, we see that $\int\limits_{\partial T} g(z)dz = 0$ for any triangle $T \subset \varOmega$. Consequently, g has a primitive in any starlike domain

$G \subset \Omega$. Hence,

$$0 = \int_{C(c,r)} g(z)dz = \int_{C(c,r)} \frac{h(z)-h(a)}{z-a}dz \text{ and finally}$$

$$\frac{1}{2\pi i}\int_{C(c,r)} \frac{h(z)}{z-a}dz = \frac{1}{2\pi i}\int_{C(c,r)} \frac{h(a)}{z-a}dz = h(a). \qquad \square$$

The main proof will be divided into several steps.

It clear that (i) $\Longleftrightarrow$ (ii) and (vi) $\Longleftrightarrow$ (vii) $\Longrightarrow$ (i).

(v) $\Longleftrightarrow$ (vii): Use the Cauchy formula (Theorem 2.3.4) and the production lemma.

(i) $\Longrightarrow$ (iii) follows from the Cauchy-Goursat theorem (Theorem 2.3.3.

(iii) $\Longrightarrow$ (iv): Suppose that G is starlike with respect to a $c \in G$. Put $F(z) := \int_{[c,z]} f(\zeta)d\zeta$, $z \in G$. Fix an $a \in G$. Then

$$\Big|\frac{F(a+h)-F(a)}{h} - f(a)\Big| = \Big|\frac{1}{h}\Big(\int_{[c,a+h]} f(z)dz - \int_{[c,a]} f(z)dz - \int_{[a,a+h]} f(a)dz\Big)\Big|$$
$$= \Big|\frac{1}{h}\int_{[a,a+h]} (f(z)-f(a))dz\Big| \le \max\{|f(z)-f(a)| : z \in [a,a+h]\} \underset{h\to 0}{\longrightarrow} 0. \qquad \square$$

(iv) $\Longrightarrow$ (v): We apply Theorem 2.3.4 to the function F. Using the production lemma we conclude that $F \in \mathcal{O}(\Omega)$ and hence $f = F' \in \mathcal{O}(\Omega)$.

Theorem 2.3.5. *Let $D \subset \mathbb{C}$ be a starlike domain with respect to a point $a \in D$ and let $f : D \longrightarrow \mathbb{C}_*$ be holomorphic. Then f has of its logarithm n D. (cf. Theorem 2.3.12).*

Proof. Put $h(z) := \int_a^z \frac{f'(\zeta)}{f(\zeta)}d\zeta + \operatorname{Log} f(a)$, $z \in D$. We know that $h' = f'/f$ in D, and so $(fe^{-h})' = f'e^{-h} - fe^{-h}h' \equiv 0$. Thus $fe^{-h} = \text{const} = f(a)e^{-h(a)} = f(a)e^{-\operatorname{Log} f(a)} = 1$, i.e. $e^h \equiv f$. $\square$

Remark 2.3.6. If f has a branch of its logarithm in D, then f has a branch of p-th root in D for every $p \in \mathbb{N}$. Indeed, let g be a branch of logarithm of f. Then $f = e^g = (e^{g/p})^p$.

Definition 2.3.7. Let $C \subset \mathbb{C}$ be a circle (proper or not). Then we denote by $S_C : \mathbb{C} \longrightarrow \mathbb{C}$ the symmetry with respect to C (i.e. for each $z \in \mathbb{C}$ the points z and $S_C(z)$ are symmetric with respect to C).

Theorem 2.3.8 (Riemann-Schwarz symmetry principle)**.** *Let $C_1, C_2 \subset \mathbb{C}$ be circles and let $D \subset \operatorname{int} C_1$ be a domain (if C_j is a line then $\operatorname{int} C_j$ is one of the half-planes of $\mathbb{C} \setminus C_j$). Assume that $(\partial D) \cap C_1$ contains an open arc $L \neq \varnothing$. Let $f \in \mathcal{O}(D) \cap \mathcal{C}(D \cup L)$ be such that $f(L) \subset C_2$ and let* $\widetilde{f}(z) := \begin{cases} f(z), & \text{if } z \in D \cup L \\ S_{C_2}(f(S_{C_1}(z))), & \text{if } S_{C_1}(z) \in D \end{cases}$. *Then $f \in \mathcal{O}(D \cup L \cup S_{C_1}(D))$.*

In particular, if $C_1 = C_2 = \mathbb{R}$, *then* $\widetilde{f}(z) := \begin{cases} f(z), & \text{if } z \in D \cup L \\ \overline{f(\overline{z})}, & \text{if } \overline{z} \in D \end{cases}$.

Proof. Using suitable homographies we reduce the problem to the case where $C_1 = C_2 = \mathbb{R}$. Now, it remains to apply the Morera theorem. $\square$

Corollary 2.3.9. [Corollary 2.3.9⟶ Exer . . . .] *Let* $D \subset \mathbb{C}$ *be a domain such* $L_1 \subset \partial D$, *where* L_1 *is an open analytic arc, i.e.* $L_1 = \gamma_1((0,1))$, $\gamma_1 : (0,1) \longrightarrow \mathbb{C}$ *is analytic, injective, and* $\gamma_1'(t) \neq 0$, $t \in (0,1)$. *Let* $f \in \mathcal{O}(D) \cap \mathcal{C}(D \cup L_1)$ *be such that* $f(L_1) \subset L_2$, *where* L_2 *is an open analytic arc,* $L_2 = \gamma_2((0,1))$. *Then* f *extends holomorphically throught* L_1, *i.e. there exist a domain* $\widetilde{D} \supset D \cup L_1$ *and* $\widetilde{f} \in \mathcal{O}(\widetilde{D})$ *such that* $\widetilde{f} = f$ *on* $D \cup L_1$.

Theorem 2.3.10. *Let* $f : D \longrightarrow \mathbb{C}$ *be holomorphic, let* $a, b \in D$, *and let* $\gamma_0, \gamma_1 : [0,1] \longrightarrow D$ *be paths joining* a *and* b, *that are homotopic in* D. *Then* $\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz$.

Proof. Let $H : [0,1] \times [0,1] \longrightarrow D$ be a homotopy joining γ_0 and γ_1. i.e. H is continuous, $H(0,\cdot) = \gamma_0$, $H(1,\cdot) = \gamma_1$, $H(s,0) = a$, $H(s,1) = b$, $s \in [0,1]$. Note that we do not assume that $H(s,\cdot)$ is a path. Since H is uniformly continuous, we find a $\delta > 0$ such that if $|s' - s''| \le \delta$ and $|t' - t''| \le \delta$, then $|H(s',t') - H(s'',t'')| < r := \operatorname{dist}(H([0,1]\times[0,1]), \partial D)$. Fix an $n \ge 1/\delta$ and let $s_j = t_j := j/n$, $j = 0, \dots, n$, $a_{j,k} = H(s_j, t_k)$, $\sigma_j := [a_{j,0}, \dots, a_{jn}]$. Observe that $G_{j,k} := B(a_{j,k}, r) \subset D$, $G_{j,k}$ is a starlike domain and $H(s,t) \in G_{j,k}$ for $|s - s_j| \le \delta$, $|t - t_k| \le \delta$, $j,k = 1, \dots, n$. Hence $\int_{\gamma_0|_{[t_{k-1},t_k]}} f(z)dz = \int_{[a_{0,k-1},a_{0,k}]} f(z)dz$, $k = 1, \dots, n$ (cf. Theorem 2.3.12). Consequently, $\int_{\gamma_0} f(z)dz = \int_{\sigma_0} f(z)dz$. Analogously, $\int_{\gamma_1} f(z)dz = \int_{\sigma_n} f(z)dz$. It remains to show that $\int_{\sigma_{j-1}} f(z)dz = \int_{\sigma_j} f(z)dz$, $j = 1, \dots, n$. Put $\rho_{j,k} := [a_{j-1,k-1}, a_{j-1,k}, a_{j,k}, a_{j,k-1}, a_{j-1,k-1}]$. We know that $\int_{\rho_{j,k}} f(z)dz = 0$, $j,k = 1, \dots, n$.

$$\begin{array}{ccc} a_{j-1,k-1} & \longrightarrow & a_{j-1,k} \\ \updownarrow & & \updownarrow \\ a_{j,k-1} & \longrightarrow & a_{j,k} \end{array}$$

Adding the above integrals with $k = 1, \dots, n$ we get the formula. $\square$

Consequently, we get

Theorem 2.3.11 (Cauchy–Goursat theorem)**.** *Let* D *be simply connected and let* $f \in \mathcal{O}(D)$. *Then* $\int_\gamma f(z)dz$ *depends only on the end-points of* γ.

Theorem 2.3.12. *Let* D *be simply connected and let* $f \in \mathcal{O}(D, \mathbb{C}_*)$. *Then* f *has a branch of its logarithm in* D.

Proof. Fix an $a \in D$ and define $h(z) := \int_a^z \frac{f'(\zeta)}{f(\zeta)} d\zeta + \operatorname{Log} f(a)$, $z \in D$ (cf. Theorem 5.4.5). We have $(fe^{-h})' = f'e^{-h} - fe^{-h}h' \equiv 0$. This means that $fe^{-h} = \text{const} = f(a)e^{-h(a)} = f(a)e^{-\operatorname{Log} f(a)} = 1$, so $e^h \equiv f$. Thus h is a branch of the logarithm of f. $\square$

2.4. Complex one-dimensional manifolds

Exercise 2.4.1. (1) We say that a Hausdorff topological space M is *a complex one-dimensional manifold* ($M \in \text{CODM}$), if M has an *atlas*, i.e. a family of pairs $\mathcal{A} = (U_\alpha, \varphi_\alpha)_{\alpha\in A}$ such that for all $\alpha \in A$:

- $U_\alpha \in \operatorname{top} M$,
- $\varphi_\alpha : U_\alpha \longrightarrow \varphi_\alpha(U_\alpha) \subset \mathbb{C}$ is homeomorphic,
- $\varphi_\alpha(U_\alpha) \in \operatorname{top}\mathbb{C}$, and
- $\bigcup_{\alpha\in A} U_\alpha = M$,
- $\varphi_\beta \circ \varphi_\alpha^{-1} \in \mathcal{O}(\varphi_\alpha(U_\alpha \cap U_\beta))$ for all $\alpha, \beta \in A$.

Each such a pair $(U_\alpha, \varphi_\alpha) \in \mathcal{A}$ is called a *map*.

(2) Connected CODMs are called Riemann surfaces.
(3) If $N \in \operatorname{top}\widehat{\mathbb{C}}$, then $N \in \text{CODM}$. In particular, $\widehat{\mathbb{C}} \in \text{CODM}$.
(4) If $M \in \text{CODM}$ and $M' \in \operatorname{top} M$, then $M' \in \text{CODM}$.
(5) We say that a map (U, ψ) *is consistent* with the atlas $\mathcal{A} = (U_\alpha, \varphi_\alpha)_{\alpha\in A}$ if $\mathcal{A} \cup \{(U, \psi)\}$ is an atlas.
(6) We say that atlases $\mathcal{A} = (U_\alpha, \varphi_\alpha)_{\alpha\in A}$, $\mathcal{B} = (V_\beta, \psi_\beta)_{\beta\in B}$ are equivalent if $\mathcal{A}\cup\mathcal{B}$ is an atlas.
(7) If M is a Lindelöf space, then for each atlas $\mathcal{A}$ there exists an equivalent atlas $\mathcal{B} = (V_\beta, \psi_\beta)_{\beta\in B}$ such that B is countable.
(8) An atlas $(U_\alpha, \varphi_\alpha)_{\alpha\in A}$ is called *maximal*, if each map that is consistent with $\mathcal{A}$ belongs to $\mathcal{A}$.
(9) Each atlas is equivalent to an atlas contained in the maximal atlas. In fact, each atlas is contained in the unique maximal atlas.
(10) Let $M \in \text{CODM}$ with an atlas $\mathcal{A} = (U_\alpha, \varphi_\alpha)_{\alpha\in A}$. We say that a mapping $f : M \longrightarrow \mathbb{C}$ is *holomorphic* ($f \in \mathcal{O}(M)$) if $f \circ \varphi_\alpha^{-1} \in \mathcal{O}(\varphi_\alpha(U_\alpha))$ for arbitrary $\alpha \in A$. If $M \in \operatorname{top}\mathbb{C}$, then the defintion coincides with the standard definition.
(11) Let $N \in \text{CODM}$ with an atlas $(V_\beta, \psi_\beta)_{\beta\in B}$. We say that a continuous mapping $f : M \longrightarrow N$ is *holomorphic* ($f \in \mathcal{O}(M, N)$), if $\psi_\beta \circ f \circ \varphi_\alpha^{-1} \in \mathcal{O}(\varphi_\alpha(U_\alpha \cap f^{-1}(V_\beta)))$, $(\alpha, \beta) \in A \times B$. In the case $N = \mathbb{C}$ the definitions coincide.
(12) Is the assumption "f continuous" necessary?
(13) If $f : M \longrightarrow N$ is holomorphic with respect to $(U_\alpha, \varphi_\alpha)_{\alpha\in A}$ and $(V_\beta, \psi_\beta)_{\beta\in B}$, then it is holomorphic with respect to the maximal atlases.
(14) The Weierstrass theorem holds for $\mathcal{O}(M)$.
(15) If M is connected, then the identity principle holds on M: if $f, g \in \mathcal{O}(M, N)$ are such that the set $A := \{x \in M : f(x) = g(x)\}$ has an accumulation point in M, then $f \equiv g$.
(16) If M is connected, then the maximum principle holds on M.
(17) If M is compact and connected, then $\mathcal{O}(M) \simeq \mathbb{C}$. For example, $\mathcal{O}(\widehat{\mathbb{C}}) \simeq \mathbb{C}$.
(18) If M is connected and separable, then the Montel and Vitali theorem hold on M.

2.5. Hyperbolic geometry of the unit disc

(1) Recall that $\boldsymbol{m}(\lambda',\lambda'') := \left|\frac{\lambda'-\lambda''}{1-\lambda'\bar{\lambda}''}\right|$, $\lambda',\lambda''\in\mathbb{D}$, $\quad \boldsymbol{\gamma}(\lambda) := \frac{1}{1-|\lambda|^2}$, $\lambda\in\mathbb{D}$.

The function $\boldsymbol{m}$ may be extended to $(\mathbb{C}\times\mathbb{C})\setminus\{(\lambda',\lambda''):\lambda'\bar{\lambda}''=1\}$.

(2) (Schwarz–Pick lemma). Let $f\in\mathcal{O}(\mathbb{D},\mathbb{D})$. Then:

(a) $\boldsymbol{m}(f(\lambda'),f(\lambda''))\le\boldsymbol{m}(\lambda',\lambda'')$, λ', $\lambda''\in\mathbb{D}$.

(b) $\boldsymbol{\gamma}(f(\lambda))|f'(\lambda)|\le\boldsymbol{\gamma}(\lambda)$, $\lambda\in\mathbb{D}$.

(c) The following statements are equivalent:

(i) $f\in\operatorname{Aut}(\mathbb{D})$;

(ii) $\boldsymbol{m}(f(\lambda'),f(\lambda''))=\boldsymbol{m}(\lambda',\lambda'')$, λ', $\lambda''\in\mathbb{D}$;

(iii) $\boldsymbol{m}(f(\lambda_0'),f(\lambda_0''))=\boldsymbol{m}(\lambda_0',\lambda_0'')$ for some λ_0', $\lambda_0''\in\mathbb{D}$ with $\lambda_0'\neq\lambda_0''$;

(iv) $\boldsymbol{\gamma}(f(\lambda))|f'(\lambda)|=\boldsymbol{\gamma}(\lambda)$, $\lambda\in\mathbb{D}$;

(v) $\boldsymbol{\gamma}(f(\lambda_0))|f'(\lambda_0)|=\boldsymbol{\gamma}(\lambda_0)$ for some $\lambda_0\in\mathbb{D}$.

Any holomorphic function $f:\mathbb{D}\longrightarrow\mathbb{D}$ is an $\boldsymbol{m}$- and a $\boldsymbol{\gamma}$-contraction. The only holomorphic $\boldsymbol{m}$- or $\boldsymbol{\gamma}$-isometries are the automorphisms of $\mathbb{D}$.

(3) Let $\varphi\in\mathcal{O}(\mathbb{D},\mathbb{D})$ and let $\varphi(z)=\sum_{s=0}^{\infty}a_sz^s$ be its power series expansion. Then $|a_k|\le 1-|a_0|^2$, $k\in\mathbb{N}$.

Fix a $k\in\mathbb{N}$ and put $\omega_s := e^{\frac{2\pi i}{k}s}$, $s=1,\dots,k$. Recall that $\sum\limits_{s=1}^{k}\omega_s^m=0$, $1\le m<k$. Put $\widetilde{\varphi}(z):=\frac{1}{k}\sum\limits_{s=1}^{k}\varphi(\omega_sz)$, $z\in\mathbb{D}$. Obviously, $\widetilde{\varphi}\in\mathcal{O}(\mathbb{D},\mathbb{D})$ and its power series expansion is given by $\widetilde{\varphi}(z)=a_0+a_kz^k+a_{2k}z^{2k}+\dots$, $\quad z\in\mathbb{D}$. Set $g:=\frac{\widetilde{\varphi}-a_0}{1-\bar{a}_0\widetilde{\varphi}}$. Then $g\in\mathcal{O}(\mathbb{D},\mathbb{D})$ and its power series expansion is given by $g(z)=b_kz^k+\dots$ with $b_k=\frac{a_k}{1-|a_0|^2}$. Using the Cauchy inequality for the coefficient b_k gives finally the inequality.

(4) (Higher order Schwarz–Pick lemma). Let $f\in\mathcal{O}(\mathbb{D},\mathbb{D})$ and $k\in\mathbb{N}$. Then

$$\frac{|f^{(k)}(\lambda)|}{1-|f(\lambda)|^2}\le k!(1+|\lambda|)^{k-1}\frac{1}{(1-|\lambda|^2)^k},\quad \lambda\in\mathbb{D}.$$

Fix a $\lambda\in\mathbb{D}$ and put $\varphi_\lambda(z):=f\big(\frac{z+\lambda}{1+\bar{\lambda}z}\big)=\sum\limits_{j=0}^{\infty}c_j(\lambda)z^j$, $z\in\mathbb{D}$. Then $f(z)=\varphi_\lambda\big(\frac{z-\lambda}{1-\bar{\lambda}z}\big)=\sum\limits_{j=1}^{\infty}c_j(\lambda)\big(\frac{z-\lambda}{1-\bar{\lambda}z}\big)^j$, $z\in\mathbb{D}$. Taking the k-th derivative of f at the point λ we get $f^{(k)}(\lambda)=\sum\limits_{j=1}^{k}c_j(\lambda)\frac{\lambda^{k-j}}{(1-|\lambda|^2)^k}\frac{k!(k-1)!}{(k-j)!(j-1)!}$. Recall that $c_0(\lambda)=f(\lambda)$ and $|c_s(\lambda)|\le 1-|c_0(\lambda)|^2=1-|f(\lambda)|^2$ if $s\in\mathbb{N}$. Hence

$$|f^{(k)}(\lambda)|\le\frac{k!(1-|f(\lambda)|^2)}{(1-|\lambda|^2)^k}\sum_{s=1}^{k}\frac{(k-1)!}{(k-s)!(s-1)!}|\lambda|^{k-s}$$
$$=k!\frac{1-|f(\lambda)|^2}{(1-|\lambda|^2)^k}\sum_{m=0}^{k-1}\frac{(k-1)!}{m!(k-m-1)!}|\lambda|^m=k!\frac{1-|f(\lambda)|^2}{(1-|\lambda|^2)^k}(1+|\lambda|)^{k-1}.$$

(5) $\boldsymbol{m}\in\mathcal{C}^\infty((\mathbb{D}\times\mathbb{D})\setminus\{(\lambda,\lambda):\lambda\in\mathbb{D}\})$, $\boldsymbol{m}^2\in\mathcal{C}^\infty(\mathbb{D}\times\mathbb{D})$, $\boldsymbol{\gamma}\in\mathcal{C}^\infty(\mathbb{D})$.

(6) For any $a \in \mathbb{D}$, $\boldsymbol{m}(\cdot, a) = |\boldsymbol{h}_a|$. In particular, $\boldsymbol{m}(\cdot, a) = 1$ on $\mathbb{T}$ and $\log \boldsymbol{m}(\cdot, a)$ is harmonic in $\mathbb{D} \setminus \{a\}$. Since $\boldsymbol{m}$ is symmetric, the same is true for $\boldsymbol{m}(a, \cdot)$.

(7) $\lim\limits_{\substack{\lambda', \lambda'' \to a \\ \lambda' \neq \lambda''}} \frac{\boldsymbol{m}(\lambda', \lambda'')}{|\lambda' - \lambda''|} = \boldsymbol{\gamma}(a)$, $a \in \mathbb{D}$.

(8) If $u := \boldsymbol{m}^2(a, \cdot)$, then $\boldsymbol{\gamma}^2(a) = \frac{1}{4}(\Delta u)(a)$.

(9) For any a, b, $c \in \mathbb{D}$, $a \neq b \neq c \neq a$, we have $\boldsymbol{m}(a, b) < \boldsymbol{m}(a, c) + \boldsymbol{m}(c, b)$. In particular, $\boldsymbol{m} : \mathbb{D} \times \mathbb{D} \longrightarrow [0, 1)$ is a distance. It is called the *Möbius distance*.

Indeed, observe that for any a, $b \in \mathbb{D}$, $a \neq b$, there exists a unique automorphism $h = \boldsymbol{h}_{a,b} \in \operatorname{Aut}(\mathbb{D})$ such that $h(a) = 0$ and $h(b) \in (0, 1)$. The function $\boldsymbol{m}$ is invariant under $\operatorname{Aut}(\mathbb{D})$, and therefore we may assume that $a = 0$, $b \in (0, 1)$. Then the inequality reduces to $b < |c| + \left|\frac{c-b}{1-cb}\right|$, $c \in \mathbb{D} \setminus \{0, b\}$.

(10) Since $\boldsymbol{m}$ is invariant under $\operatorname{Aut}(\mathbb{D})$, $B_{\boldsymbol{m}}(a, r) = \boldsymbol{h}_{-a}(B(r))$, $a \in \mathbb{D}$, $0 < r < 1$, where $B_{\boldsymbol{m}}$ stands for the $\boldsymbol{m}$-ball. In particular:
– the topology generated by $\boldsymbol{m}$ coincides with the Euclidean topology of $\mathbb{D}$,
– the space $(\mathbb{D}, \boldsymbol{m})$ is complete.

(11) The strict triangle inequality says that the $\boldsymbol{m}$-*segment*

$$[a, b]_{\boldsymbol{m}} := \{\lambda \in \mathbb{D} : \boldsymbol{m}(a, \lambda) + \boldsymbol{m}(\lambda, b) = \boldsymbol{m}(a, b)\}$$

consists only of the ends. Thus, from the geometric point of view, the space $(\mathbb{D}, \boldsymbol{m})$ is trivial.

(12) Let $\alpha : [0, 1] \longrightarrow \mathbb{D}$ be a path. We define its $\boldsymbol{\gamma}$-*length* by the formula $L_{\boldsymbol{\gamma}}(\alpha) := \int_0^1 \boldsymbol{\gamma}(\alpha(t))|\alpha'(t)|dt$.

(13) For any $f \in \mathcal{O}(\mathbb{D}, \mathbb{D})$ we have $L_{\boldsymbol{\gamma}}(f \circ \alpha) \leq L_{\boldsymbol{\gamma}}(\alpha)$. In particular, the $\boldsymbol{\gamma}$-length is invariant under $\operatorname{Aut}(\mathbb{D})$.

(14) Define $\mathbb{P}(\lambda', \lambda'') := \inf\{L_{\boldsymbol{\gamma}}(\alpha) : \alpha : [0, 1] \longrightarrow \mathbb{D}, \alpha \text{ is a path}, \ \lambda' = \alpha(0), \lambda'' = \alpha(1)\}$, $\lambda', \lambda'' \in \mathbb{D}$.

(15) $\mathbb{P} : \mathbb{D} \times \mathbb{D} \longrightarrow \mathbb{R}_+$ is a pseudodistance dominating the Euclidean distance; for any holomorphic function $f : \mathbb{D} \longrightarrow \mathbb{D}$ we have $\mathbb{P}(f(\lambda'), f(\lambda'')) \leq \mathbb{P}(\lambda', \lambda'')$, $\lambda', \lambda'' \in \mathbb{D}$. In particular, $\mathbb{P}$ is invariant under $\operatorname{Aut}(\mathbb{D})$.

(16) For $0 < s < 1$ let $\alpha_s(t) := ts$, $0 \leq t \leq 1$, i.e. α_s denotes the interval $[0, s]$ regarded as a curve. For a, $b \in \mathbb{D}$, $a \neq b$, let $\alpha_{a,b} := h_a^{-1} \circ \alpha_{h_a(b)}$. The image $I_{a,b}$ of the curve $\alpha_{a,b}$ lies on the unique circle $C_{a,b}$ that passes through a and b and is orthogonal to $\mathbb{T}$.

(17) For any a, $b \in \mathbb{D}$, $a \neq b$, we have $\mathbb{P}(a, b) = L_{\boldsymbol{\gamma}}(\alpha_{a,b}) = \tanh^{-1}(\boldsymbol{m}(a, b))$. Moreover, $\alpha_{a,b}$ is a unique geodesic joining a and b. Recall that $\tanh^{-1}(t) = \frac{1}{2} \log \frac{1+t}{1-t}$ and $(\tanh^{-1})'(t) = \frac{1}{1-t^2}$, $0 \leq t < 1$.

Indeed, all the objects are invariant under $\operatorname{Aut}(\mathbb{D})$ and so we may assume that $a = 0$, $b \in (0, 1)$, and $\alpha_{a,b} = \alpha_b$. First, observe that $\mathbb{P}(0, b) \leq L_{\boldsymbol{\gamma}}(\alpha_b) = \int_0^b \frac{dt}{1-t^2} = \frac{1}{2} \log \frac{1+b}{1-b} = \tanh^{-1}(\boldsymbol{m}(0, b))$. On the other hand, if $\alpha = u + iv : [0, 1] \longrightarrow \mathbb{D}$ is a path joining 0 and b, then $L_{\boldsymbol{\gamma}}(\alpha) \geq \int_0^1 \frac{u'(t)}{1-u^2(t)} dt = \frac{1}{2} \log \frac{1+b}{1-b}$. Thus the inequality is satisfied and, moreover, if $\mathbb{P}(0, b) = L_{\boldsymbol{\gamma}}(\alpha)$, then we have equality. This implies that $v \equiv 0$, $u : [0, 1] \longrightarrow [0, b]$, and u is increasing. Finally $\alpha \simeq \alpha_b$.

(18) $\mathbb{P}$ is a distance with $\boldsymbol{m} \leq \mathbb{P}$.

(19) For any $f \in \mathcal{O}(\mathbb{D},\mathbb{D})$ if $\mathbb{P}(f(\lambda_0'), f(\lambda_0'')) = \mathbb{P}(\lambda_0', \lambda_0'')$ for some $\lambda_0', \lambda_0'' \in \mathbb{D}$, $\lambda_0' \neq \lambda_0''$, then $f \in \operatorname{Aut}(\mathbb{D})$.

(20) $B_{\mathbb{P}}(a, r) = B_{\boldsymbol{m}}(a, \tanh(r))$, $a \in \mathbb{D}$, $r > 0$. In particular,
– the topology generated by $\mathbb{P}$ coincides with the standard topology of $\mathbb{D}$,
– $(\mathbb{D}, \mathbb{P})$ is complete.

(21) $\lim\limits_{\substack{\lambda',\lambda''\to a\\ \lambda'\neq\lambda''}} \frac{\mathbb{P}(\lambda',\lambda'')}{|\lambda'-\lambda''|} = \boldsymbol{\gamma}(a),\ a \in \mathbb{D}.$

(22) $[a, b]_{\mathbb{P}} = I_{a,b}$, i.e. the $\mathbb{P}$-segments coincide with the images of geodesics. In particular, $\mathbb{P}(0, s) = \mathbb{P}(0, t) + \mathbb{P}(t, s)$, $0 \le t \le s < 1$.

The distance $\mathbb{P}$ is called the *Poincaré (hyperbolic) distance*. Note that $(\mathbb{D}, \mathbb{P})$ is a model of a non-Euclidean geometry (the Poincaré model).

(23) Let $\alpha : [0, 1] \longrightarrow \mathbb{D}$ be a (continuous) curve. Put

$$L_{\mathbb{P}}(\alpha) := \sup\Big\{ \sum_{j=1}^{N} \mathbb{P}(\alpha(t_{j-1}), \alpha(t_j)) : N \in \mathbb{N}, 0 = t_0 < \dots < t_N = 1\Big\}.$$

The number $L_{\mathbb{P}}(\alpha) \in [0, +\infty]$ is called the $\mathbb{P}$-*length* of α. If $L_{\mathbb{P}}(\alpha) < +\infty$, then we say that α is $\mathbb{P}$-*rectifiable*. Note that $L_{\mathbb{P}}(\alpha) \ge \mathbb{P}(\alpha(0), \alpha(1))$.

(24) (a) For any $f \in \mathcal{O}(\mathbb{D},\mathbb{D})$ we have $L_{\mathbb{P}}(f \circ \alpha) \le L_{\mathbb{P}}(\alpha)$. In particular, $L_{\mathbb{P}}$ is invariant under $\operatorname{Aut}(\mathbb{D})$.
(b) $L_{\mathbb{P}}(\alpha_{a,b}) = \mathbb{P}(a, b)$.

(25) $\mathbb{P} = \mathbb{P}^i$, where $\mathbb{P}^i(a, b) := \inf\{L_{\mathbb{P}}(\alpha) : \alpha : [0, 1] \longrightarrow \mathbb{D},\ \alpha$ is a curve joining a and $b\}$, $a, b \in \mathbb{D}$.

The above corollary shows that $\mathbb{P}$ is an *inner* distance.

(26) It is clear that we can repeat the same procedure for the distance $\boldsymbol{m}$: first we define $L_{\boldsymbol{m}}(\alpha)$ and we put $\boldsymbol{m}^i(a, b) := \inf\{L_{\boldsymbol{m}}(\alpha) : \alpha : [0, 1] \longrightarrow \mathbb{D}, \alpha$ is a curve joining a and $b\}$, $a, b \in \mathbb{D}$.

(27) (a) For any curve $\alpha : [0, 1] \longrightarrow \mathbb{D}$ we have $L_{\boldsymbol{m}}(\alpha) = L_{\mathbb{P}}(\alpha)$. In particular, $\boldsymbol{m}^i = \mathbb{P}$. Moreover, α is $\boldsymbol{m}$- or $\mathbb{P}$-rectifiable iff α is rectifiable in the Euclidean sense.
(b) For any path $\alpha : [0, 1] \longrightarrow \mathbb{D}$ we have $L_{\mathbb{P}}(\alpha) = L_{\boldsymbol{\gamma}}(\alpha)$.

The above equality may be used as an alternative way to define $\mathbb{P}$. Moreover, it shows that $\boldsymbol{m}$ is not an inner distance.

Indeed (a) First observe that for any compact $K \subset \mathbb{D}$ there exists an $M > 0$ such that $\frac{1}{M}|\lambda' - \lambda''| \le \boldsymbol{m}(\lambda', \lambda'') \le \mathbb{P}(\lambda', \lambda'') \le M|\lambda' - \lambda''|,\ \lambda', \lambda'' \in K$. Hence for any curve $\alpha : [0, 1] \longrightarrow K$ one gets $\frac{1}{M} L_{\|\,\|}(\alpha) \le L_{\boldsymbol{m}}(\alpha) \le L_{\mathbb{P}}(\alpha) \le M L_{\|\,\|}(\alpha)$, where $L_{\|\,\|}(\alpha)$ denotes the length of α in the Euclidean sense. Consequently, all the notions of rectifiability coincide.

For any compact $K \subset \mathbb{D}$ and for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $0 \le \mathbb{P}(\lambda', \lambda'') - \boldsymbol{m}(\lambda', \lambda'') \le \varepsilon|\lambda' - \lambda''|,\ \lambda', \lambda'' \in K,\ |\lambda' - \lambda''| \le \delta$, which directly implies that $L_{\boldsymbol{m}}(\alpha) = L_{\mathbb{P}}(\alpha)$.

(b) We may assume that α is of class $\mathcal{C}^1$. For any $\varepsilon > 0$ there exists an $\eta > 0$ such that $\Big|\frac{\mathbb{P}(\alpha(t'),\alpha(t''))}{|t'-t''|} - \boldsymbol{\gamma}(\alpha(t'))|\alpha'(t')|\Big| \le \varepsilon,\ 0 \le t', t'' \le 1,\ |t' - t''| \le \eta,$

(28) One may also ask how close is the Poincaré geometry to the holomorphic one, i.e. what are the relations between the set $\operatorname{Isom}(\mathbb{P})$ of all $\mathbb{P}$-*isometries* of $\mathbb{D}$ and the group $\operatorname{Aut}(\mathbb{D})$. Observe that $\operatorname{Isom}(\mathbb{P}) = \operatorname{Isom}(\boldsymbol{m})$. We can also study the set $\operatorname{Isom}(\boldsymbol{\gamma})$ of all $\boldsymbol{\gamma}$-*isometries* of $\mathbb{D}$, i.e. the set of all $\mathcal{C}^1$-mappings $f : \mathbb{D} \longrightarrow \mathbb{D}$ such that $\boldsymbol{\gamma}(f(\lambda))|(d_\lambda f)(X)| = \boldsymbol{\gamma}(\lambda)|X|,\ \lambda \in \mathbb{D}, X \in \mathbb{C}$, where $d_\lambda f : \mathbb{C} \longrightarrow \mathbb{C}$ denotes the $\mathbb{R}$-differential of f at λ.

(29) For any mapping $f : \mathbb{D} \longrightarrow \mathbb{D}$ the following conditions are equivalent:
(i) $f \in \operatorname{Isom}(\mathbb{P})$,
(ii) $f \in \mathcal{C}^1$ and $f \in \operatorname{Isom}(\boldsymbol{\gamma})$,
(iii) either $f \in \operatorname{Aut}(\mathbb{D})$ or $\overline{f} \in \operatorname{Aut}(\mathbb{D})$.
Thus, $\operatorname{Isom}(\mathbb{P}) = \operatorname{Isom}(\boldsymbol{\gamma}) = \operatorname{Aut}(\mathbb{D}) \cup \overline{\operatorname{Aut}(\mathbb{D})}$.

Indeed, it is clear that (iii) $\Longrightarrow$ (i) and (iii) $\Longrightarrow$ (ii).

(i) $\Longrightarrow$ (iii). Taking $e^{i\vartheta}\boldsymbol{h}_{f(0)} \circ f$ in place of f we may assume that $f(0) = 0$ and that $f(x_0) = x_0$ for some $0 < x_0 < 1$. Then we have $|f(\lambda)| = |\lambda|$ and $\left|\frac{f(\lambda)-x_0}{1-f(\lambda)x_0}\right| = \left|\frac{\lambda-x_0}{1-\lambda x_0}\right|,\ \lambda \in \mathbb{D}$. Hence $\operatorname{Re} f(\lambda) = \operatorname{Re}\lambda,\ \lambda \in \mathbb{D}$, and consequently either $f(\lambda) \equiv \lambda$ or $f(\lambda) \equiv \overline{\lambda}$.

(ii) $\Longrightarrow$ (iii). Since f is a $\boldsymbol{\gamma}$-isometry, we have $|f'_x(\lambda)\alpha + f'_y(\lambda)\beta| = C(\lambda)|\alpha + i\beta|,\ \lambda \in \mathbb{D},\ \alpha, \beta \in \mathbb{R}$, where $C(\lambda) := \frac{\boldsymbol{\gamma}(\lambda)}{\boldsymbol{\gamma}(f(\lambda))} > 0$. Hence for each $\lambda \in \mathbb{D}$ there exists an $\varepsilon(\lambda) \in \{-1, 1\}$ such that $f'_x(\lambda) = \varepsilon(\lambda) i f'_y(\lambda) \neq 0$. Since the partial derivatives are continuous, the function ε has to be constant, and consequently f is either holomorphic or antiholomorphic. Hence, by the Schwarz–Pick lemma, $f \in \operatorname{Aut}(\mathbb{D}) \cup \overline{\operatorname{Aut}(\mathbb{D})}$.

(30) The Poincaré distance may also be introduced axiomatically. Let $d : \mathbb{D} \times \mathbb{D} \longrightarrow \mathbb{R}$ be a function such that
(i) d is invariant under $\operatorname{Aut}(\mathbb{D})$,
(ii) $d(0, s) = d(0, t) + d(t, s),\ 0 \le t \le s < 1$,
(iii) $\lim\limits_{t \to 0+} \frac{d(0,t)}{t} = 1$.
Then $d = \mathbb{P}$.

Indeed, let $\varphi(t) := d(0, t),\ 0 \le t < 1$. In view of (ii) and (iii), $\varphi(0) = 0$ and $\varphi'(0) = 1$. We shall show in the second paragraph that $\varphi'(t) = \frac{1}{1-t^2} = \boldsymbol{\gamma}(t),\ 0 \le t < 1$. Suppose for a moment that it is true. Then $\varphi(s) = \int_0^s \varphi'(t)dt = \int_0^s \frac{dt}{1-t^2} = \frac{1}{2}\log\frac{1+s}{1-s} = \mathbb{P}(0, s),\ 0 \le s < 1$, and hence by (i), $d \equiv \mathbb{P}$.

Fix $0 < t_0 < 1$ and let $t > 0$ be such that $t_0 + t < 1$. Because of (ii), we get $\varphi(t_0 + t) - \varphi(t_0) = d(t_0, t_0 + t)$. On the other hand, by (i) we have $d(t_0, t_0 + t) = d\Big(h_{t_0}(t_0), h_{t_0}(t_0 + t)\Big) = d(0, \frac{t}{1-(t_0+t)t_0})$. Finally, $\lim\limits_{t \to 0+} \frac{\varphi(t_0+t)-\varphi(t_0)}{t} = \frac{1}{1-t_0^2}$. The proof for the left derivative is analogous.

CHAPTER 3

Singularities

3.1. Laurent series

Definition 3.1.1. Any series of the form

$$\sum_{n=-\infty}^{\infty} a_n(z-a)^n = \sum_{n=1}^{\infty} a_{-n}(z-a)^{-n} + \sum_{n=0}^{\infty} a_n(z-a)^n =: S(z) + R(z),$$

is called a *Laurent* [1] *series centered at* $a \in \mathbb{C}$. The series S is called the *singular part*, the series R — the *regular part*. Power series may be identified with those Laurent series for which $S \equiv 0$, i.e. $a_{-n} = 0$ for all $n \in \mathbb{N}$. Define the numbers $R_-, R_+ \in \{-\infty\} \cup [0, +\infty]$:

$$R_- := \begin{cases} \limsup\limits_{n\to+\infty} \sqrt[n]{|a_{-n}|}, & \text{if } \exists_{n\in\mathbb{N}} : a_{-n} \neq 0 \\ -\infty, & \text{if } \forall_{n\in\mathbb{N}} : a_{-n} = 0 \end{cases}, \quad R_+ := \frac{1}{\limsup\limits_{n\to+\infty} \sqrt[n]{|a_n|}}.$$

Remark 3.1.2. Suppose that $R_- < R_+$.

(a) The series $\sum\limits_{n=-\infty}^{\infty} a_n(z-a)^n$ converges locally uniformly in $\mathbb{A}(a, R_-, R_+)$.

(b) For any compact $K \subset\subset \mathbb{A}(a, R_-, R_+)$ there exist $C > 0$, $\vartheta \in (0,1)$ such that

$$|a_n(z-a)^n| \le C\vartheta^{|n|}, \quad z \in K,\ n \in \mathbb{Z}.$$

(c) By the Weierstrass theorem the function $f(z) := \sum\limits_{n=-\infty}^{\infty} a_n(z-a)^n$, $z \in \mathbb{A}(a, R_-, R_+)$, is holomorphic.

(d) $\frac{1}{2\pi i} \int\limits_{C(a,r)} \frac{f(\zeta)}{(\zeta-a)^{k+1}} d\zeta = \sum\limits_{n=-\infty}^{\infty} a_n \frac{1}{2\pi i} \int\limits_{C(a,r)} (\zeta-a)^{n-k-1} d\zeta = a_k$, $k \in \mathbb{Z}$, $R_- < r < R_+$. Consequently, the coefficients $(a_n)_{n\in\mathbb{Z}}$ are uniquely determined by f.

Theorem 3.1.3 (Laurent series representation)**.** *Let $f \in \mathcal{O}(\mathbb{A}(a, r_-, r_+))$, $0 \le r_- < r_+ \le +\infty$. Put*

$$a_n(r) := \frac{1}{2\pi i} \int\limits_{C(a,r)} \frac{f(\zeta)}{(\zeta-a)^{n+1}} d\zeta, \quad n \in \mathbb{Z},\ r_- < r < r_+.$$

Then $a_n := a_n(r)$ is independent of r, the Laurent series $\sum\limits_{n=-\infty}^{\infty} a_n(z-a)^n$ is convergent in $\mathbb{A}(a, r_-, r_+)$, and $f(z) = \sum\limits_{n=-\infty}^{\infty} a_n(z-a)^n$, $z \in \mathbb{A}(a, r_-, r_+)$.

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[1] Pierre Laurent (1813–1854).

Proof. The independence of $a_n(r)$ from r follows from the Cauchy integral formula. Using the Cauchy integral formula for $z \in C(a,r)$ i $r_- < r_1 < r < r_2 < r_+$ we get:

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\Big(\int\limits_{C(a,r_2)} \frac{f(\zeta)}{\zeta - z}d\zeta - \int\limits_{C(a,r_1)} \frac{f(\zeta)}{\zeta - z}d\zeta\Big)\\
&= \frac{1}{2\pi i}\Big(\int\limits_{C(a,r_2)} f(\zeta)\frac{1}{\zeta - a + a - z}d\zeta - \int\limits_{C(a,r_1)} f(\zeta)\frac{1}{\zeta - a + a - z}d\zeta\Big)\\
&= \frac{1}{2\pi i}\Big(\int\limits_{C(a,r_2)} f(\zeta)\frac{1}{\zeta - a}\frac{1}{1 - \frac{z-a}{\zeta-a}}d\zeta + \int\limits_{C(a,r_1)} f(\zeta)\frac{1}{z - a}\frac{1}{1 - \frac{\zeta-a}{z-a}}d\zeta\Big)\\
&= \frac{1}{2\pi i}\Big(\int\limits_{C(a,r_2)} f(\zeta)\sum_{n=0}^{\infty}\frac{(z-a)^n}{(\zeta - a)^{n+1}}d\zeta + \int\limits_{C(a,r_1)} f(\zeta)\sum_{n=0}^{\infty}\frac{(\zeta-a)^n}{(z - a)^{n+1}}d\zeta\Big)\\
&= \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=0}^{\infty} a_{-(n+1)}(z-a)^{-(n+1)}. \qquad \square
\end{aligned}$$

Example 3.1.4. [Example 3.1.4⟶ Exer . . . .] The typical problem related to the Laurent series expansion looks as follows. We have a function $f \in \mathcal{O}(\mathbb{C} \setminus \{a_1, \dots, a_N\})$, where $|a_1| \le \dots \le |a_N|$, and we are looking for the Laurent expansion of f in the following annuli:

- $B(|a_1|)$ provided that $a_1 \ne 0$,
- $\mathbb{A}(|a_j|, |a_{j+1}|)$ provided that $|a_j| < |a_{j+1}|$, $j = 1, \dots, N-1$,
- $\mathbb{A}(|a_N|, +\infty)$,
- $\mathbb{A}(a_j, 0, r_j)$, $r_j := \min\{|a_k - a_j| : k = 1, \dots, N,\ k \ne j\}$, $j = 1, \dots, N$.

For example for the function $f(z) := \frac{1}{z-1} + \frac{1}{z-2}$ we get:

- in $B(1)$: $f(z) = -\sum\limits_{n=0}^{\infty} z^n - \frac{1}{2}\sum\limits_{n=0}^{\infty}\left(\frac{z}{2}\right)^n = -\sum\limits_{n=0}^{\infty}(1 + 1/2^{n+1})z^n$.
- in $\mathbb{A}(1,2)$: $f(z) = \frac{1}{z}\sum\limits_{n=0}^{\infty}\left(\frac{1}{z}\right)^n - \frac{1}{2}\sum\limits_{n=0}^{\infty}\left(\frac{z}{2}\right)^n = -\sum\limits_{n=0}^{\infty} 1/2^{n+1}z^n + \sum\limits_{n=1}^{\infty} z^{-n}$.
- in $\mathbb{A}(2,+\infty)$: $f(z) = \frac{1}{z}\sum\limits_{n=0}^{\infty}\left(\frac{1}{z}\right)^n + \frac{1}{z}\sum\limits_{n=0}^{\infty}\left(\frac{2}{z}\right)^n = \sum\limits_{n=1}^{\infty}(1 + 2^{n-1})z^{-n}$.
- in $\mathbb{A}(1,0,1)$: $f(z) = \frac{1}{z-1} - \frac{1}{1-(z-1)} = \frac{1}{z-1} - \sum\limits_{n=0}^{\infty}(z-1)^n$.
- in $\mathbb{A}(2,0,1)$: $f(z) = \frac{1}{1+(z-2)} + \frac{1}{z-2} = \sum\limits_{n=0}^{\infty}(-1)^n(z-2)^n + \frac{1}{z-2}$.

3.2. Isolated singularities

Definition 3.2.1. We say that a point $a \in \mathbb{C}$ is an *isolated singularity* of a holomorphic function f if f is holomorphic at least in $\mathbb{A}(a, 0, r)$ for some $r > 0$.

Obviously, we may also have non-isolated singularities, e.g. 0 for $f(z) := 1/\sin(1/z)$.

If $f \in \mathcal{O}(\mathbb{A}(a,0,r))$, then we take the Laurent expansion $f(z) = \sum\limits_{n=-\infty}^{\infty} a_n(z-a)^n$, $z \in \mathbb{A}(a,0,r)$, and we introduce the following classifications:

- *removable singularity*, if $a_{-n} = 0$ for all $n \in \mathbb{N}$; if we put $f(a) := a_0$, then we get a holomorphic function in the whole disc $B(a,r)$;
- *pole of order* d ($d \in \mathbb{N}$), if $a_{-n} = 0$ for $n > d$ and $a_{-d} \neq 0$; we write $\operatorname{ord}_a f = -d$; the rational function
$$g(z) := \sum_{n=1}^{d} a_{-n}(z-a)^{-n}$$
is called the *principal part of the pole*; observe that $g(z) = p(\frac{1}{z-a})$, where p is a polynomial of degree d; obviously, $\lim\limits_{z\to a} f(z) = \infty$;
- *essential singularity*, if $a_{-n} \neq 0$ for infinitely many $n \in \mathbb{N}$.

The point ∞ is an *isolated singularity* of f if 0 is an isolated singularity of the function $z \overset{g}{\longmapsto} f(1/z)$. We classify singularities of f at ∞ via the classification of singularities of g at 0.

Theorem 3.2.2 (Riemann theorem on removable singularities)**.** *For $f \in \mathcal{O}(\mathbb{A}(a,0,r))$ the following conditions are equivalent:*

(i) *a is a removable singularity;*
(ii) *there exists a finite limit $\lim\limits_{z\to a} f(z)$;*
(iii) *f is bounded in $\mathbb{A}(a,0,\varepsilon)$ for some $0 < \varepsilon < r$;*
(iv) *$f \in L^p_h(\mathbb{A}(a,0,\varepsilon))$ for some $p \geq 2$ and $0 < \varepsilon < r$.*

Proof. The implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) are obvious. It remains to prove that (iv) $\Longrightarrow$ (i). We may assume that $a = 0$. Since $L^p(B_*(\varepsilon)) \subset L^2(B_*(\varepsilon))$, we may assume that $p = 2$. Let $f(z) = \sum\limits_{n=-\infty}^{\infty} a_n z^n$, $z \in B_*(r)$. We have to show that $a_{-n} = 0$ for every $n \in \mathbb{N}$. Fix an $n \in \mathbb{N}$. We are going to show that
$$|a_{-n}| \leq (1/\sqrt{2\pi})\varepsilon^{n-1}\|f\|_{L^2(B_*(\eta))}. \quad 0 < \eta < \varepsilon.$$
Since $\|f\|_{L^2(B_*(\eta))} \longrightarrow 0$ when $\eta \longrightarrow 0$, the proof will be completed. For $0 < t < \eta < \varepsilon$, using the Hölder inequality we get:
$$|a_{-n}|^2 = \Big|\frac{1}{2\pi i}\int\limits_{C(t)} \frac{f(\zeta)}{\zeta^{-n+1}}d\zeta\Big|^2 \leq \Big(\frac{1}{2\pi}\int\limits_0^{2\pi} |f(te^{i\vartheta})|t^n d\vartheta\Big)^2 \leq \frac{1}{2\pi}\int\limits_0^{2\pi} |f(te^{i\vartheta})|^2 d\vartheta\, t^{2n}.$$
On the other hand,
$$\frac{1}{2\pi}\eta^{2n-2}\int\limits_{B_*(\eta)} |f|^2 d\mathcal{L}^2 \leq \frac{1}{2\pi}\frac{1}{\eta}\eta^{2n-1}\int\limits_0^{\eta}\int\limits_0^{2\pi} |f(te^{i\vartheta})|^2 t d\vartheta dt = |a_{-n}|^2. \qquad \square$$

Remark 3.2.3. $1/z \in L^p_h(\mathbb{D}_*)$, $1 \leq p < 2$.

Definition 3.2.4. We say that a function $f \in \mathcal{O}(B(a,r))$ has a *zero of multiplicity (order)* $d$at a, if $f^{(k)}(a) = 0$ for $k \le d-1$ and $f^{(d)}(a) \ne 0$. We write $\operatorname{ord}_a f = d$.

This means that $f(z) = (z-a)^d g(z)$, $z \in B(a,r)$, where $g \in \mathcal{O}(B(a,r))$ and $g(a) \ne 0$.

If $f \in \mathcal{O}(\widehat{\mathbb{C}} \setminus \overline{B}(r))$ and $g(z) := f(1/z)$, $z \in \mathbb{A}(0,1/r)$, then $\operatorname{ord}_\infty f =: \operatorname{ord}_0 g$.

Theorem 3.2.5. *For $f \in \mathcal{O}(\mathbb{A}(a,0,r))$ and $d \in \mathbb{N}$, the following conditions are equivalent:*

(i) $\operatorname{ord}_a f = -d$;

(ii) *there exists a $g \in \mathcal{O}(B(a,r))$ such that $g(a) \ne 0$ and $f(z) = (z-a)^{-d} g(z)$, $z \in B_*(a,r)$;*

(iii) $1/f$ *(defined as 0 at a) has a zero of d at a.*

Proof. Exercise. □

Theorem 3.2.6 (Sochocki [2]-Casorati [3]-Weierstrass theorem)**.** *If $f \in \mathcal{O}(\mathbb{A}(a,0,r))$ has an essential singularity at a, then for every $0 < \varepsilon < r$ the set $f(\mathbb{A}(a,0,\varepsilon))$ is dense in $\mathbb{C}$.*

Proof. Suppose that $f(\mathbb{A}(a,0,\varepsilon))$ is not dense in $\mathbb{C}$. Then $f(\mathbb{A}(a,0,\varepsilon)) \cap B(b,\delta) = \varnothing$ for some disc $B(b,\delta)$. Thus $|f(z)-b| \ge \delta$, $z \in \mathbb{A}(a,0,\varepsilon)$. Let $g(z) := \frac{1}{f(z)-b}$, $z \in \mathbb{A}(a,0,\varepsilon)$. Since $|g| \le 1/\delta$, the function g has a removable singularity at a. Its extension to $B(a,\varepsilon)$ will be denoted also by g. If $g(a) \ne 0$, then we may assume that $g(z) \ne 0$, $z \in B(a,\epsilon)$. In this case we get $f(z) = \frac{1}{g(z)} + b$, $z \in \mathbb{A}(a,0,\varepsilon)$ and consequently, f extends holomorphically to $B(a,\epsilon)$ — a contradiction.

If $g(a) = 0$, then $g(z) = (z-a)^d h(z)$, $z \in B(a,\varepsilon)$, where $d \in \mathbb{N}$, $h \in \mathcal{O}(B(a,\varepsilon))$, and $h(a) \ne 0$. We may assume that $h(z) \ne 0$, $z \in B(a,\varepsilon)$. Then $f(z) = (z-a)^{-d}\Big(\frac{1}{h(z)} + b(z-a)^d\Big)$, $z \in \mathbb{A}(a,0,\varepsilon)$, which implies that f has a pole of order d at a — a contradiction. □

In fact, the result may be strengthened.

Theorem* 3.2.7 (Big Picard [4] theorem)**.** *Let $f \in \mathcal{O}(\mathbb{A}(a,0,r))$ have an essential singularity at a. Then all except at most one complex value is assumed at infinitely many points.*

Corollary 3.2.8. *Let $f \in \mathcal{O}(\mathbb{A}(a,0,r))$. Then:*

- *f has a removable singularity at a if and only if $\lim_{z\to a} f(z)$ exists and is finite;*
- *f has a pole at a if and only if $\lim_{z\to a} f(z) = \infty$;*
- *f has an essential singularity at a if and only if a finite or infinite limit $\lim_{z\to a} f(z)$ does not exist.*

Definition 3.2.9. If $f \in \mathcal{O}(\mathbb{A}(a,0,r))$, then the number $\operatorname{res}_a f := a_{-1} = \frac{1}{2\pi i} \int\limits_{C(a,\delta)} f(\zeta)d\zeta$ $(0 < \delta < r)$ is called the *residuum of f at a*.

Theorem 3.2.10. *If an $f \in \mathcal{O}(\mathbb{A}(a,0,r))$ has a pole of order d at a, then $\operatorname{res}_a f = \frac{1}{(d-1)!} \lim_{z\to a} \big((z-a)^d f(z)\big)^{(d-1)}$ (attention: here $()^{(d-1)}$ denotes the $(d-1)$ derivative).*

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[2] Julian Sochocki (1842–1927).

[3] Felice Casorati (1835–1890).

[4] Émile Picard (1856–1941).

Example 3.2.11. [Example 3.2.11⟶ Exer . . . .] $\operatorname{res}_i \frac{1}{(1+z^2)^n} = \frac{1}{2i}\frac{(2n-3)!!}{(2n-2)!!}$.

CHAPTER 4

Meromorphic functions

4.1. Meromorphic functions

Definition 4.1.1. Let $D \subset \widehat{\mathbb{C}}$ be a domain. We say that a function $f : D \longrightarrow \widehat{\mathbb{C}}$ is *meromorphic* ($f \in \mathcal{M}(D)$), if there exists a set $S = S(f) \subset D$ such that:

- $S' \cap D = \varnothing$,
- $f \in \mathcal{O}(D \setminus S)$,
- f has a pole at each point $a \in S$.

If $\Omega \subset \widehat{\mathbb{C}}$ is open, then we say that a function $f : \Omega \longrightarrow \widehat{\mathbb{C}}$ is *meromorphic* ($f \in \mathcal{M}(\Omega)$), if $f|_D \in \mathcal{M}(D)$ for any connected component D of Ω.

Remark 4.1.2. (a) $\mathcal{O}(\Omega) \subset \mathcal{M}(\Omega)$,
(b) $\mathcal{M}(\Omega) \subset \mathcal{C}(\Omega, \widehat{\mathbb{C}})$.

Theorem 4.1.3 (Identity principle for meromorphic functions)**.** *If $f, g \in \mathcal{M}(D)$ and the set $A := \{z \in D : f(z) = g(z)\}$ has an accumulation point in D, then $f \equiv g$.*

Proof. Let $S := S(f) \cup S(g)$. Obviously, S has no accumulation points in D. Thus $A \cap (D \setminus S)$ has an accumulation point in $D \setminus S$. By the identity principle for holomorphic functions, we get $f = g$ in $D \setminus S$. Finally, using the continuity of f and g, we get $f \equiv g$. $\square$

Theorem 4.1.4. *$\mathcal{M}(D)$ is a field.*

Proof. Let $f, g \in \mathcal{M}(D)$, $f, g \not\equiv 0$. Clearly, $f + g \in \mathcal{M}(D)$ and $S(f + g) \subset S(f) + S(g)$. If $g \not\equiv 0$, then the set $A := g^{-1}(0)$ has no accumulation points in D. Moreover, $1/g \in \mathcal{O}(D \setminus (A \cup S(g)))$. By Theorem 3.2.5 for each $a \in A$ if g has a zero of multiplicity d, then $1/g$ has a pole of order d. Similarly, for each $a \in S(g)$ if g has a pole of order d, then $1/g$ has a zero of multiplicity d. Thus $S(1/g) = A$ and $1/g \in \mathcal{M}(D)$.

It remains to prove that $f \cdot g \in \mathcal{M}(D)$. Obviously, $f \cdot g \in \mathcal{O}(D \setminus A)$, where $A := S(f) \cup S(g)$. Fix an $a \in A \cap \mathbb{C}$. Let $f(z) = (z - a)^{d_f} f_1(z)$, $g(z) = (z - a)^{d_g} g_1(z)$, $z \in \mathbb{A}(a, 0, r) \subset D \setminus A$, $f_1, g_1 \in \mathcal{O}^*(B(a, r))$. Hence $f(z)g(z) = (z - a)^{d_f + d_g} f_1(z)g_1(z)$, $z \in \mathbb{A}(a, 0, r)$.

The case $a = \infty$ is left as an Exercise.

Now, using Theorem 3.2.5, we conclude that $f \cdot g \in \mathcal{M}(D)$. $\square$

Theorem 4.1.5. $\mathcal{M}(\widehat{\mathbb{C}}) = \mathcal{R}(\mathbb{C})$.

Proof. Obviously, $\mathcal{R}(\mathbb{C}) \subset \mathcal{M}(\widehat{\mathbb{C}})$. Let $f \in \mathcal{M}(\widehat{\mathbb{C}})$. The set $S(f)$ must be finite. The case $S(f) = \varnothing$ is trivial because then $f \equiv \text{const}$. If $S(f) = \{\infty\}$, then f is an entire function. Since f has a pole at ∞, it must be a polynomial. Otherwise, $S(f) \cap \mathbb{C} = \{a_1, \ldots, a_n\}$

and let $g_k(z) = p_k\big(\frac{1}{z-a_k}\big)$ be the principal part of the pole of f at a_k, $k = 1, \dots, n$. Put $g := f - (g_1 + \dots + g_n) \in \mathcal{M}(\widehat{\mathbb{C}})$. Then $S(g) \subset \{\infty\}$, and therefore g must be a polynomial. $\square$

Theorem 4.1.6. (a) $\operatorname{Aut}(\mathbb{C}) = \operatorname{Aut}_{\mathcal{H}}(\mathbb{C}) = \{\mathbb{C} \ni z \longmapsto az + b \in \mathbb{C} : a \in \mathbb{C}_*,\ b \in \mathbb{C}\} = \mathcal{G}$.
(b) $\operatorname{Aut}(\widehat{\mathbb{C}}) = \operatorname{Aut}_{\mathcal{H}}(\widehat{\mathbb{C}}) = \mathcal{H}$.

$\operatorname{Aut}(\mathbb{C})$ depends on 4 real parameters.

Proof. (a) Clearly, $\mathcal{G} \subset \operatorname{Aut}(\mathbb{C})$. Let $f \in \operatorname{Aut}(\mathbb{C})$. Since f is proper, we get $\lim\limits_{z\to\infty} f(z) = \infty$. This means that f has a pole at ∞. Thus f is a polynomial of degree d (for some $d \in \mathbb{N}$). Since f is injective, it must be $d = 1$.

(b) We know that $\mathcal{H} \subset \operatorname{Aut}(\widehat{\mathbb{C}})$. Let $f \in \operatorname{Aut}(\widehat{\mathbb{C}})$. If $f(\infty) = \infty$, then $f \in \operatorname{Aut}(\mathbb{C})$, and so (use (a)) $f(z) = az + b \in \mathcal{H}$. If $f(\infty) = w_0 \in \mathbb{C}$, then $g := \frac{1}{f-w_0} \in \operatorname{Aut}(\widehat{\mathbb{C}})$ and $g(\infty) = \infty$, which gives $f \in \mathcal{H}$. $\square$

4.2. Residue theorem

Theorem 4.2.1 (Residue theorem)**.** *Let D be a regular domain (cf. Theorem 2.1.5), $\overline{D} \subset \Omega$, where Ω is open. Let $f \in \mathcal{M}(\Omega)$ be such that $S(f) \subset D$ (observe that $S(f)$ must be finite). Then*

$$\int\limits_{\partial D} f(\zeta)d\zeta = 2\pi i \sum_{a\in S(f)} \operatorname{res}_a f.$$

Proof. If $S(f) = \varnothing$, the result is trivial ($\sum\limits_{a\in\varnothing} \dots = 0$). Suppose that $S(f) = \{a_1, \dots, a_n\}$. Let $r > 0$ be so small that $B(a_j, r) \subset\subset D$ and $\overline{B}(a_j, r) \cap \overline{B}(a_k, r) = \varnothing$, $j \neq k$. Now we apply the Cauchy formula to the domain $G := D \setminus \bigcup\limits_{j=1}^{n} \overline{B}(a_j, r)$:

$$0 = \int\limits_{\partial G} f(\zeta)d\zeta = \int\limits_{\partial D} f(\zeta)d\zeta - \sum_{j=1}^{n} \int\limits_{C(a_j,r)} f(\zeta)d\zeta = \int\limits_{\partial D} f(\zeta)d\zeta - \sum_{j=1}^{n} 2\pi i \operatorname{res}_{a_j} f. \qquad \square$$

Exercise 4.2.2 (Applications to integrals)**.** [Exercise 4.2.2⟶ Exer . . . .]

(I) $I := \int\limits_0^{2\pi} W(\cos t, \sin t)dt$, where W is a rational function of two complex variables. Then $I = 2\pi i \sum\limits_{a\in\mathbb{D}} \operatorname{res}_a f$, where $f(z) := W(\cos z, \sin z)$.

(II) $I := \int\limits_{-\infty}^{\infty} f(x)dx$, where $f \in \mathcal{M}(\Omega)$, $\overline{\mathbb{H}}^+ \subset \Omega$, $S(f) = \{a_1, \dots, a_N\} \subset \mathbb{H}^+$. Let $C^+(r)$ denote the upper half of $C(r)$ identified with the curve $[0, \pi] \ni t \longmapsto re^{it}$. By the residue theorem applied to the domain $\{x + iy \in B(R) : y > 0\}$ with $R \gg 1$, we have $I = 2\pi i \sum\limits_{j=1}^{N} \operatorname{res} a_j f - \lim\limits_{R\to+\infty} \int\limits_{C^+(R)} f(z)dz$. We are interested in those cases where $\lim\limits_{R\to+\infty} \int\limits_{C^+(R)} f(z)dz = 0$.

(*) If there exists an $\alpha > 1$ such that $|f(z)| \leq C/|z|^\alpha$ for $z \in \mathbb{H}^+$, $|z| \geq R_0$ (e.g. $f(z) = P(z)/Q(z)$ is a rational function with $\deg P \leq \deg Q - 2$), then $\lim\limits_{R\to+\infty} \int\limits_{C^+(R)} f(z)dz = 0$.

For example

$$\int\limits_{-\infty}^{\infty} \frac{1}{(1+x^2)^n}dx = 2\pi i \operatorname{res}_i \frac{1}{(1+z^2)^n} = \pi\frac{(2n-3)!!}{(2n-2)!!}, \quad n \in \mathbb{N}.$$

(**) (Jordan [1] lemma) If $f(z) = g(z)e^{i\lambda z}$, $z \in \Omega$, where $\lambda > 0$ and $M(R) := \sup\{|g(z)| : z \in C^+(R)\} \underset{R\to+\infty}{\longrightarrow} 0$ (e.g $g(z) = P(z)/Q(z)$ is a rational function with $\deg P \leq \deg Q - 1$), then $\lim\limits_{R\to+\infty} \int\limits_{C^+(R)} f(z)dz = 0$.

For example

$$\int\limits_{-\infty}^{\infty} \frac{x\sin x}{1+x^2}dx = \operatorname{Im}\Big(\int\limits_{-\infty}^{\infty} \frac{xe^{ix}}{1+x^2}dx\Big) = \operatorname{Im}\Big(2\pi i \operatorname{res}_i \frac{ze^{iz}}{1+z^2}\Big) = \operatorname{Im}\Big(2\pi i\, \frac{ie^{-1}}{2i}\Big) = \frac{\pi}{e}.$$

(III) $I := \int\limits_0^\infty \frac{\sin x}{x}dx = \frac{1}{2}\operatorname{Im}\Big(\int\limits_{-\infty}^{\infty} \frac{e^{ix}}{x}dx\Big) = -\frac{\pi}{2}$.

(IV) $I := \int\limits_0^\infty \cos x^2 dx + i\int\limits_0^\infty \sin x^2 dx = \int\limits_0^\infty e^{iz^2}dz = e^{i\pi/4}\frac{\sqrt{\pi}}{2}$.

(V) $I := \int\limits_{-\infty}^{\infty} \frac{e^{\alpha x}}{1+e^x}dx = \frac{\pi}{\sin \alpha\pi}, \quad 0 < \alpha < 1$.

4.3. Holomorphic functions given by integrals

Theorem 4.3.1 (Holomorphic functions given by integrals)**.** *Let $I \subset \mathbb{R}$, $I \in \{[a,b],\ [a,b)\}$, let $D \subset \mathbb{C}$ be a domain, and let $f : D \times I \longrightarrow \mathbb{C}$ be such that:*

(a) *$f(\cdot,t) \in \mathcal{O}(D)$, $t \in I$,*

(b) *$f(z,\cdot) \in \mathcal{C}(I)$, $z \in D$,*

(c) *f is locally bounded in $D \times I$,*

(c') *for every compact $K \subset\subset D$ there exists an integrable function $g_K : [a,b) \longrightarrow \mathbb{R}_+$ such that $|f(z,t)| \leq g_K(t)$, $(z,t) \in K \times [a,b)$ (observe that if $I = [a,b]$, then* (c') *follows from* (c)*).*

Put $F(z) := \int\limits_a^b f(z,t)dt$, $z \in D$. Then $F \in \mathcal{O}(D)$ and $F^{(k)}(z) = \int\limits_a^b \frac{\partial^k f}{\partial z^k}(z,t)dt$, $z \in D$, $k \in \mathbb{N}$.

An analogous result is true for $I = (a,b]$ or $I = (a,b)$.

---

(1) Camille Jordan (1838–1922).

Proof. First let $I = [a, b]$. Put $t_{n,j} = a + \frac{j}{n}(b-a)$, $\xi_{n,j} \in [t_{n,j-1}, t_{n,j}]$, $n \in \mathbb{N}$, $j = 0, \dots, n$,

$$F_n(z) := \sum_{j=1}^{n} f(z, \xi_{n,j}) \frac{b-a}{n}, \quad z \in D,\ n \in \mathbb{N}.$$

Obviously, $F_n \in \mathcal{O}(D)$ and $F_n \longrightarrow F$ pointwise in D. In order to prove that $F \in \mathcal{O}(D)$, in view of the Vitali theorem, it suffices to prove that $(F_n)_{n=1}^{\infty}$ is locally bounded. For any compact $K \subset\subset D$ let $|f| \le C$ on $K \times [a, b]$. Then $|F_n| \le C(b-a)$ on K, $n \in \mathbb{N}$.

Fix $k \in \mathbb{N}$ and $z \in D$. By the Weierstrass theorem we get $F_n^{(k)}(z) \longrightarrow F^{(k)}(z)$. Observe that

$$F_n^{(k)}(z) = \sum_{j=1}^{n} \frac{\partial^k f}{\partial z^k}(z, \xi_{n,j}) \frac{b-a}{n}, \quad n \in \mathbb{N}.$$

Hence the integral $\int\limits_a^b \frac{\partial^k f}{\partial z^k}(z,t)dt$ exists and we get the formula.

In the case where $I = [a, b)$ fix $b_k \nearrow b$ and let $F_k(z) := \int\limits_a^{b_k} f(z,t)dt$, $z \in D$, $k \in \mathbb{N}$. It suffices to prove that $F_k \longrightarrow F$ locally uniformly in D. Fix a compact $K \subset\subset D$. Then for $z \in K$ and $\ell \ge k$, we obtain $|F_k(z) - F_\ell(z)| = \Big| \int\limits_{b_k}^{b_\ell} f(z,t)dt \Big| \le \int\limits_{b_k}^{b_\ell} g_K(t)dt \underset{k \to +\infty}{\longrightarrow} 0.$ □

Let $\mathbb{H}_m := \{z \in \mathbb{C} : \operatorname{Re} z > m\}$, $m \in \mathbb{R}$.

Theorem 4.3.2 (Euler (2) Γ function)**.** (a)

$$\Gamma(z) := \int\limits_0^{\infty} t^{z-1} e^{-t} dt = \int\limits_0^{\infty} e^{(z-1)\log t - t} dt, \quad z \in \mathbb{H}_0,$$

is well defined, $\Gamma(1) = 1$, *and* $\Gamma(z+1) = z\Gamma(z)$.

(b) $\Gamma(z+n) = (z+n-1)\cdots z\Gamma(z)$, *which gives* $\Gamma(z) := \frac{\Gamma(z+n)}{(z+n-1)\cdots z}$, $z \in \mathbb{H}_{-n}$, *and permits to extend* Γ *holomorphically to* $\mathbb{C} \setminus \mathbb{Z}_-$.

(c) *For* $n \in \mathbb{Z}_+$, Γ *has a pole of order* 1 *at* $-n$ *and* $\operatorname{res}_{-n} \Gamma = \frac{(-1)^n}{n!}$.

Proof. (a), (b) Exercise.

(c) $\lim\limits_{z \to -n} (z+n)\Gamma(z) = \lim\limits_{z \to -n} (z+n) \frac{\Gamma(z+n+1)}{(z+n)(z+n-1)\cdots z} = \frac{\Gamma(1)}{(-1)\cdots(-n)} = \frac{(-1)^n}{n!}$. □

Exercise 4.3.3 (Laplace transform)**.** [Exercise 4.3.3⟶ Exer . . . .]

(a) Let $\mathfrak{D}(\mathcal{L})$ denote the family of all functions $f : \mathbb{R}_+ \longrightarrow \mathbb{C}$ such that:
- there exist points $0 = t_0 < t_1 < \dots < t_N$ for which $f|_{(t_{j-1}, t_j)} \in \mathcal{C}([t_{j-1}, t_j])$, $j = 1, \dots, N$, and $f|_{(t_N, +\infty)} \in \mathcal{C}([t_N, +\infty))$,
- there exist $M, m \ge 0$ such that $|f(t)| \le M e^{mt}$, $t \in \mathbb{R}_+$.

We put $m(f) := \inf\{m \ge 0 : \exists_{M \ge 0} : \ |f(t)| \le M e^{mt},\ t \in \mathbb{R}_+\}$. If f is bounded, then $m(f) = 0$.

(b) $\mathfrak{D}(\mathcal{L})$ is an algebra.

(c) For $f \in \boldsymbol{\mathfrak{D}}(\mathcal{L})$ define the *Laplace transform* $F(s) = \mathcal{L}(f)(s) := \int_0^\infty f(t)e^{-st}dt$, $s \in \mathbb{H}_{m(f)}$. Observe that F is well-defined. Indeed, for any $m > m(f)$ if $|f(t)| \leq Me^{mt}$, $t \in \mathbb{R}_+$, for some constant $M \geq 0$, then $|f(t)e^{-st}| \leq Me^{(m-\operatorname{Re} s)t}$, $t \in \mathbb{R}_+$. Moreover, $F \in \mathcal{O}(\mathbb{H}_{m(f)})$ and $|F(s)| \leq \frac{M}{\operatorname{Re} s - m} \underset{\mathbb{H}_m \ni s \to \infty}{\longrightarrow} 0$. The operator $\mathcal{L}$ is obviously linear.

(d) We have:

$f(t)$	$F(s)$
1	$\frac{1}{s}$
$e^{\lambda t}$ $(\lambda \in \mathbb{C})$	$\frac{1}{s-\lambda}$
$\sin t$	
$\cos t$	
$\sinh t$	
$\cosh t$	
$f(at)$ $(a > 0)$	$\frac{1}{a}F(\frac{s}{a})$
$f(t+\omega) = f(t)$, $t \in \mathbb{R}_+$ $(\omega > 0)$	$\frac{1}{1-e^{-\omega s}} \int_0^\omega f(t)e^{-st}dt$
$f(t-b)$ $(b > 0)$	$e^{-bs}F(s)$
$f(t+b)$ $(b > 0)$	$e^{bs}(F(s) - \int_0^b f(t)e^{-st}dt)$
t^α $(\alpha \geq 0)$	$\frac{\varGamma(\alpha+1)}{s^{\alpha+1}}$
$e^{-\lambda t}f(t)$ $(\lambda \in \mathbb{C})$	$F(s+\lambda)$
$\frac{e^{ct}t^{k-1}}{(k-1)!}$	$\frac{1}{(s-c)^k}$
$(-t)^k f(t)$	$F^{(k)}(s)$
$f^{(k)}(t)$ $(f^{(j)} \in \boldsymbol{\mathfrak{D}}(\mathcal{L}) \cap \mathcal{C}(\mathbb{R}_{>0})$, $j = 1, \dots, k)$	$s^k F(s) - \sum_{j=0}^{k-1} s^j f^{(k-j-1)}(0+)$

(e) For $s \in \mathbb{H}_0$ we have $\mathcal{L}(t^\alpha)(s) = \frac{\varGamma(\alpha+1)}{s^{\alpha+1}}$.

(f) Consider the equation $a_n y^{(n)} + \dots + a_1 y' + a_0 y = f(t)$, where $y \in \boldsymbol{\mathfrak{D}}(\mathcal{L})$, $y^{(j)} \in \boldsymbol{\mathfrak{D}}(\mathcal{L}) \cap \mathcal{C}(\mathbb{R}_{>0})$, $j = 1, \dots, n$, $f \in \boldsymbol{\mathfrak{D}}(\mathcal{L})$. Let $\mathcal{L}(f) = F$, $\mathcal{L}(y) = Y$, $p_j := y^{(j)}(0+)$, $j = 0, \dots, n$, $P(s) := a_n s^n + \dots + a_1 s + a_0$. Then

$$F = \sum_{k=0}^{n} a_k \mathcal{L}(y^{(k)}) = \sum_{k=0}^{n} a_k \Big(s^k Y - \sum_{j=0}^{k-1} s^j p_{k-j-1}\Big) = PY - Q, \text{ where } Q \in \mathcal{P}_{n-1}(\mathbb{C}). \qquad \square$$

4.4. Residues of the logarithmic derivative. Rouché theorem, Hurwitz theorem

Theorem 4.4.1 (Residues of the logarithmic derivative). *Let D be a regular domain, $\overline{D} \subset \Omega$, where Ω is open, and let $f \in \mathcal{M}(\Omega)$, $f \not\equiv 0$ on D, be such that $f^{-1}(0) \cup S(f) \subset D$ ($f^{-1}(0) \cup S(f)$ must be finite). Let $\alpha(z) := \operatorname{ord}_z f$, $z \in f^{-1}(0)$, $\beta(p)$ denote the order of pole* *of f at $p \in S(f)$. Then, for an arbitrary function $\varphi \in \mathcal{O}(\Omega)$ we have*

$$\frac{1}{2\pi i}\int\limits_{\partial D} \varphi(\zeta)\frac{f'(\zeta)}{f(\zeta)}d\zeta = \sum_{z\in f^{-1}(0)} \alpha(z)\varphi(z) - \sum_{p\in S(f)} \beta(p)\varphi(p).$$

In particular, if $\varphi = 1$, then $\frac{1}{2\pi i}\int\limits_{\partial D} \varphi(\zeta)\frac{f'(\zeta)}{f(\zeta)}d\zeta = Z - P$, where Z (resp. P) denotes the number of zeros (resp. poles) of f counted with multiplicities.

Proof. By the residue theorem we obtain

$$\frac{1}{2\pi i}\int\limits_{\partial D} \varphi(\zeta)\frac{f'(\zeta)}{f(\zeta)}d\zeta = \sum_{z\in f^{-1}(0)} \operatorname{res}_z\Big(\varphi\frac{f'}{f}\Big) + \sum_{p\in S(f)} \operatorname{res}_p\Big(\varphi\frac{f'}{f}\Big) = \sum_{z\in f^{-1}(0)} \alpha(z)\varphi(z) - \sum_{p\in S(f)} \beta(p)\varphi(p),$$

because if $f(z) = (z-a)^k g(z)$, $z \in \mathbb{A}(a,0,r) \subset\subset D$, where $k \in \mathbb{Z}$ and $g \in \mathcal{O}(B(a,r))$, $g(a) \neq 0$, then

$$\varphi(z)\frac{f'(z)}{f(z)} = \varphi(z)\frac{k(z-a)^{k-1}g(z) + (z-a)^k g'(z)}{(z-a)^k g(z)} = \varphi(z)\frac{k}{z-a} + \varphi(z)\frac{g'(z)}{g(z)}, \quad z \in \mathbb{A}(a,0,r).$$

□

Theorem 4.4.2 (Rouché [3] theorem). *Let $D \subset \mathbb{C}$ be a bounded domain and let $f, g \in \mathcal{O}(D) \cap \mathcal{C}(\overline{D})$ be such that $|g(\zeta)| < |f(\zeta)|$, $\zeta \in \partial D$. Then $f+g$ and f have the same number of zeros in D, counted with multiplicities.*

Proof. Observe that the functions $f+g$ and f have no zeros on ∂D. Consequently, the number of zeros in D is finite. Let $G \subset\subset D$ be regular such that $(f+g)^{-1}(0) \cup f^{-1}(0) \subset G$ and $|g(\zeta)| < |f(\zeta)|$, $\zeta \in \partial G$. To get G we may use square nets.

Observe that for $\zeta \in \partial G$ and $t \in [0,1]$ we have $|f(\zeta) + tg(\zeta)| \geq |f(\zeta)| - t|g(\zeta)| \geq |f(\zeta)| - |g(\zeta)| > 0$. In particular, the function $f + tg$ has no zeros on ∂G. Let $Z(t)$ denote the number of zeros in G of $f + tg$ counted with multiplicities. By the theorem on residues of the logarithmic derivative, we know that

$$Z(t) = \frac{1}{2\pi i}\int\limits_{\partial G} \frac{f'(\zeta) + tg'(\zeta)}{f(\zeta) + tg(\zeta)}d\zeta, \quad t \in [0,1].$$

It remains to note that the function Z is continuous. □

Corollary 4.4.3. *Every polynomial $P \in \mathcal{P}_n(\mathbb{C})$, $\deg P = n \geq 1$, has exactly n roots counted with multiplicities.*

Proof. Let $P(z) = a_n z^n + \dots + a_1 z + a_0$, $f(z) := a_n z^n$, $g(z) := a_{n-1}z^{n-1} + \dots + a_1 z + a_0$. Then $|g(\zeta)| < |f(\zeta)|$, $\zeta \in C(R)$, for $R \gg 1$. It remains to use Rouché theorem. □

Theorem 4.4.4 (Hurwitz [4] theorem). *Let $D \subset \mathbb{C}$ be domain, $(f_k)_{k=1}^\infty \subset \mathcal{O}(D)$, $f_k \longrightarrow f$ locally uniformly in D, $f \not\equiv 0$. Then for an $a \in D$ and a $d \in \mathbb{Z}_+$ the following conditions are equivalent:*

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[3] Eugene Rouché (1832–1910).
[4] Adolf Hurwitz (859–1919).

(i) *$a \in D$ is a zero of f with multiplicity d*
(ii) *there exists an $\varepsilon > 0$ such that for every $0 < \delta < \varepsilon$ there exists a $k_0 \in \mathbb{N}$ such that for $k \geq k_0$ the function f_k has exactly d zeros in $B(a, \delta)$, counted with multiplicities.*

Proof. (i) $\Longrightarrow$ (ii): Take an $\varepsilon > 0$ such that $f(z) \neq 0$, $z \in \overline{B}(a, \varepsilon) \setminus \{a\}$. Let $0 < \delta < \varepsilon$ and let $\eta := \frac{1}{2}\min\{|f(z)| : z \in C(a, \delta)\} > 0$. Choose $k_0 \in \mathbb{N}$ such that $|f_k(z) - f(z)| \leq \eta$, $z \in \overline{B}(a, \delta)$, $k \geq k_0$. Then for $z \in C(a, \delta)$ and $k \geq k_0$ we get $|f_k(z) - f(z)| \leq \eta < 2\eta \leq |f(z)|$. Now, by the Rouché theorem the functions $f_k = (f_k - f) + f$ and the same number of zeros in $B(a, \delta)$, counted with multiplicities.

(ii) $\Longrightarrow$ (i): In view of the previous argument, f must have a zero of multiplicity d at a. □

Corollary 4.4.5. *Let $D \subset \mathbb{C}$ be a domain, $(f_k)_{k=1}^{\infty} \subset \mathcal{O}(D)$, $f_k \longrightarrow f$ locally uniformly in D, $f \not\equiv \mathrm{const}$. Assume that each function f_k is injective. Then f is injective.*

Proof. Suppose that $f(a) = f(b) =: c$ for some $a, b \in D$, $a \neq b$. Let $B(a, r) \cap B(b, r) = \varnothing$. By the Hurwitz theorem applied to $(f_k - c)_{k=1}^{\infty}$ and $f - c$, we conclude that there exists a $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$ the function $f_k - c$ has at least one zero in $B(a, r)$ and in $B(b, r)$, say a_k, b_k. Thus $f_k(a_k) = f_k(b_k)$, $k \geq k_0$ — a contradiction. □

4.4.1. Multiplicity at a point.

Definition 4.4.6. Let $D \subset \mathbb{C}$ be a domain, $a \in D$, and let $f \in \mathcal{O}(D)$. We say that f has *multiplicity d at a* ($d \in \mathbb{N}$), if there exists a neighborhood $U_0 \subset D$ of a such that for every neighborhood $U \subset U_0$ of a there exists a neighborhood V of $f(a)$ such that for every $w \in V \setminus \{f(a)\}$ the function $f - w$ has exactly d zeros in U, counted with multiplicities.

Corollary 4.4.7. *Let $D \subset \mathbb{C}$ be a domain, $a \in D$, and let $f \in \mathcal{O}(D)$. Then the following conditions are equivalent:*

(i) *f has multiplicity d at a;*
(ii) *a is a zero of $f - f(a)$ of order d.*

Proof. (ii) $\Longrightarrow$ (i): Let $r > 0$ be such that the function $f - f(a)$ has exactly one zero in $B(a, r) \subset D$. Let $0 < \delta < r$ and $\eta := \min\{|f(z) - f(a)| : z \in C(a, \delta)\}$. Let $0 < |w - f(a)| < \eta$. Then $|f(a) - w| < |f(z) - f(a)|$, $z \in C(a, \delta)$. Hence, by the Rouché theorem the functions $f(z) - w = (f(z) - f(a)) + (f(a) - w)$ and $f(z) - f(a)$ have in $B(a, \delta)$ the same number of zeros counted with multiplicities.

(i) $\Longrightarrow$ (ii): By the above proof, if a is a zero of $f - f(a)$ of multiplicity k, then f has multiplicity k at a. Thus $k = d$. □

Corollary 4.4.8. *Let $D \subset \mathbb{C}$ be a domain and let $f \in \mathcal{M}(D)$, $f \not\equiv \mathrm{const}$. Then f is an open mapping.*

Remark 4.4.9. If $f : D \longrightarrow \mathbb{C}$ is open, then $|f| : D \longrightarrow \mathbb{R}_+$ is open and $|f|$ satisfies the maximum principle.

CHAPTER 5

Biholomorphic mappings

5.1. Biholomorphic mappings

5.2. Biholomorphisms of annuli

Theorem 5.2.1. *For $f \in \mathcal{O}(D)$ the following conditions are equivalent:*

(i) $G := f(D)$ *is open and* $f \in \operatorname{Bih}(D, G)$*;*
(ii) f *is injective and* $f'(z) \neq 0$, $z \in D$*;*
(iii) f *is injective.*

Proof. Indeed, the implications (i) $\Longleftrightarrow$ (ii) $\Longrightarrow$ (iii) are elementary.

(iii) $\Longrightarrow$ (i): By Corollary 4.4.8, f is an open mapping. By Corollary 4.4.7 f satisfies (ii). $\square$

Theorem 5.2.2 (Hadamard [1] three circles theorem)**.** *Let $f \in \mathcal{O}(\mathbb{A}(r_1, r_2))$, $0 < r_1 < r_2 < +\infty$, and let $M_j := \sup\{\limsup_{z\to\zeta} |f(z)| : \zeta \in C(r_j)\}$, $j = 1, 2$. Then*

$$|f(z)| \le M_1^{\frac{\log\frac{|z|}{r_2}}{\log\frac{r_1}{r_2}}} M_2^{\frac{\log\frac{|z|}{r_1}}{\log\frac{r_2}{r_1}}}, \quad z \in \mathbb{A}(r_1, r_2).$$

Proof. We may assume that M_1, $M_2 < +\infty$, $f \not\equiv \text{const}$. Let $u(z) := |z|^\alpha |f(z)|$, $z \in \mathbb{A}(r_1, r_2)$. Observe that u is an open mapping because locally $u = |e^{\alpha\ell} f|$, where ℓ is a local branch of the logarithm. Since all open mappings satisfy the maximum principle we get $|z|^\alpha |f(z)| \le \max\{r_1^\alpha M_1,\ r_2^\alpha M_2\}$, $z \in \mathbb{A}(r_1, r_2)$. Taking α so that $r_1^\alpha M_1 = r_2^\alpha M_2$ we get the result (Exercise). $\square$

Remark 5.2.3. If $f \in \mathcal{O}(\mathbb{A}(r_1, r_2)) \cap \mathcal{C}(\overline{\mathbb{A}}(r_1, r_2))$ and $M(r) := \max\{|f(z)| : z \in C(r)\}$, then the function $[\log r_1, \log r_2] \ni t \longmapsto \log M(e^t)$ is convex.

Theorem 5.2.4. *If $f \in \operatorname{Bih}(\mathbb{A}(r_1, R_1), \mathbb{A}(r_2, R_2))$, $0 < r_j < R_j < +\infty$, $j = 1, 2$, then $R_1/r_1 = R_2/r_2$ and $f(z) = (r_2/r_1)z$ or $f(z) = r_1 R_2/z$ up to a rotation.*

In particular, for $0 < r < R < +\infty$, $\operatorname{Aut}(\mathbb{A}(r, R)) = \{z \longmapsto e^{i\vartheta} z : \vartheta \in \mathbb{R}\} \cup \{z \longmapsto e^{i\vartheta} rR/z : \vartheta \in \mathbb{R}\}$; the group $\operatorname{Aut}(\mathbb{A}(r, R))$ depends on one real parameter and does not act transitively.

Proof. We may assume that $r_1 = r_2 = 1$. Let $g := f^{-1}$. The mapping f is proper so

$$\lim_{\operatorname{dist}(z, \partial\mathbb{A}(1, R_1)) \to 0} \operatorname{dist}(f(z), \partial\mathbb{A}(1, R_2)) = 0.$$

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[1] Jacques Hadamard (1865–1963).

We will show that either

$$\lim_{|z|\to 1}|f(z)|=1 \text{ and } \lim_{|z|\to R_1}|f(z)|=R_2, \tag{†}$$

or

$$\lim_{|z|\to 1}|f(z)|=R_2 \text{ and } \lim_{|z|\to R_1}|f(z)|=1. \tag{‡}$$

Suppose for a moment that (†) is true. Then, by the Hadamard theorem,

$$|f(z)|\le R_2^{\frac{\log|z|}{\log R_1}}=|z|^{\frac{\log R_2}{\log R_1}},\ z\in\mathbb{A}(1,R_1),\quad\text{ and }\quad |g(w)|\le R_1^{\frac{\log|w|}{\log R_2}}=|w|^{\frac{\log R_1}{\log R_2}},\ w\in\mathbb{A}(1,R_2).$$

Hence $|f(z)|=|z|^{\frac{\log R_2}{\log R_1}}=:|z|^\alpha$, $z\in\mathbb{A}(1,R_1)$. Our aim is to show that $\alpha=1$. We have $f(z)=e^{i\vartheta}e^{\alpha\operatorname{Log}z}$, $z\in\mathbb{A}(1,R_1)\setminus\mathbb{R}_-$ (for a $\vartheta\in\mathbb{R}$). Since f is continuous, we must have $e^{i\vartheta}e^{\alpha(\log t+i\pi)}=e^{i\vartheta}e^{\alpha(\log t-i\pi)}$, $t\in(1,R_1)$. Hence $e^{2\alpha\pi i}=1$, and therefore $\alpha\in\mathbb{Z}$. Since f is injective we get $\alpha=\pm1$. The condition (†) implies that $\alpha=1$.

The case (‡) reduces to the above after the composition with the inversion

$$\mathbb{A}(1,R_2)\ni w\longmapsto R_2/w\in\mathbb{A}(1,R_2). \tag{*}$$

It remains to check (†), (‡). Let $r:=\sqrt{R_2}$, $B_-:=\mathbb{A}(1,r)$, $B_+:=\mathbb{A}(r,R_2)$. Since $g(C(r))$ is compact there exist $1<s_1<s_2<R_1$ such that $g(C(r))\subset\mathbb{A}(s_1,s_2)$. Consider domains $A_+:=f(\mathbb{A}(s_2,R_1))$ and $A_-:=f(\mathbb{A}(1,s_1))$. Since $A_+\cap C(r)=\varnothing$, the domain A_+ is contained in B_+ or B_-. We may assume that $A_+\subset B_+$ (use the inversion (*)). This means that $\lim_{|z|\to R_1}|f(z)|=R_2$. It remains to show that $A_-\subset B_-$. Suppose that $A_-\subset B_+$. Then we can joint an arbitrary point $a_+\in A_+$ with any $a_-\in A_-$ by a curve γ in B_+. Then the curve $g(\gamma)$ connects $g(a_+)\in\mathbb{A}(s_2,R_1)$ and $g(a_-)\in\mathbb{A}(1,s_1)$ and is disjoint with $g(C(r))$ — a contradiction. □

Exercise 5.2.5. Describe all biholomorphisms $f:\mathbb{A}(r_1,R_1)\longrightarrow\mathbb{A}(r_2,R_2)$, $0\le r_j<R_j\le+\infty$, $j=1,2$, in all the cases not covered by Theorem 5.2.4.

5.3. Riemann theorem

Theorem 5.3.1 (Riemann theorem)**.** *Let $D\subset\widehat{\mathbb{C}}$ be a simply connected domain with $\#\partial D\ge 2$. Then there exists a biholomorphism $f:D\longrightarrow\mathbb{D}$.*

Proof. The case $\infty\in D$ reduces to a $D\subset\mathbb{C}$ via an inversion. Let $a,b\in\partial D$, $a\ne b$. Fix a $z_0\in D$ and let $\boldsymbol{\mathfrak{R}}:=\{f\in\mathcal{O}(D,\mathbb{D}): f(z_0)=0,\ f \text{ is injective}\}$.

First we prove that $\boldsymbol{\mathfrak{R}}\ne\varnothing$. Observe that it suffices to find an injective $g:D\longrightarrow\mathbb{C}$ such that $B(c,r)\cap g(D)=\varnothing$ for some $c\in\mathbb{C}$ and $r>0$. In fact, if we have g, then we put $f:=\frac{r}{g-c}$.

We move to the construction of g. We may assume that $a\in\mathbb{C}\setminus D$. Let g be a branch of $z\longmapsto\sqrt{z-a}$ (cf. Theorem 2.3.12). It is an injective function in D and $g(D)\cap(-g(D))=\varnothing$. In fact, if $g(z_1)=-g(z_2)$, then $g^2(z_1)=g^2(z_2)$, so $z_1=z_2$. Hence $g(z_1)=-g(z_1)=0$ and therefore $z_1=z_2=a$ — a contradiction. Now we can take an arbitrary $B(c,r)\subset -g(D)$.

Let $M:=\sup\{|f'(z_0)|: f\in\boldsymbol{\mathfrak{R}}\}$. Since each $f\in\boldsymbol{\mathfrak{F}}$ is injective we must have $M>0$. Let $(f_k)_{k=1}^\infty\subset\boldsymbol{\mathfrak{R}}$, $f_k'(z_0)\longrightarrow M$. By the Montel theorem we may assume that $f_k\longrightarrow f_0$ locally uniformly in D. Obviously, $f_0 \in \mathcal{O}(D, \overline{\mathbb{D}})$, $f_0'(z_0) = M > 0$. In particular, $f_0 \not\equiv \text{const}$. Since $f_0(z_0) = 0$, we conclude that $f \in \mathcal{O}(D, \mathbb{D})$. By the Hurwitz theorem we get $f_0 \in \boldsymbol{\mathcal{R}}$. We will show that $f_0(D) = \mathbb{D}$ and therefore f_0 is the required mapping.

Suppose that $G := f_0(D) \subsetneq \mathbb{D}$. We need the following lemma.

Lemma 5.3.2. *Let $G \subsetneq \mathbb{D}$ be a simply connected domain with $0 \in G$. Then there exists an injective mapping $\psi \in \mathcal{O}(G, \mathbb{D})$ such that $\psi(0) = 0$, and $|\psi'(0)| > 1$.*

Proof. Fix a $c \in \mathbb{D} \setminus G$ and let $G_1 := h_c(G)$. Then $G_1 \subset \mathbb{D}$ is a simply connected domain with $0 \notin G_1$. In particular, there exists a branch of the square root G_1. Let $d := g(h_c(0))$ and let $\psi := h_d \circ g \circ h_c$. Then $\psi : G \longrightarrow \mathbb{D}$ is injective and $\psi(0) = 0$. Observe that $\psi^{-1} = h_{-c} \circ (z \longmapsto h_{-d}^2(z)) \in \mathcal{O}(\mathbb{D}, \mathbb{D})$ (in the sense of the extension from $\psi(G)$ to $\mathbb{D}$). The Schwarz lemma implies that $|\psi^{-1}(w)| \leq |w|$, $w \in \mathbb{D}_*$, $|(\psi^{-1})'(0)| \leq 1$. The equality would imply that $\psi^{-1}(w) = e^{i\alpha} w$, and hence $(h_{-d}(z))^2 = h_c(e^{i\alpha} z)$, $z \in \mathbb{D}$ — a contradiction. $\square$

Now let $\psi \in \mathcal{O}(G, \mathbb{D})$ be as in the lemma. Put $f := \psi \circ f_0$. Then $f \in \boldsymbol{\mathcal{R}}$ and $|f'(z_0)| = |\psi'(0) f_0'(z_0)| = |\psi'(0)| M > M$ – a contradiction. $\square$

Corollary 5.3.3. *Let $D \subset \widehat{\mathbb{C}}$ be a simply connected domain with $\#\partial D \geq 2$. Let $z_0 \in D \cap \mathbb{C}$, $\vartheta \in \mathbb{R}$. Then there exists exactly one $f \in \operatorname{Bih}(D, \mathbb{D})$ such that $f(z_0) = 0$ and $\vartheta \in \arg f'(z_0)$.*

Proof. By the Riemann theorem there exists a biholomorphic mapping $f : D \longrightarrow \mathbb{D}$. Taking $h_{f(z_0)} \circ f \in \operatorname{Aut}(\mathbb{D})$ we get $f(z_0) = 0$. Now it remains to use a suitable rotation to get $\vartheta \in \arg f'(z_0)$.

If $f_1, f_2 : D \longrightarrow \mathbb{D}$ are two mappings with the above property, then $\varphi = f_2 \circ f_1^{-1} \in \operatorname{Aut}(\mathbb{D})$, $\varphi(0) = 0$ and $\varphi'(0) \in \mathbb{R}_{>0}$. Hence $\varphi = \mathrm{id}$ and so $f_1 \equiv f_2$. $\square$

5.4. Index

Definition 5.4.1. Let $\gamma : [0, 1] \longrightarrow \mathbb{C}$ be a closed path. For $a \in \mathbb{C} \setminus \gamma^*$ the integral

$$\operatorname{Ind}_\gamma(a) := \frac{1}{2\pi i} \int_\gamma \frac{1}{z - a} dz$$

is called the *index of a with respect to γ*.

Theorem 5.4.2. $\operatorname{Ind}_\gamma(a) \in \mathbb{Z}$ *and* $\operatorname{Ind}_\gamma$ *is zero in the unbounded component of* $\mathbb{C} \setminus \gamma^*$.

Proof. Obviously, $\operatorname{Ind}_\gamma$ is continuous and $|\operatorname{Ind}_\gamma(a)| \leq \frac{1}{2\pi} \frac{\boldsymbol{\ell}(\gamma)}{\operatorname{dist}(a, \gamma^*)} \underset{a \to \infty}{\longrightarrow} 0$. It remains to prove that $\operatorname{Ind}_\gamma(a) \in \mathbb{Z}$, $a \in \mathbb{C} \setminus \gamma^*$. Fix an a and let $h(x) := \int_0^x \frac{\gamma'(t)}{\gamma(t) - a} dt$, $0 \leq x \leq 1$. The function h is continuous, differentiable in $(0, 1)$ except a finite number of points, $h(0) = 0$, $h(1) = 2\pi i \operatorname{Ind}_\gamma(a)$. Observe that $(e^{-h}(\gamma - a))' = e^{-h}(-h'(\gamma - a) + \gamma') = 0$ except for a finite number of points. Hence $e^{-h}(\gamma - a) = \text{const} = \gamma(0) - a$. Consequently, $e^h = \frac{\gamma - a}{\gamma(0) - a}$, and therefore $e^{h(1)} = 1$. Thus $h(1) = 2\pi i \operatorname{Ind}_\gamma(a) = 2\pi i\ k$ for a $k \in \mathbb{Z}$. $\square$

Exercise 5.4.3. Let $\gamma : [0, 1] \longrightarrow \mathbb{C}$ be a Jordan path with positive orientation with respect to $\operatorname{int} \gamma$. Then $\operatorname{Ind}_\gamma(z) = \begin{cases} 1, & \text{if } z \in \operatorname{int} \gamma \\ 0, & \text{if } z \in \operatorname{ext} \gamma \end{cases}$.

Theorem 5.4.4. *Let $\gamma : [0,1] \longrightarrow \mathbb{C}$ be a closed curve, let $a \in \mathbb{C} \setminus \gamma^*$, and let $r := \operatorname{dist}(a, \gamma^*)$. Let $\sigma_j : [0,1] \longrightarrow \mathbb{C}$ be a closed path such that $\|\sigma_j - \gamma\|_{[0,1]} \le r/4$, $j = 1, 2$. Then $\operatorname{Ind}_{\sigma_1}(a) = \operatorname{Ind}_{\sigma_2}(a)$. Consequently, the formula $\operatorname{Ind}_\gamma(a) := \lim_{\substack{\sigma-\text{closed path}\\ \|\sigma-\gamma\|_{[0,1]}\longrightarrow 0}} \operatorname{Ind}_\sigma(a)$, $a \in \mathbb{C} \setminus \gamma^*$, defines $\operatorname{Ind}_\gamma : \mathbb{C} \setminus \gamma^* \longrightarrow \mathbb{Z}$ for arbitrary closed curve $\gamma : [0,1] \longrightarrow \mathbb{C}$.*

[Theorem 5.4.4⟶ Exer . . . .]

Theorem 5.4.5 (Cauchy–Goursat)**.** *Let $D \subset \mathbb{C}$ be simply connected and let $f \in \mathcal{O}(D)$. Then*

$$\int_\gamma f(z)dz = 0 \quad \text{and} \quad f(a)\operatorname{Ind}_\gamma(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}dz, \quad a \in D \setminus \gamma^*,$$

for every closed path $\gamma : [0,1] \longrightarrow D$ (cf. Theorem 2.3.12).

Theorem 5.4.6 (Cauchy–Dixon theorem)**.** *Let D be a domain and let γ be a closed path in D. Then the following conditions are equivalent:*

(i) *for every $f \in \mathcal{O}(D)$ we have $f(a)\operatorname{Ind}_\gamma(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}dz$, $a \in D \setminus \gamma^*$;*

(ii) *for every $f \in \mathcal{O}(D)$ we have $\int_\gamma f(z)dz = 0$;*

(iii) *for $\operatorname{Ind}_\gamma(a) = 0$, for every $a \in \mathbb{C} \setminus D$.*

Proof. (i) $\Longrightarrow$ (ii): We apply (i) to the function $z \longmapsto (z-a)f$.

(ii) $\Longrightarrow$ (iii): We apply (ii) to the function $z \longmapsto \frac{1}{z-a}$.

(iii) $\Longrightarrow$ (i): Fix an f. We have to check that $\frac{1}{2\pi i}\int_\gamma \frac{f(z)-f(a)}{z-a}dz = 0$, $a \in \mathbb{C} \setminus \gamma^*$. Define $g(z,w) := \begin{cases} \frac{f(z)-f(w)}{z-w}, & \text{if } z \ne w \\ f'(z), & \text{if } z = w \end{cases}$, $(z,w) \in D \times D$. We know that g is separately holomorphic ([2]). The continuity of G out of the diagonal is trivial. For $(a,a) \in D \times D$ and $B(a,r) \subset\subset D$ we have

$$g(z,w) - g(a,a) = \frac{1}{2\pi i}\int_{C(a,r)} \Big(\frac{1}{z-w}\Big(\frac{f(\zeta)}{\zeta-z} - \frac{f(\zeta)}{\zeta-w}\Big) - \frac{f(\zeta)}{(\zeta-a)^2}\Big)d\zeta$$
$$= \frac{1}{2\pi i}\int_{C(a,r)} f(\zeta)\Big(\frac{1}{(\zeta-z)(\zeta-w)} - \frac{1}{(\zeta-a)^2}\Big)d\zeta \underset{(z,w)\to(a,a)}{\longrightarrow} 0$$

because the function under the integral is uniformly continuous with respect to ζ when $(z,w) \longrightarrow (a,a)$. Let

$$h(w) = \begin{cases} h_1(z) \\ h_2(z) \end{cases} := \begin{cases} \frac{1}{2\pi i}\int_\gamma g(z,w)dz, & \text{if } w \in D \\ \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w}dz, & \text{if } w \in \mathbb{C} \setminus D \end{cases}.$$

We are going to prove that $h \in \mathcal{O}(\mathbb{C})$. Since $h(w) \longrightarrow 0$ when $w \longrightarrow \infty$, the maximum principle implies that $h \equiv 0$. In particular, $\frac{1}{2\pi i}\int_\gamma \frac{f(z)-f(a)}{z-a}dz = 0$, $a \in \mathbb{C} \setminus \gamma^*$.

By the production lemma, the function $\mathbb{C} \setminus D \subset \mathbb{C} \setminus \gamma^* \ni w \overset{h_0}{\longmapsto} \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w}dz$ is holomorphic.

([2]) In fact, every separately holomorphic function is holomorphic with respect to all variables – at the moment this result is beyond our lecture.

The function h is continuous on D. For every triangle $T \subset\subset D$, using the Fubini ([3]) theorem, we get

$$\int_{\partial T} h(w)dw = \frac{1}{2\pi i}\int_\gamma \Big(\int_{\partial T} g(z,w)dw\Big)dz = 0.$$

Consequently, by the Morera theorem $h \in \mathcal{O}(D)$.

In view of (iii) $\operatorname{Ind}_\gamma = 0$ in each connected component of $\mathbb{C} \setminus \gamma^*$ that intersects $\mathbb{C} \setminus D$, i.e. $h = h_0 = 0$ in each connected component of $\mathbb{C} \setminus \gamma^*$ that intersects $\mathbb{C} \setminus D$.

Let $C := \{z \in \mathbb{C} \setminus \gamma^* : \operatorname{Ind}_\gamma(z) = 0\}$. We have $\mathbb{C} \setminus D \subset C$. Moreover, $h_1 = h_2$ on $D \setminus C$. Hence, by the identity principle, $h \in \mathcal{O}(\mathbb{C})$. □

Theorem 5.4.7. *The following conditions are equivalent:*

(i) *every $f \in \mathcal{O}(D)$ has a primitive;*
(ii) *every $f \in \mathcal{O}^*(D)$ has a branch of its logarithm in D;*
(iii) *for every $f \in \mathcal{O}^*(D)$ there exists a $p = p(f) \in \mathbb{N}_2$ such that f has a branch of its p-th root in D;*
(iv) *$\int_\gamma f(z)dz = 0$ for every closed path $\gamma : [0,1] \longrightarrow D$;*
(v) *the set $\widehat{\mathbb{C}} \setminus D$ is connected.*

Proof. (i) $\Longrightarrow$ (ii): Let $g \in \mathcal{O}(D)$ be such that $g' = f'/f$. We may assume that $e^{g(a)} = f(a)$ for an $a \in D$. We have $\left(\frac{e^g}{f}\right)' = \frac{g'e^g f - e^g f'}{f^2} = 0$ and therefore $e^g = f$ (cf. Theorem 2.3.12).

(ii) $\Longrightarrow$ (iii): $f = e^g = (e^{g/p})^p$ (cf. Remark 2.3.6).

(iii) $\Longrightarrow$ (ii): It suffices to show that f'/f has a primitive. We already know (cf. Lemma 2.3.1) that we only need to show that that $\int_\gamma \frac{f'(z)}{f(z)}dz = 0$ for every closed path γ in D. Let

$p_1 := p(f)$, $g_1 \in \mathcal{O}^*(D)$, $g_1^{p_1} = f$,
$p_2 := p(g_1)$, $g_2 \in \mathcal{O}^*(D)$, $g_2^{p_2} = g_1$, $g_2^{p_1p_2} = f$, ...,
$p_k := p(g_{k-1})$, $g_k \in \mathcal{O}^*(D)$, $g_k^{p_k} = g_{k-1}$, $g_k^{p_1\cdots p_k} = f$, ...

Put $q_k := p_1 \cdots p_k \nearrow +\infty$. Hence $\frac{f'}{f} = \frac{q_k g_k^{q_k-1} g_k'}{g_k^{q_k}} = q_k \frac{g_k'}{g_k}$, and therefore

$$\operatorname{Ind}_{f\circ\gamma}(0) = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = q_k \frac{1}{2\pi i}\int_\gamma \frac{g_k'(z)}{g_k(z)}dz = q_k \operatorname{Ind}_{g_k\circ\gamma}(0), \quad k \in \mathbb{N}.$$

Thus $q_k | \operatorname{Ind}_{f\circ\gamma}(0)$ for every $k \in \mathbb{N}$. It is only possible if $\operatorname{Ind}_{f\circ\gamma}(0) = 0$.

(ii) $\Longrightarrow$ (iv): Fix an $a \notin D$ and let $g \in \mathcal{O}(D)$ be such that $e^g = z - a$. Then $e^g g' = 1$. hence $g' = \frac{1}{z-a}$. Thus the function $z \longmapsto \frac{1}{z-a}$ has a primitive. Now, using Lemma 2.3.1, we get $\operatorname{Ind}_\gamma(a) = 0$. –Where is f?

(iv) $\Longrightarrow$ (i): It follows from the Cauchy-Dixon Theorem 5.4.6 and Lemma 2.3.1.

(iv) $\Longrightarrow$ (v): Suppose that $\widehat{\mathbb{C}} \setminus D$ is not connected. Let K be a compact component of $\widehat{\mathbb{C}} \setminus D$ such that $U := D \cup K$ is open. Let $G := \operatorname{int} Q$ be an open set based on a net $Q_{j,k} := [\frac{j}{m}, \frac{j+1}{m}] \times [\frac{k}{m}, \frac{k+1}{m}]$ $(m \gg 1)$

([3]) Guido Fubini (1879–1943).

such that $K \subset G \subset\subset U$, $Q := \bigcup_{\substack{Q_{j,k}\colon Q_{j,k} \subset D, \\ Q_{j,k} \cap K \neq \varnothing}} Q_{j,k}$, G is open and its boundary may be identified with a finite number of Jordan piecewise linear curves $\gamma_1, \dots, \gamma_N$. Then $\operatorname{Ind}_\gamma(a) = 1$, $a \in K$. In particular, $\operatorname{Ind}_{\gamma_j}(a) \neq 0$ for some $a \in K \subset \mathbb{C} \setminus D$ and $j \in \{1, \dots, N\}$ – a contradiction.

(v) $\Longrightarrow$ (iv): We know that $\operatorname{Ind}_\gamma(a) = 0$, $a \in D_\infty$, where D_∞ is the unbounded component of $\widehat{\mathbb{C}} \setminus \gamma^*$ ($\operatorname{Ind}_\gamma(\infty) := 0$). Clearly, $(\widehat{\mathbb{C}} \setminus D) \cap D_\infty \neq \varnothing$. It remains to use the fact that $\operatorname{Ind}_\gamma$ is constant on $\widehat{\mathbb{C}} \setminus D$. $\square$

CHAPTER 6

Runge theorem

6.1. Runge theorem

Exercise 6.1.1. [Exercise 6.1.1⟶ Exer] For every open set $\Omega \subset \widehat{\mathbb{C}}$ there exists a sequence of compact sets $(K_k)_{k=1}^{\infty} \subset \Omega$ such that

- $K_k \subset \operatorname{int} K_{k+1}$,
- every connected component of $\widehat{\mathbb{C}} \setminus K_k$ intersects $\widehat{\mathbb{C}} \setminus \Omega$, $k \in \mathbb{N}$,
- $\Omega = \bigcup_{k=1}^{\infty} K_k$.

Theorem 6.1.2 (Runge ([1]) Theorem)**.** (a) *Let $\Omega \subset \widehat{\mathbb{C}}$ be open and let $f \in \mathcal{O}(\Omega)$. Then there exists a sequence $(f_k)_{k=1}^{\infty}$ of rational functions with poles in $\widehat{\mathbb{C}} \setminus \Omega$ such that $f_k \longrightarrow f$ locally uniformly in Ω.*

Equivalently: for every compact set $K \subset\subset \Omega$ and $\varepsilon > 0$ there exists a rational function g with poles in $\widehat{\mathbb{C}} \setminus \Omega$ such that $|g - f| \le \varepsilon$ on K.

(b) *Let $\Omega \subset \mathbb{C}$ be an open set such that $\widehat{\mathbb{C}} \setminus \Omega$ is connected and let $f \in \mathcal{O}(\Omega)$. The there exists a sequence $(f_k)_{k=1}^{\infty} \subset \mathcal{P}(\mathbb{C})$ such that $f_k \longrightarrow f$ locally uniformly in Ω.*

Equivalently: for every compact set $K \subset\subset \Omega$ and $\varepsilon > 0$ there exists a polynomial $g \in \mathcal{P}(\mathbb{C})$ such that $|g - f| \le \varepsilon$ on K.

Exercise 6.1.3. The polynomial version of the Runge theorem does not hold for $\Omega = \mathbb{A}(r, R)$, $0 < r < R < +\infty$.

[Exercise 6.1.3⟶ Exer . . . .]

Proof. (a) The case $\Omega = \widehat{\mathbb{C}}$ is trivial because $f \equiv \text{const}$. If $\infty \in \Omega \subsetneq \widehat{\mathbb{C}}$, then fix a point $z_0 \in \mathbb{C} \setminus \Omega$. Define h. If g_1 is a rational function with poles in $\widehat{\mathbb{C}} \setminus h(\Omega)$ such that $|g_1 - f \circ h^{-1}| \le \varepsilon$ on $h(K)$, then $g := g_1 \circ h$ solves our problem. Thus we may assume that $\infty \notin \Omega$.

Let $(K_k)_{k=1}^{\infty}$ be as in Exercise 6.1.1. We only need to approximate f on each K_k. Fix $K := K_{k_0}$ and ε. Let G be an open set based on a square net $[\frac{j}{m}, \frac{j+1}{m}] \times [\frac{k}{m}, \frac{k+1}{m}]$ ($m \gg 1$) so that $K \subset G \subset\subset \Omega$. The Cauchy integral formula gives

$$f(z) = \frac{1}{2\pi i} \int_{\partial G} \frac{f(\zeta)}{\zeta - z} d\zeta = \sum_{s=1}^{N} \frac{1}{2\pi i} \int_{L_s} \frac{f(\zeta)}{\zeta - z} d\zeta =: \sum_{s=1}^{N} f_s(z), \quad z \in G,$$

where each L_s is a single vertical or horizontal segment from our net. Now, it suffices to approximate each function f_s uniformly on K by rational functions with poles in $\widehat{\mathbb{C}} \setminus \Omega$. Fix an s.

First, we will find an approximation by rational functions with poles in $L_s =: [a, b]$. Let $\zeta(t) := a + t(b - a)$, $\zeta_{n,j} := \zeta(\frac{j}{n})$, $n \in \mathbb{N}$, $j = 0, \dots, n$. For $z \in K$ we obtain:

$$\Big|f_s(z) - \frac{1}{2\pi i}\sum_{j=1}^{n} \frac{f(\zeta_{n,j})}{\zeta_{n,j} - z}\frac{|b-a|}{n}\Big| = \Big|\frac{1}{2\pi i}\sum_{j=1}^{n}\int_{\frac{j-1}{n}}^{\frac{j}{n}} \frac{f(\zeta(t))}{\zeta(t) - z}(b-a)dt - \frac{1}{2\pi i}\sum_{j=1}^{n}\frac{f(\zeta_{n,j})}{\zeta_{n,j} - z}\frac{b-a}{n}\Big|$$

$$\leq \frac{|b-a|}{2\pi}\sum_{j=1}^{n}\int_{\frac{j-1}{n}}^{\frac{j}{n}}\Big|\frac{f(\zeta(t))}{\zeta(t) - z} - \frac{f(\zeta_{n,j})}{\zeta_{n,j} - z}\Big|dt.$$

Now, using the uniform continuity of the function $K \times [a, b] \ni (z, \zeta) \longmapsto \frac{f(\zeta)}{\zeta - z}$, we conclude that for $n \gg 1$ we get

$$\Big|f_s(z) - \frac{b-a}{2\pi i\ n}\sum_{j=1}^{n}\frac{f(\zeta_{n,j})}{\zeta_{n,j} - z}\Big| \leq \frac{|b-a|}{2\pi}\varepsilon, \quad z \in K.$$

Thus, it remains to prove that for every $c \in [a, b]$, the function $\frac{1}{z-c}$ may be approximated uniformly on K by rational functions with poles in $\widehat{\mathbb{C}} \setminus \Omega$. It follows from the following general result.

Lemma 6.1.4 (Pole transport lemma)**.** *Let $K \subset\subset \mathbb{C}$ be compact and let $f = P(\frac{1}{z-a})$, where $P \in \mathcal{P}(\mathbb{C})$, $\deg P \geq 1$. Let $b \in \widehat{\mathbb{C}} \setminus K$ be in the same connected component of $\widehat{\mathbb{C}} \setminus K$ as a. Then for every $\varepsilon > 0$ there exists a $Q \in \mathcal{P}(\mathbb{C})$ such that $|f - g| \leq \varepsilon$ on K, where $g := Q(\frac{1}{z-b})$. If $b = \infty$, then $g = Q$.*

Proof. Let G be a connected component of $\widehat{\mathbb{C}} \setminus K$ with a, $b \in G$. Note that $G \cap \mathbb{C}$ is connected. Let G_0 be the set of all $c \in G \cap \mathbb{C}$ for which for every $\varepsilon > 0$ there exists a polynomial R such that $|h - f| \leq \varepsilon$ on K, where $h = R(\frac{1}{z-c})$. Obviously, $a \in G_0$. We will show that G_0 is open and closed in $G \cap \mathbb{C}$, which will prove that $G_0 = G \cap \mathbb{C}$.

Openness: Let $c \in G_0$ and let $h = R(\frac{1}{z-c})$ be such that $|f - h| \leq \varepsilon/2$ on K. Let $r := \operatorname{dist}(c, K)$, $d \in B(c, r/3) \subset\subset G$. We only need to approximate uniformly on K the function $\frac{1}{z-c}$ by functions of the form $S(\frac{1}{z-d})$. It suffices to observe that for $z \in K$ we get $|\frac{c-d}{z-d}| \leq 1/2$ and

$$\frac{1}{z-c} = \frac{1}{z - d + d - c} = \frac{1}{z-d}\frac{1}{1 - \frac{c-d}{z-d}} = \sum_{n=0}^{\infty}\frac{(c-d)^n}{(z-d)^{n+1}}$$

and the series is uniformly convergent on K.

Closedness: Let $d \in G_0' \cap G \cap \mathbb{C}$. Take a $c \in G_0 \cap B(d, r/2)$, where $r := \operatorname{dist}(d, K)$. Then $|\frac{c-d}{z-d}| \leq 1/2$ and we may repeat the above argument.

It remains to consider the case where $\infty \in G$. Take a $c \in G_0 \setminus B(2r)$, where $K \subset B(r)$. Then $|\frac{z}{c}| \leq 1/2$, $z \in K$, and

$$\frac{1}{z-c} = -\frac{1}{c}\frac{1}{1-\frac{z}{c}} = -\sum_{n=0}^{\infty} \frac{z^n}{c^{n+1}}$$

and the series is uniformly convergent on K. □

(b) follows from (a) and the lemma. □

The Runge theorem may be essentially strengthened.

Theorem* 6.1.5 (Mergeljan ([2]) theorem)**.** *Let $K \subset \mathbb{C}$ be a compact set such that the set $\mathbb{C} \setminus K$ is connected and let $f \in \mathcal{C}(K) \cap \mathcal{O}(\operatorname{int} K)$. Then there exists a sequence $(f_k)_{k=1}^{\infty} \subset \mathcal{P}(\mathbb{C})$ such that $f_k \longrightarrow f$ uniformly on K.*

Exercise 6.1.6. The assumptions in the Mergeljan theorem are also necessary.

([2]) Sergey Mergelyan (1928–2008).

CHAPTER 7

Mittag-Leffler theorem

7.1. Mittag-Leffler theorem

Theorem 7.1.1 (Mittag-Leffler ([1]) theorem)**.** *For arbitrary open set $\varOmega \subsetneq \widehat{\mathbb{C}}$, for arbitrary set $B \subset \varOmega$ without accumulation points in $\varOmega$, and for arbitrary family $(P_a)_{a\in B} \subset \mathcal{P}(\mathbb{C})$ of polynomials of degree ≥ 1 with $P_a(0) = 0$, $a \in B$, there exists an $f \in \mathcal{M}(\varOmega) \cap \mathcal{O}(\varOmega \setminus B)$ such that for each $a \in B$ the function $f - P_a\big(\frac{1}{z-a}\big)$ has a removable singularity at a, i.e. $P_a(\frac{1}{z-a})$ is the principal part of pole of f at a. If $\infty \in B$, then we mean that P_∞ is the principal part of pole of f at ∞.*

Proof. If $\infty \in B$, $B_1 := B \setminus \{\infty\}$ and $f_1 \in \mathcal{M}(\varOmega) \cap \mathcal{O}(\varOmega \setminus B_1)$ is such that for each $a \in B_1$ the principal part of pole of f_1 at a equals $P_a(\frac{1}{z-a})$, then $f := f_1 + P_\infty$ is a solution of the initial problem. Thus we may assume that $\infty \notin B$.

If B is finite, then we may take $f := \sum_{a\in B} P_a(\frac{1}{z-a})$.

Assume that B is infinite. Let $(K_k)_{k=1}^{\infty}$ be as in Remark 6.1.1 an let

$$f_k(z) := \sum_{a\in B\cap(K_k\setminus K_{k-1})} P_a\Big(\frac{1}{z-a}\Big), \quad k \in \mathbb{N},$$

where $K_0 := \varnothing$ and $\sum_{a\in\varnothing} \cdots := 0$. Each set $B \cap (K_k \setminus K_{k-1})$ is finite. Thus f_k is a well-defined rational function with poles in $\mathbb{C} \setminus K_{k-1}$. By the pole transport lemma, there exists a rational function g_k with poles in $\widehat{\mathbb{C}} \setminus \varOmega$ such that $|f_k - g_k| \leq 1/2^k$ in K_{k-1}. In particular, the series $\sum_{n=k}^{\infty}(f_n - g_n)$ is uniformly convergent in K_{k-1}. Let $f := \sum_{n=1}^{\infty}(f_n - g_n)$. Clearly, $f \in \mathcal{M}(\varOmega) \cap \mathcal{O}(\varOmega \setminus B)$. Moreover, for $a \in B \cap (K_{k_0} \setminus K_{k_0-1})$, we have

$$f - P_a\Big(\frac{1}{z-a}\Big) = \sum_{n=1}^{k_0-1}(f_n - g_n) + \Big(f_{k_0} - P_a\Big(\frac{1}{z-a}\Big)\Big) - g_{k_0} + \sum_{n=k_0+1}^{\infty}(f_n - g_n) =: A + B - g_{k_0} + C,$$

where
- A has poles in K_{k_0-1},
- B is holomorphic in a neighborhood of a,
- C has poles outside K_{k_0}. □

The Mittag-Leffler theorem may be also formulated in the following sheaf-theory form.

([1]) Magnus Mittag–Leffler (1846–1927).

Theorem 7.1.2 (Mittag-Leffler theorem)**.** *For every open covering $(\Omega_\alpha)_{\alpha\in A}$ of an open set Ω and for every family $f_\alpha \in \mathcal{M}(\Omega_\alpha)$, $\alpha \in A$ such that $f_\alpha - f_\beta \in \mathcal{O}(\Omega_\alpha \cap \Omega_\beta)$, $\alpha, \beta \in A$, there exists an $f \in \mathcal{M}(\Omega)$ such that $f - f_\alpha \in \mathcal{O}(\Omega_\alpha)$, $\alpha \in A$.*

Theorem 7.1.2 $\Longrightarrow$ Theorem 7.1.1. Let Ω, B, and $(P_a)_{a\in B}$ be as in Theorem 7.1.1. Let $r_a > 0$, $a \in B$, be such that $B(a, r_a) \cap B(b, r_b) = \varnothing$, $a \neq b$, $a, b \in B$. If $\infty \in B$, then by $B(\infty, r_\infty)$ we mean a suitable neighborhood of ∞. Set

$$A := \{*\} \cup B, \quad \Omega_* := \Omega \setminus B, \quad \Omega_a := B(a, r_a), \quad f_* := 0, \quad f_a := P_a\Big(\frac{1}{z-a}\Big), \quad a \in B;$$

if $\infty \in B$, then $f_\infty := P_\infty$. One can easily check that all the assumptions of Theorem 7.1.2 are satisfied. Let $f \in \mathcal{M}(\Omega)$ be as in Theorem 7.1.2. Then

$$f = f - f_* \in \mathcal{O}(\Omega_*) = \mathcal{O}(\Omega \setminus B),\ f - P_a\Big(\frac{1}{z-a}\Big) = f - f_a \in \mathcal{O}(\Omega_a) = \mathcal{O}(B(a, r_a)),\ a \in B.$$

$\square$

Theorem 7.1.1 $\Longrightarrow$ Theorem 7.1.2. Let Ω, $(\Omega_\alpha)_{\alpha\in A}$, and $(f_\alpha)_{\alpha\in A}$ be as in Theorem 7.1.2. Set

$$B_\alpha := S(f_\alpha), \quad B := \bigcup_{\alpha\in A} B_\alpha.$$

Since $f_\alpha - f_\beta \in \mathcal{O}(\Omega_\alpha \cap \Omega_\beta)$ we conclude that, $B_\alpha \cap \Omega_\beta \subset B_\beta$, $\alpha, \beta \in A$. In particular, B has no accumulation points in Ω. For $a \in B_\alpha$, let $P_{\alpha,a} \in \mathcal{P}(\mathbb{C})$ be polynomial of degree ≥ 1 such that $P_{\alpha,a}(0) = 0$ and $f_\alpha - P_{\alpha,a}(\frac{1}{z-a})$ extends holomorphically to a neighborhood of a (i.e. $P_{\alpha,a}(\frac{1}{z-a})$ is the principal part of pole of f_α at a), with the standard change if $\infty \in B_\alpha$. Since $f_\alpha - f_\beta \in \mathcal{O}(\Omega_\alpha \cap \Omega_\beta)$, we conclude that $P_{\alpha,a}$ is independent of α. Put $P_a := P_{\alpha,a}$. Let $f \in \mathcal{M}(\Omega)$ be as in Theorem 7.1.1. Then $S(f) = B$ and for any $\alpha \in A$ and $a \in B_\alpha$, the function

$$f - f_\alpha = \Big(f - P_a\Big(\frac{1}{z-a}\Big)\Big) - \Big(f_\alpha - P_a\Big(\frac{1}{z-a}\Big)\Big)$$

extends holomorphically to a neighborhood of a (if $\infty \in B$, then $f - f_\infty = (f - P_\infty) - (f_\alpha - P_\infty)$). $\square$

7.2. Weierstrass theorem

Theorem 7.2.1 (Weierstrass theorem)**.** *For every open set $\Omega \subsetneq \widehat{\mathbb{C}}$, for every set $S \subset \Omega$ without accumulation points in Ω, and for every function $k : S \longrightarrow \mathbb{N}$, there exists a function $f \in \mathcal{O}(\Omega) \cap \mathcal{O}^*(\Omega \setminus S)$ such that* $\operatorname{ord}_a f = k(a)$, $a \in S$.

Proof. If $\infty \notin \Omega$, then we choose an arbitrary $z_0 \in \Omega \setminus S$ and use the transform $h(z) := \frac{1}{z-z_0}$. Then $\infty \in \Omega_1 := h(\Omega)$. Let $S_1 := h(S)$. Suppose that $f_1 \in \mathcal{O}(\Omega_1) \cap \mathcal{O}^*(\Omega_1 \setminus S_1)$ is such that f_1 has a zero of multiplicity $k(a)$ at $h(a)$, $a \in S$. then $f := f_1 \circ h$ solves our problem. Thus we may assume that $\infty \in \Omega$.

If S is finite, then we may take $f(z) := \prod_{a\in S}(z-a)^{k(a)}$. Thus assume that S is infinite. Write $S = \{s_1, s_2, \dots\}$ and let $a_1, a_2, \dots$ be the sequence obtained from $(s_j)_{j=1}^\infty$ by repeating each s_j $k(s_j)$ times. Let $c_k \in \partial\Omega$ be such that $|a_k - c_k| = \operatorname{dist}(a_k, \partial\Omega)$, $k \in \mathbb{N}$. Observe that $|a_k - c_k| \longrightarrow 0$ (Exercise).

Assume for a moment that, the following two lemmas are true.

Lemma 7.2.2. *Let $\Omega \subsetneq \widehat{\mathbb{C}}$ be open and let $f_k \in \mathcal{O}(\Omega)$, $k \in \mathbb{N}$. Assume that the series $\sum\limits_{k=1}^{\infty} |f_k|$ is convergent locally uniformly in Ω. Put $I_n := \prod_{k=1}^{n}(1+f_k) \in \mathcal{O}(\Omega)$, $n \in \mathbb{N}$. Then the sequence $(I_n)_{n=1}^{\infty}$ is convergent locally uniformly in Ω. Let $I := \lim\limits_{n\to+\infty} I_n =: \prod_{k=1}^{\infty}(1+f_k)$. Moreover for an $a \in \Omega$ we get $I(a) = 0 \iff \exists_{k\in\mathbb{N}} :\ 1 + f_k(a) = 0$.*

Lemma 7.2.3. *For a $k \in \mathbb{N}$, let $E_k(u) := (1-u)\exp\Big(u + \frac{u^2}{2} + \dots + \frac{u^k}{k}\Big)$. Then $|1 - E_k(u)| \le |u|^{k+1}$ for $u \in \mathbb{D}$.*

First, let us finish the proof of the Weierstrass theorem. Let

$$f(z) := \prod_{k=1}^{\infty} E_k\Big(\frac{a_k - c_k}{z - c_k}\Big), \quad z \in \Omega$$

By Lemma 7.2.2, it suffices to prove that for $f_k(z) := E_k(\frac{a_k-c_k}{z-c_k}) - 1$, the series $\sum\limits_{k=1}^{\infty} |f_k|$ is convergent locally uniformly in Ω. Fix a compact $K \subset\subset \Omega$ and let $k_0 \in \mathbb{N}$ be such that $2|a_k - c_k| \le \operatorname{dist}(K, \partial\Omega)$, $k \ge k_0$. Then $|\frac{a_k-c_k}{z-c_k}| \le 1/2$ for $z \in K$ i $k \ge k_0$. Now using Lemma 7.2.3, we conclude that $|f_k| \le (1/2)^{k+1}$ on K for $k \ge k_0$. The proof is completed. $\square$

Proof of Lemma 7.2.2. It suffices to prove that the series $\sum\limits_{n=2}^{\infty}(I_n - I_{n-1})$ is locally uniformly convergent. Observe that $|I_n| \le \prod_{k=1}^{n}(1+|f_k|) \le \prod_{k=1}^{n} e^{|f_k|} = \exp\Big(\sum\limits_{k=1}^{n} |f_k|\Big)$, which shows that the sequence $(I_n)_{n=1}^{\infty}$ is locally uniformly bounded. The equality $|I_n - I_{n-1}| = |I_{n-1}||f_n|$ implies now the locally uniform convergence.

To prove the second part it suffices to prove that there exists a $C > 0$ such that $|\prod_{k=k_0}^{n}(1+f_k(a))| \ge C$, $n \gg k_0$. Fix a neighborhood $U \subset\subset \Omega$ of a and let $k_0 \in \mathbb{N}$ be such that $|f_k| \le 1/2$ on U for $k \ge k_0$. Then for $k \ge k_0$ on U we have $\Big|\frac{f_k}{1+f_k}\Big| \le \frac{|f_k|}{1-|f_k|} \le 2|f_k|$. This means that the series $\sum\limits_{k=k_0}^{\infty} \frac{f_k}{1+f_k}$ is convergent locally uniformly in U. Hence, by the first part of the proof, the product

$$\prod_{k=k_0}^{\infty}\Big(1 - \frac{f_k}{1+f_k}\Big) = \prod_{k=k_0}^{\infty} \frac{1}{1+f_k} = \frac{1}{\prod_{k=k_0}^{\infty}(1+f_k)}$$

is convergent on U. $\square$

Proof of Lemma 7.2.3. We have

$$\begin{aligned} E_k'(u) &= -\exp\Big(u + \frac{u^2}{2} + \dots + \frac{u^k}{k}\Big) + (1-u)\exp\Big(u + \frac{u^2}{2} + \dots + \frac{u^k}{k}\Big)(1 + u + \dots + u^{k-1}) \\ &= -u^k \exp\Big(u + \frac{u^2}{2} + \dots + \frac{u^k}{k}\Big) = -u^k \sum_{j=0}^{\infty} c_j u^j. \end{aligned}$$

Observe that $c_j \geq 0$, $j \in \mathbb{Z}_+$. In particular, $\operatorname{ord}_0(1 - E_k) \geq k + 1$. Let

$$f(u) := \frac{1 - E_k(u)}{u^{k+1}} = \sum_{j=0}^{\infty} a_j u^j.$$

Looking at the coefficient (EXERCISE) we get $a_j = \frac{c_j}{k+j+1}$ and hence $a_j \geq 0$, $j \in \mathbb{Z}_+$. Thus for $u \in \mathbb{D}$ we obtain $|f(u)| \leq f(1) = 1$. □

Corollary 7.2.4. *For every domain $D \subsetneq \widehat{\mathbb{C}}$ and for every function $f \in \mathcal{M}(D)$ there exist $g, h \in \mathcal{O}(D)$ such that $h \in \mathcal{O}^*(D \setminus S(f))$ and $f = g/h$. Consequently, $\mathcal{M}(D)$ is the field of fractions of $\mathcal{O}(D)$.*

PROOF. By the Weierstrass theorem there exists $h \in \mathcal{O}(D)$ having zeros at poles of f such that the multiplicity of zero equals to the order of pole and without zeros elsewhere. It suffices to take $g := f \cdot h$. □

Theorem 7.2.5 (Weierstrass-Mittag-Leffler theorem)**.** *For every open set $\Omega \subsetneq \widehat{\mathbb{C}}$, for every $S \subset \Omega$ without accumulation points in Ω. and for every function $k : S \longrightarrow \mathbb{Z}_*$, there exists an $f \in \mathcal{M}(\Omega) \cap \mathcal{O}^*(\Omega \setminus S)$ such that $\operatorname{ord}_a f = k(a)$, $a \in S$.*

PROOF. Let $S_\pm := \{a \in S : \pm h(a) > 0\}$ and let $f_\pm$ be a function from the Weierstrass theorem for $S_\pm$ and $\pm k|_{S_\pm}$: $f_\pm \in \mathcal{O}^*(\Omega \setminus S_\pm)$, f has a zero of multiplicity $\pm h(a)$ at $a \in S_\pm$. Now we may put $f := f_+/f_-$. □

Theorem 7.2.5 may be formulated in the sheaf theory language.

Theorem 7.2.6. *For every open covering of an open set $\Omega \subsetneq \widehat{\mathbb{C}}$ and for every family $f_\alpha \in \mathcal{M}(\Omega_\alpha)$, $\alpha \in A$ such that $f_\alpha/f_\beta \in \mathcal{O}^*(\Omega_\alpha \cap \Omega_\beta)$, $\alpha, \beta \in A$, there exists an $f \in \mathcal{M}(\Omega)$ such that $f/f_\alpha \in \mathcal{O}^*(\Omega_\alpha)$, $\alpha \in A$.*

Theorem 7.2.6 $\Longrightarrow$ Theorem 7.2.5. Let Ω, S and $k : S \longrightarrow \mathbb{Z}_*$ be as in Theorem 7.2.5. Let $r_a > 0$, $a \in S$, be such that $B(a, r_a) \cap B(b, r_b) = \varnothing$. $a \neq b$, $a, b \in S$. If $\infty \in S$, then $B(\infty, r_\infty)$ is a neighborhood of ∞. Put

$$A := \{*\} \cup S, \quad \Omega_* := \Omega \setminus S, \quad \Omega_a := B(a, r_a),\ a \in S, \quad f_* := 1, \quad f_a := (z - a)^{k(a)}, \quad a \in S;$$

if $\infty \in S$, then $f_\infty := z^{-B(\infty)}$. It is clear that all the assumptions of Theorem 7.2.6 are satisfied. Let $f \in \mathcal{M}(\Omega)$ be as in Theorem 7.2.6. Then

$$f = f/f_* \in \mathcal{O}^*(\Omega_*) = \mathcal{O}^*(\Omega \setminus S), \quad f \cdot (z - a)^{-k(a)} = f/f_a \in \mathcal{O}^*(\Omega_a) = \mathcal{O}^*(B(a, r_a)), \quad a \in S.$$

□

Theorem 7.2.5 $\Longrightarrow$ Theorem 7.2.6. Let Ω, $(\Omega_\alpha)_{\alpha \in A}$ and $(f_\alpha)_{\alpha \in A}$ be as in Theorem 7.2.6. Put $S_\alpha := S(f_\alpha) \cup f_\alpha^{-1}(0)$, $S := \bigcup_{\alpha \in A} S_\alpha$. Since $f_\alpha/f_\beta \in \mathcal{O}^*(\Omega_\alpha \cap \Omega_\beta)$ we get $S_\alpha \cap \Omega_\beta \subset S_\beta$, $\alpha, \beta \in A$. In particular, S has no accumulation points in Ω. For $a \in S_\alpha$ let $B(\alpha, a) := \operatorname{ord}_a f_\alpha$. Since $f_\alpha/f_\beta \in \mathcal{O}^*(\Omega_\alpha \cap \Omega_\beta)$, we see that $B(\alpha, a)$ is independent of α. Put $k(a) := B(\alpha, a)$. Let

$f \in \mathcal{M}(\Omega)$ be as in Theorem 7.2.5. Then f has neither zeros nor poles outside S and for any $\alpha \in A$ and $a \in S_\alpha$ the function

$$\frac{f}{f_\alpha} = \frac{f \cdot (z-a)^{-k(a)}}{f_\alpha \cdot (z-a)^{-k(a)}}$$

extends holomorphically to a. The extension has no zeros in a neighborhood of a; if $\infty \in S_\alpha$, then

$$\frac{f}{f_\alpha} = \frac{f \cdot z^{B(\infty)}}{f_\alpha \cdot z^{B(\infty)}}. \qquad \square$$

Remark 7.2.7. [Remark 7.2.7⟶ Exer . . . .]

(a) For $(a_n)_{n=1}^\infty \mathbb{C}_*$, $a_n \longrightarrow \infty$, and $(\alpha_n)_{n=1}^\infty \subset \mathbb{N}$ let $(z_k)_{k=1}^\infty$ be generated by $(a_n)_{n=1}^\infty$ in such a way that a_n is repeated α_n–times. For $\alpha \in \mathbb{Z}_+$, define

$$f(z) := z^\alpha \prod_{k=1}^\infty E_k\Big(\frac{z}{z_k}\Big), \quad z \in \mathbb{C}.$$

Then
- $f \in \mathcal{O}(\mathbb{C})$,
- f has a zero of multiplicity α at $z = 0$,
- f has a zero of multiplicity α_n at $z = a_n$,
- there are no other zeros.

Indeed, the only problem is to prove that the product is locally uniformly convergent. Let $K \subset\subset \mathbb{C}$. Then $|z/z_k| \le 1/2$, $z \in K$, $k \ge k_0 \gg 1$. Hence

$$\Big|E_k\Big(\frac{z}{z_k}\Big) - 1\Big| \le \Big(\frac{1}{2}\Big)^{k+1}, \quad z \in K,\ k \ge k_0.$$

(b) Every entire function $f \in \mathcal{O}(\mathbb{C})$ having infinitely many zeros may be written in the form

$$f(z) = e^{g(z)} z^\alpha \prod_{k=1}^\infty E_k\Big(\frac{z}{z_k}\Big), \quad z \in \mathbb{C},$$

where $g \in \mathcal{O}(\mathbb{C})$.

(c) One can take in (a)

$$f(z) := z^\alpha \prod_{k=1}^\infty E_{n_k}\Big(\frac{z}{z_k}\Big), \quad z \in \mathbb{C},$$

where the sequence $(n_k)_{k=1}^\infty$ is such that the series $\sum_{k=1}^\infty |z/z_k|^{n_k+1}$ is locally uniformly convergent.

(d) For example $z_k := -k$, $n_k := 1$, $\alpha := 1$, $f(z) = z\prod_{k=1}^\infty \Big(1 + \frac{z}{k}\Big) \exp\Big(-\frac{z}{k}\Big)$, $z \in \mathbb{C}$.

(e) $\sin \pi z = \pi z \prod_{k=1}^\infty \Big(1 - \frac{z^2}{k^2}\Big)$, $\quad z \in \mathbb{C}$.

Indeed, we know that

$$\sin \pi z = e^{g(z)} z \prod_{k=1}^\infty \Big(1 - \frac{z^2}{k^2}\Big), \quad z \in \mathbb{C}, \tag{7.2.1}$$

for a $g \in \mathcal{O}(\mathbb{C})$. We must prove that $e^g \equiv \pi$. We have

$$\pi \operatorname{ctg} \pi z = \frac{(\sin \pi z)'}{\sin \pi z} = g'(z) + \frac{1}{z} + \sum_{k=1}^{\infty} \Big(\frac{1}{z-k} + \frac{1}{z+k} \Big). \tag{7.2.2}$$

In particular, g' is an odd function.

$$\frac{\pi^2}{\sin^2 \pi z} = g''(z) - \frac{1}{z^2} - \sum_{k=1}^{\infty} \Big(\frac{1}{(z-k)^2} + \frac{1}{(z+k)^2} \Big) = g''(z) - \sum_{k=-\infty}^{\infty} \frac{1}{(z-k)^2}.$$

In particular, $g''(z+1) = g''(z)$, $z \in \mathbb{C}$. Let $A := \{x+iy : 0 \le x \le 1\}$. For $z = x+iy \in A$, $|y| \ge 1$, we get:

$$\Big| \sum_{k=-\infty}^{\infty} \frac{1}{(z-k)^2} \Big| \le \sum_{k=-\infty}^{\infty} \frac{1}{(x-k)^2 + y^2} \le 2 \sum_{k=0}^{\infty} \frac{1}{k^2 + y^2},$$

$$\Big| \frac{\pi^2}{\sin^2 \pi z} \Big| = \frac{4\pi^2}{|e^{\pi i z} - e^{-\pi i z}|^2} = \frac{4\pi^2}{|(e^{-\pi y} - e^{\pi y}) \cos \pi x + i(e^{-\pi y} + e^{\pi y}) \sin \pi x|^2}$$
$$= \frac{4\pi^2}{e^{2\pi y} + e^{-2\pi y} - 2\cos 2\pi x} \le \frac{4\pi^2}{e^{2\pi |y|} - 2}.$$

This means that $\lim_{A \ni z \to \infty} g''(z) = 0$. Thus g'' is bounded on A. Since g'' is periodic, we conclude that g'' is bounded on $\mathbb{C}$. Consequently, $g' \equiv \text{const}$. Since g' is odd, we must have $g' \equiv 0$, so $g \equiv \text{const} = c$. By (7.2.1) we obtain $\pi = \lim_{z \to 0} \frac{\sin \pi z}{z} = e^c$.

(f) We have $\pi \operatorname{ctg} \pi z = \frac{1}{z} + \sum_{k=1}^{\infty} \Big(\frac{1}{z-k} + \frac{1}{z+k} \Big)$, $z \in \mathbb{C}$.

(g)

$$1/\Gamma(z) = e^{\gamma z} z \prod_{k=1}^{\infty} \Big(1 + \frac{z}{k}\Big) \exp\Big(-\frac{z}{k}\Big), \quad z \in \mathbb{C}, \text{ where} \tag{7.2.3}$$

$$\gamma := \lim_{n \to +\infty} \gamma_n = \lim_{n \to +\infty} \Big(\sum_{k=1}^{n} \frac{1}{k} - \log n \Big) = 0,577\ldots$$

is the *Euler constant*. In particular, $\frac{1}{\Gamma(z)\Gamma(-z)} = -\frac{z}{\pi} \sin \pi z$, $z \in \mathbb{C}$.

(7.2.3) follows for the formula

$$\Gamma(z) = \lim_{n \to +\infty} \frac{n! e^{z \log n}}{z(z+1)\cdots(z+n)}, \quad z \in \mathbb{C} \setminus \mathbb{Z}_-. \tag{7.2.4}$$

Indeed, (7.2.4) implies that

$$1/\Gamma(z) = z \lim_{n \to +\infty} e^{-z \log n} (1 + z/1) \cdots (1 + z/n) = z \lim_{n \to +\infty} e^{\gamma_n z} \prod_{k=1}^{n} \Big(1 + \frac{z}{k}\Big) \exp\Big(-\frac{z}{k}\Big). \tag{7.2.5}$$

Let $\widehat{\varGamma}$ be given by the right side of (7.2.4). Observe that $\widehat{\varGamma}$ is well defined and $\widehat{\varGamma} \in \mathcal{O}(\mathbb{C} \setminus \mathbb{Z}_-)$. It suffices to show that $\widehat{\varGamma} = \varGamma$ on $(0,1]$, i.e.

$$\varGamma(x)\frac{x(x+1)\cdots(x+n)}{n!n^x} \longrightarrow 1, \quad x \in (0,1].$$

It is equivalent to proving that $\frac{\varGamma(x+n+1)}{n!n^x} \longrightarrow 1$, $x \in (0,1]$. For $x \in (0,1]$ we get

$$\begin{aligned}\varGamma(x+n+1) &= \int_0^\infty t^{x+n}e^{-t}dt \le n^x\int_0^n t^n e^{-t}dt + n^{x-1}\int_n^\infty t^{n+1}e^{-t}dt\\ &= n^x\int_0^n t^n e^{-t}dt + n^{x-1}\Big(-t^{n+1}e^{-t}|_n^\infty + (n+1)\int_n^\infty t^n e^{-t}dt\Big)\\ &= n^x\int_0^\infty t^n e^{-t}dt + n^{x-1}\int_n^\infty t^n e^{-t}dt + n^{x+n}e^{-n}\\ &= n^x n! + n^{x-1}\int_n^\infty t^n e^{-t}dt + n^{x+n}e^{-n} \le n^x n! + n^{x-1}n! + n^{x+n}e^{-n}.\end{aligned}$$

Analogously,

$$\begin{aligned}\varGamma(x+n+1) &\ge n^{x-1}\int_0^n t^{n+1}e^{-t}dt + n^x\int_n^\infty t^n e^{-t}dt\\ &= n^{x-1}\Big(-t^{n+1}e^{-t}|_0^n + (n+1)\int_0^n t^n e^{-t}dt\Big) + n^x\int_n^\infty t^n e^{-t}dt\\ &= n^x n! + n^{x-1}\int_0^n t^n e^{-t}dt - n^{x+n}e^{-n} \ge n^x n! - n^{x+n}e^{-n}.\end{aligned}$$

Consequently, $1 - \frac{n^n e^{-n}}{n!} \le \frac{\varGamma(x+n+1)}{n!n^x} \le 1 + \frac{1}{n} + \frac{n^n e^{-n}}{n!}$. It remains to use the Stirling (2) formula $n! \approx \frac{n^{n+1/2}\sqrt{2\pi}}{e^n}$.

Theorem 7.2.8 (Weierstrass-Mittag-Leffler theorem)**.** *For every open set $\varOmega \subsetneq \widehat{\mathbb{C}}$, for every set $S \subset \varOmega$ without accumulation points in $\varOmega$, and for every family of polynomials $(P_a)_{a\in S} \subset \mathcal{P}(\mathbb{C})$, there exists an $f \in \mathcal{O}(\varOmega)$ such that for every $a \in S$ the Taylor series f begins from $P_a(z-a)$; if $\infty \in S$, then we mean that the Taylor series of $z \longmapsto f(1/z)$ at 0 starts from $P_\infty(z)$.*

Observe that $\operatorname{ord}_a(f - P_a) \ge \deg P_a + 1$, $a \in S$.

Proof. By the Weierstrass theorem there exists a $g \in \mathcal{O}^*(\varOmega \setminus S)$ such that $\operatorname{ord}_a g = \deg P_a + 1$, $a \in S$. By the Mittag-Leffler theorem there exists an $h \in \mathcal{M}(\varOmega) \cap \mathcal{O}(\varOmega \setminus S)$ such that $h_a := h - \frac{P_a(z-a)}{g}$ is holomorphic in a neighborhood of a for every $a \in S$; if $a = \infty$, then $h_\infty := h - \frac{P_\infty(1/z)}{g}$ is holomorphic in a neighborhood of ∞. Define $f := h \cdot g$. In a neighborhood of each point $a \in S$ we get

$$f - P_a(z-a) = h\cdot g - P_a(z-a) = g\Big(h - \frac{P_a(z-a)}{g}\Big) = g\cdot h_a,$$

(2) James Stirling (1692–1770).

which implies that $\operatorname{ord}_a(f - P_a(z-a)) \geq \operatorname{ord}_a g = \deg P_a + 1$. This means that the Taylor series of f at a starts with $P_a(z-a)$. $\square$

7.2.1. ζ Riemann function. Let

$$\zeta(z) := \sum_{n=1}^{\infty} \frac{1}{n^z} = \sum_{n=1}^{\infty} \frac{1}{e^{z \log n}}, \quad z \in \mathbb{H}_1 = \{z \in \mathbb{R} : \operatorname{Re} z > 1\}.$$

Since $|n^z| = |e^{z \log n}| = e^{(\operatorname{Re} z)\log n} = n^{\operatorname{Re} z}$, the series is locally uniformly convergent in $\mathbb{H}_1$ and defines a holomorphic function called ζ *Riemann function*.

Theorem 7.2.9 (Euler theorem)**.** *Let $(p_k)_{k=1}^{\infty} \subset \mathbb{N}$ be a sequence of all prime numbers. Then*

$$\zeta(z) = \prod_{k=1}^{\infty} \frac{1}{1 - p_k^{-z}}, \quad z \in \mathbb{H}_1.$$

Proof. Fix a $z \in \mathbb{H}_1$. Since $|p_k^{-z}| = p_k^{-\operatorname{Re} z} < 1$, we get

$$\prod_{k=1}^{n} \frac{1}{1 - p_k^{-z}} = \prod_{k=1}^{n} \sum_{m=0}^{\infty} (p_k^{-z})^m = \sum_{m_1,\dots,m_n=0}^{\infty} (p_1^{m_1} \cdots p_n^{m_n})^{-z}.$$

It remains to use the uniqueness of the decomposition into prime numbers. $\square$

Theorem* 7.2.10. *The function ζ extends to a meromorphic function on $\mathbb{C} \setminus \{1\}$ so that:*

- *ζ has a single pole with $\operatorname{res}_1 \zeta = 1$ at 1,*
- *ζ satisfies the* Riemann equation $\zeta(z) = 2e^{(z-1)\log(2\pi)}\Gamma(1-z)\zeta(1-z)\sin(\frac{\pi}{2}z)$,
- $\zeta(-2k) = 0$, $k \in \mathbb{N}$*; they are called* trivial zeros*;*

Indeed, by the Riemann equation ζ has no zeros in $\mathbb{H}_1$. If z_0 is a zero of ζ such that $\operatorname{Re} z_0 < 0$ and $\sin(\frac{\pi}{2}z_0) \neq 0$ (i.e. $z_0 \notin -2\mathbb{N}$), then the Riemann equation gives $\Gamma(1-z_0) = 0$ — a contradiction.

- *ζ has no non-trivial zeros outside $\{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$.*
- *(Riemann Conjecture)* All non-trivial zeros of the Riemann function are on the line $\operatorname{Re} z = \frac{1}{2}$.

CHAPTER 8

Subharmonic functions

8.1. Harmonic functions

Definition 8.1.1. Let $\Omega \in \mathrm{top}(\mathbb{R}^2)$ and let $h \in \mathcal{C}^2(\Omega, \mathbb{R})$. We say that h is *harmonic* on Ω ($h \in \mathcal{H}(\Omega)$), if

$$\Delta h = \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} \equiv 0 \quad \text{ on } \Omega.$$

Remark 8.1.2. (a) $\mathcal{H}(\Omega)$ is a vector space.

(b) $\Delta = 4\frac{\partial^2}{\partial z \partial \bar{z}}$.

(c) Harmonic functions may be defined in any open set $\Omega \subset \mathbb{R}^n$: we say that a function $h \in \mathcal{C}^2(\Omega, \mathbb{R})$ is *harmonic* on Ω ($h \in \mathcal{H}(\Omega)$), if

$$\Delta h = \sum_{j=1}^{n} \frac{\partial^2 h}{\partial x_j^2} \equiv 0 \quad \text{ on } \Omega.$$

(d) For $n = 1$, if $\Omega \subset \mathbb{R}$ is a segment, then a function $h \in \mathcal{C}^2(\Omega, \mathbb{R})$ is harmonic if and only if h is linear.

Theorem 8.1.3. *Let $D \subset \mathbb{C}$ be a starlike domain and let $h : D \longrightarrow \mathbb{R}$. Then $h \in \mathcal{H}(D)$ if and only if there exists an $f \in \mathcal{O}(D)$ with $h = \operatorname{Re} f$.*

Proof. Let $f = u + iv \in \mathcal{O}(D)$. Then

$$\Delta u = \frac{\partial}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial}{\partial y}\frac{\partial u}{\partial y} = \frac{\partial}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial}{\partial y}\frac{\partial v}{\partial x} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = 0.$$

Now let $h \in \mathcal{H}(D)$. Then the form $Pdx + Qdy := -h'_y dx + h'_x dy$ ic closed because $P'_y - Q'_x = -h''_{xx} - h''_{yy} = -\Delta h = 0$. Thus there exists a $v \in \mathcal{C}^1(D, \mathbb{R})$ such that $v'_x = P = -h'_y$, $v'_y = Q = h'_x$, which means that $h + iv \in \mathcal{O}(D)$. □

Definition 8.1.4. Let $D \subset \mathbb{C}$ be a domain. If $h \in \mathcal{H}(D)$ and $h + iv \in \mathcal{O}(D)$, then we say that v is a *conjugate harmonic function* to h.

Corollary 8.1.5. *Let $\Omega \subset \mathbb{C}$ be open.*

((a) $\mathcal{H}(\Omega) \subset \mathcal{C}^\omega(\Omega)$.

((b) If $f \in \mathcal{O}(\Omega)$ and $0 \notin f(\Omega)$, then $\log|f| \in \mathcal{H}(\Omega)$.

((c) Let Ω, $\Omega' \subset \mathbb{C}$ be open, $h \in \mathcal{H}(\Omega')$, $f \in \mathcal{O}(\Omega, \Omega')$. Then $h \circ f \in \mathcal{H}(\Omega)$.

Remark 8.1.6. The conjugate harmonic function is unique up to a constant.

Theorem 8.1.7 (Identity principle). *Let $D \subset \mathbb{C}$ be a domain and let $h \in \mathcal{H}(D)$ be such that $h = 0$ on a non-empty open subset $U \subset D$. Then $h \equiv 0$ on D. Consequently, if $h_1, h_2 \in \mathcal{H}(D)$ are equal on a non-empty open set, then $h_1 \equiv h_2$.*

Proof. Let $D_0 := \{a \in D : h = 0 \text{ in an open neighborhood } U \subset D \text{ of } a\}$. Obviously, $D_0 \neq \varnothing$ and D_0 is open. Let $b \in D \cap D_0'$ and let $U \subset D$ be a starlike domain with $b \in U$. Let $f \in \mathcal{O}(U)$ be such that $\operatorname{Re} f = h$ (Theorem 8.1.3). Then $\operatorname{Re} f = h = 0$ on $U \cap D_0 \neq \varnothing$. Hence $h = \operatorname{Re} f = 0$ na U. $\square$

Theorem 8.1.8 (Maximum principle). *Let $D \subset \mathbb{C}$ be a domain and let $h \in \mathcal{H}(D)$, $h \not\equiv$ const. Then h does not have local maxima. Moreover, if D is bounded, then*

$$h(z) < \sup\{\limsup_{D \ni z \to \zeta} h(z) : \zeta \in \partial D\}, \quad z \in D.$$

If we substitute h by $-h$, we can get the *minimum principle*.

Proof. Suppose that h has a local maximum at $a \in D$. Let $U \subset D$ be a starlike domain with $a \in U$ such that $h(z) \leq h(a)$, $z \in U$. Let $h = \operatorname{Re} f$, where $f \in \mathcal{O}(U)$. Then $|e^f| = e^h$ has a local maximum at a. Consequently, $e^h =$ const and therefore $h =$ const in U. Using the identity principle we conclude we get a contradiction.

If D is bounded, then we argue as in Theorem 2.1.8. $\square$

Remark 8.1.9. Let $u : C(a,r) \longrightarrow [-\infty, +\infty)$ be a measurable function (i.e. the function $[0, 2\pi) \ni \vartheta \longmapsto u(a + re^{i\vartheta})$ is $\mathcal{L}_1$ measurable). Then $\frac{1}{2\pi}\int\limits_0^{2\pi} u(a+re^{i\vartheta})d\vartheta = \frac{1}{2\pi i}\int_{\mathbb{T}} u(a+r\zeta)\frac{d\mathcal{L}^{\mathbb{T}}}{\zeta}$.

Definition 8.1.10. Let $u : C(a,r) \longrightarrow [-\infty, +\infty)$ be an upper bounded measurable function, e.g. u is upper semicontinuous. Put

$$\mathbb{P}(u; a, r; z) := \frac{1}{2\pi}\int\limits_0^{2\pi} \frac{r^2 - |z-a|^2}{|re^{i\vartheta} - (z-a)|^2}\, u(a + re^{i\vartheta})d\vartheta, \quad z \in B(a,r),$$

$$\mathbb{J}(u; a, r) := \mathbb{P}(u; a, r; a) = \frac{1}{2\pi}\int\limits_0^{2\pi} u(a + re^{i\vartheta})d\vartheta;$$

$\mathbb{J}(u; a, r)$ is the *integral mean value* of u on $C(a,r)$. The function $P(z,\zeta) := \frac{|\zeta|^2 - |z|^2}{|\zeta - z|^2}$ is called the *Poisson kernel* [1]. Thus $\mathbb{P}(u; a, r; z) = \frac{1}{2\pi}\int\limits_0^{2\pi} P(z-a, re^{i\vartheta})u(a+re^{i\vartheta})d\vartheta$.

Remark 8.1.11. Observe that $\frac{|\zeta|^2 - |z|^2}{|\zeta - z|^2} = \operatorname{Re}\frac{\zeta + z}{\zeta - z}$, $\quad z \in \mathbb{C} \setminus \{\zeta\}$. Thus $P(\cdot, \zeta) \in \mathcal{H}(\mathbb{C} \setminus \{\zeta\})$ and therefore $\mathbb{P}(u; a, r; \cdot) \in \mathcal{H}(B(a,r))$.

Theorem 8.1.12 (Poisson formula). *Let $h \in \mathcal{C}(\overline{B}(a,r)) \cap \mathcal{H}(B(a,r))$. Then $h(z) = \mathbb{P}(h; a, r; z)$, $z \in B(a,r)$. In particular,*

- $h(a) = \boldsymbol{J}(h; a, r)$ *(*mean value theorem*),*

---

[1] Siméon Poisson (1781–1840).

- $1 = \frac{1}{2\pi}\int\limits_0^{2\pi} P(z-a, re^{i\vartheta})d\vartheta,\ z \in B(a,r)$.

Proof. We may assume that $a = 0$. Let $f \in \mathcal{O}(B(r))$, $h = \operatorname{Re} f$. Then for $|z| < s < r$ we get $s^2/\overline{z} \notin \overline{B}(s)$, and therefore, using the Cauchy integral formula, we have $0 = \frac{1}{2\pi i}\int\limits_{C(s)} \frac{f(\zeta)}{\zeta - \frac{s^2}{\overline{z}}}d\zeta$. Now

$$\begin{aligned}
h(z) &= \operatorname{Re} f(z) = \operatorname{Re}\Big(\frac{1}{2\pi i}\int\limits_{C(s)} \frac{f(\zeta)}{\zeta - z}d\zeta - \frac{1}{2\pi i}\int\limits_{C(s)} \frac{f(\zeta)}{\zeta - \frac{s^2}{\overline{z}}}d\zeta\Big) \\
&= \operatorname{Re}\Big(\frac{1}{2\pi i}\int\limits_{C(s)} \frac{-\frac{s^2}{\overline{z}} + z}{(\zeta - z)(\zeta - \frac{s^2}{\overline{z}})}f(\zeta)d\zeta\Big) = \operatorname{Re}\Big(\frac{1}{2\pi i}\int\limits_{C(s)} \frac{-s^2 + |z|^2}{(\zeta - z)(\zeta\overline{z} - s^2)}f(\zeta)d\zeta\Big) \\
&= \operatorname{Re}\Big(\frac{1}{2\pi i}\int\limits_{C(s)} \frac{s^2 - |z|^2}{\zeta|\zeta - z|^2}f(\zeta)d\zeta\Big) = \operatorname{Re}\Big(\frac{1}{2\pi}\int\limits_0^{2\pi} \frac{s^2 - |z|^2}{|se^{i\vartheta} - z|^2}f(se^{i\vartheta})d\vartheta\Big) \\
&= \frac{1}{2\pi}\int\limits_0^{2\pi} \frac{s^2 - |z|^2}{|se^{i\vartheta} - z|^2}h(se^{i\vartheta})d\vartheta.
\end{aligned}$$

It remains to allow $s \nearrow r$. $\square$

Corollary 8.1.13 (Schwarz formula). *For $h \in \mathcal{H}(B(a,r)) \cap \mathcal{C}(\overline{B}(a,r))$ let*

$$f(z) := \frac{1}{2\pi}\int\limits_0^{2\pi} \frac{re^{i\vartheta} + (z-a)}{re^{i\vartheta} - (z-a)}h(a + re^{i\vartheta})d\vartheta, \quad z \in B(a,r).$$

Then $f \in \mathcal{O}(B(a,r))$, $\operatorname{Re} f = h$.

Corollary 8.1.14 (Poisson-Jensen ([2]) formula). [Corollary 8.1.14⟶ Exer] *Let $f \in \mathcal{M}(\Omega)$, where $\Omega \supset \overline{\mathbb{D}}$. Assume that f has neither zeros nor poles on $\mathbb{T}$ and let $a_1, \dots, a_p$ denote the zeros of f in $\mathbb{D}$, $b_1, \dots, b_q$ –the poles of f in $\mathbb{D}$ counted with multiplicities. Then*

$$\log\Big|f(z)\frac{\prod_{j=1}^q h_{b_j}(z)}{\prod_{j=1}^p h_{a_j}(z)}\Big| = \mathbb{P}(\log|f|; 0, 1; z) = \frac{1}{2\pi}\int\limits_0^{2\pi} P(z, e^{i\vartheta})\log|f(e^{i\vartheta})|d\vartheta, \quad z \in \mathbb{D},$$

where $\prod\limits_{\varnothing} := 1$. In particular:

- $\log\Big|f(0)\frac{b_1\cdots b_q}{a_1\cdots a_p}\Big| = \mathbb{J}(\log|f|; 0, 1) = \frac{1}{2\pi}\int\limits_0^{2\pi} \log|f(e^{i\vartheta})|d\vartheta$.

- *If $q = 0$ then $\log|f(z)| \le \mathbb{P}(\log|f|; 0, 1; z) = \frac{1}{2\pi}\int\limits_0^{2\pi} P(z, e^{i\vartheta})\log|f(e^{i\vartheta})|d\vartheta$, $z \in \mathbb{D}$; $\log|f(0)| \le \mathbb{J}(\log|f|; 0, 1)$.*

([2]) Johan Jensen (1859–1925).

Exercise 8.1.15. [Exercise 8.1.15⟶ Exer] What is the Poisson-Jensen formula for $B(a,r)$?

Definition 8.1.16. For a bounded domain $D \subset \mathbb{C}$ and $b \in \mathcal{C}(\partial D, \mathbb{R})$, the *Dirichlet* ([3]) *problem* is to find an $h \in \mathcal{H}(D) \cap \mathcal{C}(\overline{D})$ such that $h = b$ on ∂D.

Observe that the Dirichlet problem has at most one solution.

Exercise 8.1.17. Show that the Dirichlet problem for $\mathbb{D}_*$ and a b may be without any solution.

Theorem 8.1.18 (Dirichlet problem for a disc)**.** *For $b \in \mathcal{C}(C(a,r), \mathbb{R})$ define*

$$h(z) := \begin{cases} b(z), & \text{if } z \in C(a,r) \\ \mathbb{P}(b; a, r; z), & \text{if } z \in B(a,r) \end{cases}.$$

Then $h \in \mathcal{C}(\overline{B}(a,r)) \cap \mathcal{H}(B(a,r))$.

Proof. We may assume that $a = 0$. We already know that $h \in \mathcal{H}(B(r))$. It remains to prove that for each $\zeta \in C(r)$ we have $\lim\limits_{z \to \zeta_0} \mathbb{P}(b; 0, r; z) = b(\zeta_0)$.

Let $C > 0$ be such that $|b(z)| \le C$, $z \in C(r)$. Fix a $\zeta_0 = re^{i\vartheta_0} \in C(r)$. First, assume that $0 < \vartheta_0 < 2\pi$. For $\varepsilon > 0$ let $0 < \delta < \min\{\vartheta_0, 2\pi - \vartheta_0\}$ be such that $|b(re^{i\vartheta}) - b(re^{i\vartheta_0})| \le \varepsilon$ for all $|\vartheta - \vartheta_0| \le \delta$. Then:

$$|\mathbb{P}(b; 0, r; z) - b(\zeta_0)| = \Big| \frac{1}{2\pi} \int_0^{2\pi} P(z, re^{i\vartheta}) b(re^{i\vartheta}) d\vartheta - \frac{1}{2\pi} \int_0^{2\pi} P(z, re^{i\vartheta}) b(\zeta_0) d\vartheta \Big|$$

$$\le \frac{1}{2\pi} \Big(\int_{[0,2\pi] \setminus [\vartheta_0 - \delta, \vartheta_0 + \delta]} P(z, re^{i\vartheta}) |b(re^{i\vartheta}) - b(re^{i\vartheta_0})| d\vartheta + \int_{[\vartheta_0 - \delta, \vartheta_0 + \delta]} P(z, re^{i\vartheta}) |b(re^{i\vartheta}) - b(re^{i\vartheta_0})| d\vartheta \Big)$$

$$\le \frac{1}{2\pi} \Big(2C \int_{[0,2\pi] \setminus [\vartheta_0 - \delta, \vartheta_0 + \delta]} P(z, re^{i\vartheta}) d\vartheta + \varepsilon \int_{[\vartheta_0 - \delta, \vartheta_0 + \delta]} P(z, re^{i\vartheta}) d\vartheta \Big)$$

$$\le \frac{C}{\pi} \int_{[0,2\pi] \setminus [\vartheta_0 - \delta, \vartheta_0 + \delta]} \frac{r^2 - |z|^2}{|re^{i\vartheta} - z|^2} d\vartheta + \varepsilon \underset{z \to \zeta_0}{\longrightarrow} \varepsilon.$$

□

1.11.8, The case $\zeta_0 = r$ is left as an Exercise.

Exercise 8.1.19. [Exercise 8.1.19⟶ Exer .] Prove that if $b : C(a,r) \longrightarrow \mathbb{R}$ is a bounded measurable function that is continuous at a point $\zeta_0 \in C(a,r)$, then $\lim\limits_{z \to \zeta_0} \mathbb{P}(b; a, r; z) = b(\zeta_0)$.

Corollary 8.1.20. *The Dirichlet problem has the solution in any bounded Jordan domain.*

Proof. Let $f : D \longrightarrow \mathbb{D}$ be biholomorphic that is homeomorphic $\overline{D} \longrightarrow \overline{\mathbb{D}}$ (Osgood-Carathéodory theorem). Let h be the solution of the Dirichlet problem for $\mathbb{D}$ and the function $b \circ f^{-1}$. Then $h \circ f$ is the solution of the initial Dirichlet problem. □

([3]) Peter Dirichlet (1805–1859).

Theorem 8.1.21 (1-st Harnack's theorem). *Let $\Omega \subset \mathbb{C}$ be open and let $(h_\nu)_{\nu=1}^\infty \subset \mathcal{H}(\Omega)$. If $h_\nu \longrightarrow h$ locally uniformly in Ω, then $h \in \mathcal{H}(\Omega)$.*

Proof. Fix $a \in \Omega$ and $r > 0$ such that $\overline{B}(a,r) \subset \Omega$. Then, by Theorem **??**, we get

$$h_\nu(z) = \mathbb{P}(h_\nu; a, r; z), \quad z \in B(a,r),\ \nu \in \mathbb{N}.$$

Since $h_\nu \longrightarrow h$ uniformly on $C(a,r)$, we get $\mathbb{P}(h_\nu; a, r; z) \longrightarrow \mathbb{P}(h; a, r; z)$. On the other hand $h_\nu(z) \longrightarrow h(z)$. Thus

$$h(z) = \mathbb{P}(h; a, r; z), \quad z \in B(a,r).$$

Now, by Theorem **??**, $h \in \mathcal{H}(B(a,r))$. □

Theorem 8.1.22 (2-nd Harnack's theorem). *Let D be a domain in $\mathbb{C}$, $(h_\nu)_{\nu=1}^\infty \subset \mathcal{H}(D)$, and $h_\nu \le h_{\nu+1}$, $\nu \ge 1$. If there exists $a \in D$ such that $\lim_{\nu\to+\infty} h_\nu(a)$ exists and is finite, then $(h_\nu)_{\nu=1}^\infty$ converges locally uniformly in D.*

Proof. Let

$$D_0 = \{z \in D : (h_\nu)_{\nu=1}^\infty \text{ is convergent uniformly in a neighborhood of } z\}.$$

If we show that D_0 is non-empty open and closed in D, then $D_0 = D$, which will end the proof.

The set D_0 is open by definition. To prove that $D_0 \neq \varnothing$ we show that $a \in D_0$. Choose $r > 0$ such that $\overline{B}(a,r) \subset D$. Note that

$$\frac{r^2 - |z-a|^2}{|re^{i\vartheta} - (z-a)|^2} \le \frac{r^2 - |z-a|^2}{(r - |z-a|)^2} = \frac{r + |z-a|}{r - |z-a|}, \quad z \in B(a,r). \tag{8.1.6}$$

Moreover, for $z \in B(a,r)$ and $\nu, \mu \in \mathbb{N}$, we have

$$0 \le h_{\nu+\mu}(z) - h_\nu(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{r^2 - |z-a|^2}{|re^{i\vartheta} - (z-a)|^2}(h_{\nu+\mu}(a + re^{i\vartheta}) - h_\nu(a + re^{i\vartheta}))\, d\vartheta$$
$$\le \frac{1}{2\pi}\int_0^{2\pi} \frac{r + |z-a|}{r - |z-a|}(h_{\nu+\mu}(a + re^{i\vartheta}) - h_\nu(a + re^{i\vartheta}))\, d\vartheta = \frac{r + |z-a|}{r - |z-a|}(h_{\nu+\mu}(a) - h_\nu(a)).$$

For $|z - a| < r/2$ this last expression is not greater than $3(h_{\nu+\mu}(a) - h_\nu(a))$. Therefore the sequence $(h_\nu)_{\nu=1}^\infty$ satisfies the uniform Cauchy condition in $B(a, r/2)$, and hence converges uniformly there. Thus $a \in D_0$.

Suppose now that $z_0 \in D$ is an accumulation point of the set D_0. Choose $r > 0$ such that $\overline{B}(z_0, r) \subset D$. There exists $b \in D_0 \cap K(z_0, r/3)$. Hence $\overline{B}(b, 2r/3) \subset D$. Since $b \in D_0$, the sequence $(h_\nu(b))_{\nu=1}^\infty$ is convergent. Similarly as above we prove that $(h_\nu)_{\nu=1}^\infty$ is convergent uniformly in $K(b, r/3)$. Hence $(h_\nu)_{\nu=1}^\infty$ is convergent uniformly in a neighborhood of z_0, and so $z_0 \in D_0$, which proves that D_0 is relatively closed. □

Theorem 8.1.23. *Any annulus*

$$A := \{z \in \mathbb{C} : r^- < |z| < r^+\}, \quad 0 < r^- < r^+ < +\infty,$$

is regular with respect to the Dirichlet problem.

Theorem 8.1.24 ([Schwartz]). *Let $u \in L^1(\Omega, \mathrm{loc})$ (4) be such that $\Delta u = 0$ in the sense of distribution, i.e.*

$$\int_\Omega u \cdot (\Delta\varphi)\, d\mathcal{L}^2 = 0, \quad \varphi \in \mathcal{C}_0^\infty(\Omega).$$

Then there exists $h \in \mathcal{H}(\Omega)$ such that $u = h$ $\mathcal{L}^2$-a.e. on Ω.

8.2. Subharmonic functions

Definition 8.2.1. Let $\Omega \subset \mathbb{C}$ be open. A function $u : \Omega \longrightarrow [-\infty, +\infty)$ is called *subharmonic* in Ω (we write $u \in \mathcal{SH}(\Omega)$) if:
- u is upper semicontinuous in Ω ($u \in \mathcal{C}^\uparrow(\Omega)$),
- for every domain $D \subset\subset \Omega$ and for every function $h \in \mathcal{C}(\overline{D}) \cap \mathcal{H}(D)$, if $u \le h$ on ∂D, then $u \le h$ in D.

In particular, the function $u \equiv -\infty$ is subharmonic.

The following properties are immediate consequences of the above definition and of the maximum principle for harmonic functions:
$\mathcal{H}(\Omega) \subset \mathcal{SH}(\Omega)$,
$\mathcal{SH}(\Omega) + \mathcal{H}(\Omega) = \mathcal{SH}(\Omega)$,
$\mathbb{R}_{>0} \cdot \mathcal{SH}(\Omega) = \mathcal{SH}(\Omega)$.

Theorem 8.2.2 (Mean value property). *If $u \in \mathcal{SH}(\Omega)$, then*

$$u(a) \le \mathbb{J}(u; a, r) = \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\vartheta})\, d\vartheta, \quad a \in \Omega,\ 0 < r < d_\Omega(a).$$

Proof. Fix an $a \in \Omega$ and $0 < r < d_\Omega(a)$. Let $b_\nu : C(a, r) \longrightarrow \mathbb{R}$, $\nu \in \mathbb{N}$, be a sequence of continuous functions such that $b_\nu \searrow u$ pointwise on $C(a, r)$. Let h_ν be the solution of the Dirichlet problem for $B(a, r)$ with $h_\nu = b_\nu$ on $C(a, r)$. Then $u \le h_\nu$ on $C(a, r)$ and hence on $B(a, r)$. Consequently, we get

$$u(a) \le h_\nu(a) = \mathbb{J}(h_\nu; a, r) = \mathbb{J}(b_\nu; a, r), \quad \nu \ge 1.$$

Since $b_\nu \searrow u$ on $C(a, r)$, the monotone convergence theorem implies that

$$\mathbb{J}(b_\nu; a, r) \longrightarrow \mathbb{J}(u; a, r). \qquad \square$$

Lemma 8.2.3. *Let $D \subset \mathbb{C}$ be a domain and let $v \in \mathcal{C}^\uparrow(D, [-\infty, +\infty))$, $v \not\equiv \mathrm{const}$. Assume that for every $a \in D$ there exists a number $0 < R(a) \le d_D(a)$ such that*

$$v(a) \le \mathbb{J}(v; a, r), \quad 0 < r < R(a).$$

Then v does not attain its global maximum in D.

Proof. Suppose that $v(z) \le v(z_0)$, $z \in D$ (for some $z_0 \in D$). Let $D_0 := v^{-1}(v(z_0))$. Then $D_0 \ne \varnothing$. Note that for every accumulation point $a \in D$ of D_0 we have

$$v(z_0) = \limsup_{D_0 \ni z \to a} v(z) \le \limsup_{D \ni z \to a} v(z) = v(a) \le v(z_0).$$

(4) $L^1(\Omega, \mathrm{loc}) := \{u : \forall_{K \subset\subset \Omega} : u|_K \in L^1(K, \mathcal{L}^2)\}$.

Hence $a \in D_0$, which means that D_0 is relatively closed in D. On the other hand, if $a \in D_0$, then

$$v(z_0) = v(a) \le \mathbb{J}(v; a, r) \le v(z_0), \quad 0 < r < R(a).$$

Now, since v is upper semicontinuous, we conclude that $v = v(z_0)$ on $C(a, r)$ with $0 < r < R(a)$. This implies that $B(a, R(a)) \subset D_0$, and therefore D_0 is open. Since D is connected, we have $D_0 = D$, which shows that $v \equiv v(z_0)$; contradiction. $\square$

From Theorem 8.2.2 and Lemma 8.2.3 we immediately obtain

Corollary 8.2.4 (Maximum principle)**.** *Let $D \subset \mathbb{C}$ be a domain and let $u \in \mathcal{SH}(D)$, $u \not\equiv \mathrm{const}$. Then u does not attain its global maximum in D. Moreover, if D is bounded, then*

$$u(z) < \sup_{\zeta \in \partial D} \{\limsup_{D \ni w \to \zeta} u(w)\}, \quad z \in D.$$

Notice that a subharmonic function can attain its global minimum.

Theorem 8.2.5. *Let $u : \Omega \longrightarrow [-\infty, +\infty)$. Then $u \in \mathcal{SH}(\Omega)$ iff $u \in \mathcal{C}^{\uparrow}(\Omega)$ and for every $a \in \Omega$ there exists an $R(a)$, $0 < R(a) \le d_\Omega(a)$, such that*

$$u(a) \le \mathbb{J}(u; a, r), \quad 0 < r < R(a). \tag{8.2.7}$$

Proof. The implication $\Longrightarrow$ follows from Theorem 8.2.2.

To prove the opposite, fix a domain $D \subset\subset \Omega$ and a function $h \in \mathcal{C}(\overline{D}) \cap \mathcal{H}(D)$ such that $u \le h$ on ∂D. Put $v(z) := u(z) - h(z)$, $z \in \overline{D}$. By Theorem **??** and (8.2.7) we have

$$v(a) \le \mathbb{J}(v; a, r), \quad 0 < r < \min\{R(a), d_D(a)\},\ a \in D.$$

Using Lemma 8.2.3, we conclude that $v \le 0$ in D, which shows that $u \le h$ in D. $\square$

Corollary 8.2.6. (a) *Let $u : \Omega \longrightarrow [-\infty, +\infty)$. Then $u \in \mathcal{SH}(\Omega)$ iff every point $a \in \Omega$ admits an open neighborhood $U_a \subset \Omega$ such that $u|_{U_a} \in \mathcal{SH}(U_a)$. In other words, subharmonicity is a local property.*
(b) $\mathcal{SH}(\Omega) + \mathcal{SH}(\Omega) = \mathcal{SH}(\Omega)$.

Theorem 8.2.7. *Let $u : \Omega \longrightarrow [-\infty, +\infty)$. Then $u \in \mathcal{SH}(\Omega)$ iff $u \in \mathcal{C}^{\uparrow}(\Omega)$ and for any $a \in \Omega$, $0 < r < d_\Omega(a)$, and $p \in \mathcal{P}(\mathbb{C})$, if $u \le \operatorname{Re} p$ on $C(a, r)$, then $u \le \operatorname{Re} p$ in $B(a, r)$.*

Proof. Since the function $\operatorname{Re} p$ is harmonic, the implication $\Longrightarrow$ is obvious.

We prove now the opposite. Fix $a \in \Omega$ and $0 < r < d_\Omega(a)$. In virtue of Theorem 8.2.5 and the proof of Theorem 8.2.2, it is sufficient to prove that for every continuous function $b : C(a, r) \longrightarrow \mathbb{R}$ such that $u \le b$ we have $u(a) \le \mathbb{J}(b; a, r)$. Fix a function b and let $\varphi_\nu : \mathbb{R} \longrightarrow \mathbb{R}$, $\nu \ge 1$, be a sequence of trigonometric polynomials ([5]) such that

$$|b(a + re^{i\vartheta}) + \frac{1}{\nu} - \varphi_\nu(\vartheta)| < \frac{1}{\nu}, \quad \vartheta \in \mathbb{R}$$

([5]) Recall that $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ is a *trigonometric polynomial* if

$$\varphi(\vartheta) = \alpha_0 + \sum_{j=1}^{k} (\alpha_j \cos j\vartheta + \beta_j \sin j\vartheta), \quad \vartheta \in \mathbb{R},$$

(cf. [*Rudin*], the Fejèr theorem). Let $p_\nu \in \mathcal{P}(\mathbb{C})$ be such that $\varphi_\nu(\vartheta) = \operatorname{Re} p_\nu(a + re^{i\vartheta})$, $\vartheta \in \mathbb{R}$, $\nu \geq 1$. Then $u \leq \operatorname{Re} p_\nu$ on $C(a, r)$ and hence

$$u(a) \leq \operatorname{Re} p_\nu(a) = \mathbb{J}(\operatorname{Re} p_\nu; a, r) \leq \mathbb{J}(b; a, r) + \frac{2}{\nu}, \quad \nu \geq 1.$$

(the first equality follows from the fact that the function $\operatorname{Re} p_\nu$ is harmonic). Letting $\nu \longrightarrow +\infty$, we end the proof. $\square$

Theorem 8.2.8. *If $f \in \mathcal{O}(\Omega)$, then $\log|f| \in \mathcal{SH}(\Omega)$.*

Proof. Let $u := \log|f|$. Then $u \in \mathcal{C}^\uparrow(\Omega)$. By Theorem 8.2.5, it is enough to check that $u(a) \leq \mathbb{J}(u; a, r)$, $a \in \Omega$, $0 < r < R(a)$. This is evident if $f(a) = 0$. If $f(a) \neq 0$, then $u \in \mathcal{H}(B(a, R(a)))$, where $R(a) := d_{\Omega \setminus f^{-1}(0)}(a)$ (cf. Remark 8.1.2(e)). $\square$

Theorem 8.2.9. (a) *If $\mathcal{SH}(\Omega) \ni u_\nu \searrow u$, then $u \in \mathcal{SH}(\Omega)$.*
(b) *If $\mathcal{SH}(\Omega) \ni u_\nu \longrightarrow u$ locally uniformly in Ω, then $u \in \mathcal{SH}(\Omega)$.*

Proof. It is clear that in both cases $u \in \mathcal{C}^\uparrow(\Omega)$. For each ν we have

$$u_\nu(a) \leq \mathbb{J}(u_\nu; a, r), \quad a \in \Omega,\ 0 < r < d_\Omega(a).$$

Letting $\nu \longrightarrow +\infty$ proves that u satisfies (8.2.7). $\square$

Theorem 8.2.10. *If a family $(u_\iota)_{\iota \in I} \subset \mathcal{SH}(\Omega)$ is locally bounded from above* (6)*, then the function*

$$u := (\sup_{\iota \in I} u_\iota)^*,$$

is subharmonic, where * denotes the upper regularization. (7)

In particular, $\max\{u_1, \dots, u_N\} \in \mathcal{SH}(\Omega)$ *for any $u_1, \dots, u_N \in \mathcal{SH}(\Omega)$.*

Proof. It is clear that u is upper semicontinuous. Let $D \subset\subset \Omega$, $h \in \mathcal{C}(\overline{D}) \cap \mathcal{H}(D)$, $u \leq h$ on ∂D. Then $u_\iota \leq h$ on ∂D for every $\iota \in I$, and hence $\sup_{\iota \in I} u_\iota \leq h$ in D. Finally, since h is continuous, we get $u \leq h$ in D. $\square$

Theorem 8.2.11. *Let $G \subset \Omega \subset \mathbb{C}$ be open and let $v \in \mathcal{SH}(G)$, $u \in \mathcal{SH}(\Omega)$. Assume that*

$$\limsup_{G \ni z \to \zeta} v(z) \leq u(\zeta), \quad \zeta \in (\partial G) \cap \Omega.$$

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for some $\alpha_0, \dots, \alpha_k, \beta_1, \dots, \beta_k \in \mathbb{R}$. Observe that $\varphi(\vartheta) = \operatorname{Re} p(a + re^{i\vartheta})$, where

$$p(z) := q(\frac{z-a}{r}), \quad q(z) := \alpha_0 + \sum_{j=1}^{k} (\alpha_j - i\beta_j) z^j.$$

(6) Note that in general the function $\sup_{\iota \in I} u_\iota$ need not be upper semicontinuous.
(7) If $v : \Omega \longrightarrow [-\infty, +\infty)$ is locally bounded from above, then (cf. [*Lojasiewicz*])

$$v^*(z) := \limsup_{z' \to z} v(z') = \inf\{\varphi(z) : \varphi \in \mathcal{C}(\Omega, \mathbb{R}),\ v \leq \varphi\}, \quad z \in \Omega.$$

Let

$$\widetilde{u}(z) := \begin{cases} \max\{v(z), u(z)\}, & z \in G \\ u(z), & z \in \Omega \setminus G \end{cases}.$$

Then $\widetilde{u} \in \mathcal{SH}(\Omega)$.

Proof. It is evident that $\widetilde{u} \in \mathcal{C}^{\uparrow}(\Omega)$ and $\widetilde{u} \in \mathcal{SH}(\Omega \setminus \partial G)$. For $a \in \Omega \cap \partial G$ we have

$$\widetilde{u}(a) = u(a) \le \mathbb{J}(u; a, r) \le \mathbb{J}(\widetilde{u}; a, r), \quad 0 < r < d_\Omega(a).$$ □

Theorem 8.2.12. *Let* $u : \Omega \longrightarrow [-\infty, +\infty)$*. Then* $u \in \mathcal{SH}(\Omega)$ *iff* $u \in \mathcal{C}^{\uparrow}(\Omega)$ *and for every* $a \in \Omega$ *there exists an* $R(a)$, $0 < R(a) \le d_\Omega(a)$, *such that*

$$u(z) \le \mathbb{P}(u; a, r; z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{r^2 - |z-a|^2}{|re^{i\vartheta} - (z-a)|^2} u(a + re^{i\vartheta})\, d\vartheta, \quad 0 < r < R(a),\ z \in B(a, r). \tag{8.2.8}$$

Proof. Since $\mathbb{P}(u; a, r; a) = \mathbb{J}(u; a; r)$, the implication $\Longleftarrow$ follows from Theorem 8.2.5.

To prove the opposite, it is sufficient to argue as in the proof of Theorem 8.2.2 and use the Poisson formula

$$u(z) \le h_\nu(z) = \mathbb{P}(h_\nu; a, r; z) = \mathbb{P}(b_\nu; a, r; z) \searrow \mathbb{P}(u; a, r, z).$$ □

By Theorems **??** and 8.2.12 we get

Corollary 8.2.13. $\mathcal{SH}(\Omega) \cap (-\mathcal{SH}(\Omega)) = \mathcal{H}(\Omega)$.

Theorem 8.2.14. *If a sequence* $(u_\nu)_{\nu=1}^\infty \subset \mathcal{SH}(\Omega)$ *is locally bounded from above, then the function*

$$u := (\limsup_{\nu \to +\infty} u_\nu)^*.$$

is subharmonic. (8)

Proof. Of course, the function u is upper semicontinuous. Fix $a \in \Omega$ and $0 < r < d_\Omega(a)$. By Theorem 8.2.12 and Fatou's lemma we get

$$\limsup_{\nu \to +\infty} u_\nu(z) \le \limsup_{\nu \to +\infty} \mathbb{P}(u_\nu; a, r; z) \le \mathbb{P}(\limsup_{\nu \to +\infty} u_\nu; a, r; z) \le \mathbb{P}(u; a, r; z), \quad z \in B(a, r).$$

Since the right–hand side is a continuous function of z, we get $u(z) \le \mathbb{P}(u; a, r; z)$, $z \in B(a, r)$. □

Let $u : B(a, r) \longrightarrow [-\infty, +\infty)$ be bounded from above and measurable. Define

$$\mathbb{A}(u; a, r) := \frac{1}{\pi r^2} \int_{B(a,r)} u \, d\mathcal{L}^2;$$

$\mathbb{A}(u; a, r)$ is the mean value of u on the disc $B(a, r)$.

Theorem 8.2.15 (Mean value property)**.** *Let* $u : \Omega \longrightarrow [-\infty, +\infty)$*. Then* $u \in \mathcal{SH}(\Omega)$ *iff* $u \in \mathcal{C}^{\uparrow}(\Omega)$ *and for every* $a \in D$ *there exists an* $R(a)$, $0 < R(a) \le d_D(a)$, *such that*

$$u(a) \le \mathbb{A}(u; a, r), \quad 0 < r < R(a).$$

(8) Note that in general the function $\limsup_{\nu \to +\infty} u_\nu$ need not be upper semicontinuous.

Proof. Let $u \in \mathcal{SH}(\Omega)$. Using polar coordinates, we have by Theorem 8.2.2

$$\begin{aligned}\mathbb{A}(u;a,r) &= \frac{1}{\pi r^2}\int_0^r \int_0^{2\pi} u(a+\tau e^{i\vartheta})\tau\, d\vartheta\, d\tau \\ &= \frac{2}{r^2}\int_0^r \mathbb{J}(u;a,\tau)\tau\, d\tau \geq \frac{2}{r^2}\int_0^r u(a)\tau\, d\tau = u(a), \quad a\in\Omega,\ 0<r<d_\Omega(a).\end{aligned}$$

To prove the opposite we check first that u does not attain its maximum (like in the proof of Lemma 8.2.3), and then we proceed as in the proof of Theorem 8.2.5. $\square$

Theorem 8.2.16. *Let $D \subset \mathbb{C}$ be a domain and let $u \in \mathcal{SH}(D)$, $u \not\equiv -\infty$. Then $u \in L^1(D, \mathrm{loc})$. In particular, $\mathcal{L}^2(u^{-1}(-\infty)) = 0$.*

Proof. Suppose that for some $z_0 \in D$ we have $\int_U u\, d\mathcal{L}^2 = -\infty$ for any neighborhood U of z_0. Let $2r := d_D(z_0)$. By Theorem 8.2.15

$$u(z) \leq \mathbb{A}(u; z, r) = -\infty, \quad z \in K(z_0, r).$$

Let $D_0 := \{z \in D : u = -\infty \text{ in a neighborhood of } z\}$. The set D_0 is clearly open. We have already shown that it is non-empty ($z_0 \in D_0$). To obtain a contradiction, it is sufficient to note that proceeding exactly as above, we can prove that D_0 is relatively closed in D. $\square$

Theorem 8.2.17 (Removable singularities)**.** *Let $D \subset \mathbb{C}$ be a domain and let $M \subset D$ be a relatively closed subset of D such that for every point $a \in M$ there exist a connected open neighborhood $U_a \subset D$ of a and a function $v_a \in \mathcal{SH}(U_a)$, $v_a \not\equiv -\infty$, such that $M \cap U_a = v_a^{-1}(-\infty)$. Let $u \in \mathcal{SH}(D \setminus M)$ be locally bounded from above in D ([9]). Define*

$$\widetilde{u}(z) := \limsup_{D\setminus M \ni z' \to z} u(z'), \quad z \in D.$$

Then $\widetilde{u} \in \mathcal{SH}(D)$. In particular, the set $D \setminus M$ is connected.

Proof. By Theorem 8.2.16 the set M is nowhere dense and hence the function $\widetilde{u}$ is well defined for every $z \in D$. Note that $\widetilde{u} = (u_0)^*$, where $u_0 := u$ on $D \setminus M$ and $u_0 := -\infty$ on M. In particular, $\widetilde{u} \in \mathcal{C}^{\uparrow}(D)$. Moreover, $\widetilde{u} = u$ on $D \setminus M$.

It remains to prove that $\widetilde{u}$ is subharmonic. We may assume that $M = v^{-1}(-\infty)$, where $v \in \mathcal{SH}(D)$, $v \not\equiv -\infty$ and $v \leq 0$ in D. For $\varepsilon > 0$ let

$$u_\varepsilon(z) := \begin{cases} u(z) + \varepsilon v(z), & z \in D \setminus M \\ -\infty, & z \in M \end{cases}.$$

It is easy to see that $u_\varepsilon \in \mathcal{SH}(D)$ and that the family $(u_\varepsilon)_{\varepsilon>0}$ is locally bounded from above in D. Observe that $u_0 = \sup_{\varepsilon>0} u_\varepsilon$. Hence, by Theorem 8.2.10, $\widetilde{u} = (u_0)^* \in \mathcal{SH}(D)$.

To prove that $D \setminus M$ is connected, suppose that $D \setminus M = U_1 \cup U_2$, where U_1 and U_2 are disjoint and non–empty open sets. Then the function $u(z) := j$ for $z \in U_j$ would extend to a subharmonic function on D; contradiction. $\square$

([9]) That is, every point $a \in D$ admits an open neighborhood $V_a \subset D$ such that u is bounded from above in $V_a \setminus M$.

The above result can be generalized in the following way:

We say that a set $M \subset \mathbb{C}$ is *polar* if for every point $a \in M$ there exist a connected open neighborhood U_a and a function $v_a \in \mathcal{SH}(U_a)$, $v_a \not\equiv -\infty$, such that $M \cap U_a \subset v_a^{-1}(-\infty)$.

Note that the set M from Theorem 8.2.17 is polar. Every polar set has measure zero (by Theorem 8.2.16).

Lemma 8.2.18. *Let $M \subset \mathbb{C}$ be a polar set. Then for every $a \in \mathbb{C}$ there exists an $R(a) > 0$ such that*

$$\mathcal{L}^1(\{\vartheta \in [0, 2\pi) : a + re^{i\vartheta} \in M\}) = 0, \quad 0 < r < R(a).$$

Proof. Suppose that for some $a \in \mathbb{C}$ it is not the case. Fix a disc $B(a, R)$ and a function $v \in \mathcal{SH}(B(a, R))$, $v \not\equiv -\infty$, such that $M \cap B(a, R) \subset v^{-1}(-\infty)$. Let $0 < r < R$ be such that

$$\mathcal{L}^1(\{\vartheta \in [0, 2\pi) : a + re^{i\vartheta} \in M\}) > 0.$$

This means that $v(a + re^{i\vartheta}) = -\infty$ for ϑ in a set of positive measure. In particular, $v(z) \le \mathbb{P}(v; a, r; z) = -\infty$ for $z \in B(a, r)$, and so $v \equiv -\infty$ in $B(a, r)$; contradiction. □

Theorem 8.2.19 (Removable singularities)**.** *Let $D \subset \mathbb{C}$ be a domain and let $M \subset D$ be a polar set. Assume that $u \in \mathcal{C}^{\uparrow}(D \setminus M)$ is locally bounded from above in D and for arbitrary $a \in D \setminus M$ there exists an $R(a)$, $0 < R(a) \le d_D(a)$, such that*

$$u(a) \le \mathbb{J}(u; a, r), \quad 0 < r < R(a). \quad (^{10})$$

Put

$$\widetilde{u}(z) := \limsup_{D \setminus M \ni z' \to z} u(z'), \quad z \in D.$$

Then $\widetilde{u} \in \mathcal{SH}(D)$. In particular, if M is closed in D, then $D \setminus M$ is a domain.

Proof. The function $\widetilde{u}$ is upper semicontinuous and $\widetilde{u} = u$ in $D \setminus M$. Let $G \subset\subset D$ be an arbitrary domain and let $h \in \mathcal{H}(G) \cap \mathcal{C}(\overline{G})$ be such that $\widetilde{u} \le h$ on ∂G. It is sufficient to check that $\widetilde{u} \le h$ in $G \setminus M$. Fix an $a \in G \setminus M$. One can prove (see for instance [*Hay-Ken*], Th. 5.11), that there exists a function v subharmonic in the neighborhood of $\overline{G}$ and such that $M \cap G \subset v^{-1}(-\infty)$, $v \le 0$, and $v(a) > -\infty$. Define $h_\varepsilon := \widetilde{u} + \varepsilon v - h$, $\varepsilon > 0$. Then $h_\varepsilon \in \mathcal{C}^{\uparrow}(\overline{G})$ and $h_\varepsilon \le 0$ on ∂G. One can easily check that $h_\varepsilon \in \mathcal{SH}(G)$ $(^{11})$. By the maximum principle (Corollary 8.2.4) it follows that $h_\varepsilon \le 0$ in G, $\varepsilon > 0$. In particular, $\widetilde{u}(a) - h(a) = \sup_{\varepsilon > 0}\{h_\varepsilon(a)\} \le 0$.

□

Theorem 8.2.20 (Hartogs lemma)**.** *Let $(u_\nu)_{\nu=1}^{\infty} \subset \mathcal{SH}(\Omega)$ be locally bounded from above. Assume that for some $m \in \mathbb{R}$*

$$\limsup_{\nu \to +\infty} u_\nu \le m.$$

Then for any compact $K \subset \Omega$ and $\varepsilon > 0$ there exists a ν_0 such that

$$\max_K u_\nu \le m + \varepsilon, \quad \nu \ge \nu_0; \quad \textit{cf. Lemma } \textbf{??}.$$

$(^{10})$ Note that if M is a closed subset of D, then every function $u \in \mathcal{SH}(D \setminus M)$ satisfies this condition (with $R(a) := d_{D \setminus M}(a)$). Moreover, by Lemma 8.2.18, the integral $\mathbb{J}(u; a, r)$ is well defined for small r.

$(^{11})$ We apply for instance Theorem 8.2.5: since $h_\varepsilon = -\infty$ on M, it is sufficient to observe that $h_\varepsilon(z_0) \le \mathbb{J}(h_\varepsilon; z_0, r)$ for $z_0 \in G \setminus M$.

Proof. It is sufficient to show that for every $a \in \Omega$ the assertion holds for $K := \overline{K}(a, \delta(a))$, where $\delta(a) > 0$ is sufficiently small. Fix a and $0 < R < d_\Omega(a)/2$. We may assume that $u_\nu \le 0$ in $\overline{K}(a, 2R)$, $\nu \ge 1$, and $m < 0$. By Fatou's lemma we have

$$\limsup_{\nu\to+\infty} \mathbb{A}(u_\nu; a, R) \le \mathbb{A}(\limsup_{\nu\to+\infty} u_\nu; a, R) \le \mathbb{A}(m; a, R) = m.$$

Let $0 < \delta < R/2$. By the above inequality, since $u_\nu \le 0$ on $\overline{K}(a, 2R)$, we get

$$\limsup_{\nu\to+\infty} \max_{z\in\overline{K}(a,\delta)} u_\nu(z) \le \limsup_{\nu\to+\infty} \sup_{z\in\overline{K}(a,\delta)} \mathbb{A}(u_\nu; z, R+\delta) \le \limsup_{\nu\to+\infty} \frac{R^2}{(R+\delta)^2}\mathbb{A}(u_\nu; a, R) \le \frac{R^2}{(R+\delta)^2}m.$$

Now it is sufficient to take a $\delta = \delta(a)$ so small that the last term is smaller than $m + \varepsilon$. $\square$

Theorem 8.2.21. *Let $I \subset \mathbb{R}$ be an open interval and let $\varphi : I \longrightarrow \mathbb{R}$ be non-decreasing and convex. Then $\varphi \circ u \in \mathcal{SH}(\Omega)$ for any subharmonic function $u : \Omega \longrightarrow I$. In particular,*
$e^u \in \mathcal{SH}(\Omega)$ for any function $u \in \mathcal{SH}(\Omega)$ (12),
$u^p \in \mathcal{SH}(\Omega)$ for any subharmonic function $u : \Omega \longrightarrow \mathbb{R}_+$ and $p \ge 1$ (13).

Proof. Since φ is convex, it is continuous (cf. [*Schwartz:Analiza*]), and therefore $\varphi\circ u \in \mathcal{C}^\uparrow(\Omega)$. Fix $a \in \Omega$ and $0 < r < d_\Omega(a)$. By the monotonicity and convexity of φ and by Jensen's inequality (cf. [*Rudin*]), we obtain

$$\varphi(u(a)) \le \varphi(\mathbb{J}(u; a, r)) \le \mathbb{J}(\varphi \circ u; a, r). \qquad \square$$

Theorem 8.2.22. *Let $u \in \mathcal{SH}(\Omega)$, $a \in \Omega$. Then the functions*

$$(-\infty, \log d_\Omega(a)) \ni t \longmapsto \mathbb{J}(u; a, e^t), \qquad (-\infty, \log d_\Omega(a)) \ni t \longmapsto \mathbb{A}(u; a, e^t)$$

are non–decreasing and convex. Moreover,

$$\mathbb{J}(u; a, r) \searrow u(a) \text{ when } r \searrow 0, \qquad \mathbb{A}(u; a, r) \searrow u(a) \text{ when } r \searrow 0.$$

Proof. We show first that it is sufficient to consider only the function $\mathbb{J}$. Note that if the function $\mathbb{J}(u; a, \cdot)$ is convex with respect to $\log r$, then it is continuous, and therefore we have

$$\mathbb{A}(u; a, r) = \frac{2}{r^2}\int_0^r \mathbb{J}(u; a, \tau)\tau\, d\tau = \lim_{N\to+\infty} \frac{2}{N^2}\sum_{j=1}^N j\mathbb{J}(u; a, \frac{jr}{N}) =: \lim_{N\to+\infty} \varphi_N(r).$$

If the function $\mathbb{J}(u; a, \cdot)$ is non–decreasing and convex with respect to $\log r$, then the same properties has every function φ_N, and so also the limit function $\mathbb{A}(u; a, .)$. Moreover,

$$u(a) \le \mathbb{A}(u; a, r) = \frac{2}{r^2}\int_0^r \mathbb{J}(u; a, \tau)\tau\, d\tau \le \sup_{0<\tau<r} \mathbb{J}(u; a, \tau) \le \mathbb{J}(u; a, r).$$

Therefore, if $\mathbb{J}(u; a, r) \longrightarrow u(a)$, then the same property has the function $\mathbb{A}$.

---

(12) First we consider $u : \Omega \longrightarrow \mathbb{R}$ and next we observe that in the general case we have $e^{\max\{u,-\nu\}} \searrow e^u$ when $\nu \nearrow +\infty$.

(13) First we consider $u : \Omega \longrightarrow \mathbb{R}_{>0}$ and next we observe that in the general case we have $(u+\varepsilon)^p \searrow u^p$ when $\varepsilon \searrow 0$.

Now consider the function $\mathbb{J}$. Let $0 < r_1 < r_2 < d_\Omega(a)$, let $b_\nu \in \mathcal{C}(C(a, r_2), \mathbb{R})$, $b_\nu \searrow u$, and denote by h_ν the solution of the Dirichlet problem for $B(a, r_2)$ with boundary condition b_ν (cf. Theorem **??**). Then

$$\mathbb{J}(u; a, r_1) \le \mathbb{J}(h_\nu; a, r_1) = h_\nu(a) = \mathbb{J}(h_\nu; a, r_2) = \mathbb{J}(b_\nu; a, r_2).$$

The last integral converges to $\mathbb{J}(u; a, r_2)$ when $\nu \longrightarrow +\infty$. Letting $\nu \longrightarrow +\infty$ we get the monotonicity of the function $\mathbb{J}(u; a, \cdot)$.

Note that by Fatou's lemma we have

$$u(a) \le \lim_{r\to 0} \mathbb{J}(u; a, r) \le \frac{1}{2\pi} \int_0^{2\pi} \limsup_{r\to 0} u(a + re^{i\vartheta})\, d\vartheta \le u(a).$$

This proves that $\mathbb{J}(u; a, r) \searrow u(a)$ when $r \searrow 0$.

It remains to check the convexity with respect to $\log r$, i.e. we want to prove the inequality

$$\mathbb{J}(u; a, r) \le \mathbb{J}(u; a, r_1) + \frac{\mathbb{J}(u; a, r_2) - \mathbb{J}(u; a, r_1)}{\log \frac{r_2}{r_1}} \log \frac{r}{r_1}, \quad 0 < r_1 < r < r_2 < d_\Omega(a).$$

Fix $0 < r_1 < r_2 < d_\Omega(a)$. Let $A := \{z \in \mathbb{C} : r_1 < |z| < r_2\}$, let $b_\nu \in \mathcal{C}(\partial A, \mathbb{R})$, $b_\nu \searrow u$, and let h_ν be the solution of the Dirichlet problem for the annulus A with boundary condition b_ν (cf. Theorem **??**). Differentiating under the integral sign, we obtain

$$\begin{aligned}\frac{d}{dt}\mathbb{J}(h_\nu; a, e^t) = \frac{d}{dt}\frac{1}{2\pi}\int_0^{2\pi} h_\nu(a+e^t e^{i\vartheta})\, d\vartheta &= \frac{1}{2\pi}\int_0^{2\pi} \Big(\frac{\partial h_\nu}{\partial x}(a+e^t e^{i\vartheta})e^t \cos\vartheta + \frac{\partial h_\nu}{\partial y}(a+e^t e^{i\vartheta})e^t \sin\vartheta\Big) d\vartheta \\ &= \frac{1}{2\pi}\int_{C(a,e^t)} -\frac{\partial h_\nu}{\partial y}dx + \frac{\partial h_\nu}{\partial x}dy = \mathrm{const}(\nu).\end{aligned}$$

The last equality follows from the fact that the form

$$-\frac{\partial h_\nu}{\partial y}dx + \frac{\partial h_\nu}{\partial x}dy$$

is closed. Consequently, there exist $\alpha_\nu, \beta_\nu \in \mathbb{R}$ such that

$$\mathbb{J}(h_\nu; a, r) = \alpha_\nu \log r + \beta_\nu, \quad r_1 < r < r_2.$$

Finally,

$$\begin{aligned}\mathbb{J}(u; a, r) \le \mathbb{J}(h_\nu; a, r) &= \mathbb{J}(h_\nu; a, r_1) + \frac{\mathbb{J}(h_\nu; a, r_2) - \mathbb{J}(h_\nu; a, r_1)}{\log \frac{r_2}{r_1}} \log \frac{r}{r_1} \\ &= \mathbb{J}(b_\nu; a, r_1) + \frac{\mathbb{J}(b_\nu; a, r_2) - \mathbb{J}(b_\nu; a, r_1)}{\log \frac{r_2}{r_1}} \log \frac{r}{r_1}, \quad r_1 < r < r_2.\end{aligned}$$

Letting $\nu \longrightarrow +\infty$ we end the proof. □

Corollary 8.2.23. *Let $u_1, u_2 \in \mathcal{SH}(\Omega)$. If $u_1 = u_2$ $\mathcal{L}^2$-almost everywhere in Ω, then $u_1 \equiv u_2$ in Ω.*

Corollary 8.2.24. *Let D and M be as in Theorem 8.2.17 or 8.2.19. Then for every function $u \in \mathcal{SH}(D)$ we have*

$$u(z) = \limsup_{D\setminus M \ni z' \to z} u(z'), \quad z \in D.$$

Fix a function $\Psi \in \mathcal{C}_0^\infty(\mathbb{C}, \mathbb{R}_+)$ such that
- $\operatorname{supp} \Psi = \overline{\mathbb{D}}$,
- $\Psi(z) = \Psi(|z|)$, $z \in \mathbb{C}$,
- $\int \Psi \, d\mathcal{L}^2 = 1$.

Let
$$\Psi_\varepsilon(z) := \frac{1}{\varepsilon^2}\Psi(\frac{z}{\varepsilon}), \quad z \in \mathbb{C},\ \varepsilon > 0.$$

For every function $u \in L^1(\Omega, \mathrm{loc})$, we put
$$u_\varepsilon(z) := \int_\Omega u(w)\Psi_\varepsilon(z-w)\, d\mathcal{L}^2(w) = \int_{\mathbb{D}} u(z+\varepsilon w)\Psi(w)\, d\mathcal{L}^2(w), \quad z \in \Omega_\varepsilon := \{z \in \Omega : d_\Omega(z) > \varepsilon\}.$$
The function u_ε is called the *ε-regularization* of u.

Theorem 8.2.25. *If $u \in \mathcal{SH}(\Omega) \cap L^1(\Omega, \mathrm{loc})$, then $u_\varepsilon \in \mathcal{SH}(\Omega_\varepsilon) \cap \mathcal{C}^\infty(\Omega_\varepsilon)$ and $u_\varepsilon \searrow u$ when $\varepsilon \searrow 0$.*

Proof. Since we can differentiate under the integral sign in the first integral above, it is clear that $u_\varepsilon \in \mathcal{C}^\infty(\Omega_\varepsilon)$. For $a \in \Omega_\varepsilon$ and $0 < r < d_{\Omega_\varepsilon}(a)$ we have
$$\begin{aligned}\mathbb{J}(u_\varepsilon; a, r) &= \frac{1}{2\pi}\int_0^{2\pi}\int_{\mathbb{D}} u(a + re^{i\vartheta} + \varepsilon w)\Psi(w)\, d\mathcal{L}^2(w)\, d\vartheta \\ &= \int_{\mathbb{D}} \mathbb{J}(u; a + \varepsilon w, r)\Psi(w)\, d\mathcal{L}^2(w) \geq \int_{\mathbb{D}} u(a + \varepsilon w)\Psi(w)\, d\mathcal{L}^2(w) = u_\varepsilon(a),\end{aligned}$$
which shows that $u_\varepsilon \in \mathcal{SH}(\Omega_\varepsilon)$. Note that
$$u_\varepsilon(a) = \int_{\mathbb{D}} u(a+\varepsilon w)\Psi(w)\, d\mathcal{L}^2(w) = \int_0^1\int_0^{2\pi} u(a+\varepsilon\tau e^{i\vartheta})\Psi(\tau)\tau\, d\vartheta\, d\tau = 2\pi\int_0^1 \mathbb{J}(u; a, \varepsilon\tau)\Psi(\tau)\tau\, d\tau.$$
Now, by Theorem 8.2.22 and monotone convergence theorem, we get $u_\varepsilon(a) \searrow u(a)$ when $\varepsilon \searrow 0$ for every $a \in \Omega$. □

Remark 8.2.26. It follows from the proof of Theorem 8.2.25 that for an arbitrary function $\Psi \in \mathcal{C}_0^\infty(\mathbb{C}, \mathbb{R}_+)$ such that $\operatorname{supp}\Psi = \overline{\mathbb{D}}$ and for every function $u \in \mathcal{SH}(\Omega)$, the functions
$$u_\varepsilon(z) := \int_{\mathbb{D}} u(z + \varepsilon w)\Psi(w)\, d\mathcal{L}^2(w), \quad z \in \Omega_\varepsilon, \quad \varepsilon > 0,$$
are subharmonic.

Theorem 8.2.27. *Let $u \in \mathcal{C}^2(\Omega, \mathbb{R})$. Then $u \in \mathcal{SH}(\Omega)$ iff $\Delta u \geq 0$ in Ω.*

Proof. $\Longleftarrow$. Assume first that $\Delta u > 0$ in Ω. Let $D \subset\subset \Omega$, $h \in \mathcal{C}(\overline{D}) \cap \mathcal{H}(D)$, $u \leq h$ on ∂D. Put $v := u - h$ and let $z_0 \in \overline{D}$ be such that $v(z_0) = \max_{\overline{D}} v$. Suppose that $v(z_0) > 0$ (in particular, $z_0 \in D$). Then $(\Delta u)(z_0) \leq 0$; contradiction.

For arbitrary u, take the sequence $v_\varepsilon(z) := u(z) + \varepsilon|z|^2$, $z \in \Omega$, $\varepsilon > 0$, and note that $\Delta v_\varepsilon = \Delta u + 4\varepsilon > 0$ and $v_\varepsilon \searrow u$.

$\Longrightarrow$. Suppose that $\Delta u < 0$ on some domain $D \subset \Omega$. Then, by the previous part of the proof, $-u \in \mathcal{SH}(D)$. Hence $u \in \mathcal{H}(D)$; contradiction. □

Theorem 8.2.28. *If $u \in \mathcal{SH}(D)$ (D is a domain in $\mathbb{C}$), $u \not\equiv -\infty$, then $\varDelta u \ge 0$ in D in the distribution sense, i.e. for every function $\varphi \in \mathcal{C}_0^\infty(D, \mathbb{R}_+)$ we have*

$$\int_D u \cdot (\varDelta\varphi)\, d\mathcal{L}^2 \ge 0.$$

Conversely, if $u \in L^1(D, \mathrm{loc})$ is such that $\varDelta u \ge 0$ in D in the distribution sense, then there exists a function $v \in \mathcal{SH}(D)$ such that $u = v$ $\mathcal{L}^2$-almost everywhere in D; cf. Theorem **??**.

Proof. Note first that if $u \in \mathcal{C}^2(D)$, then, by the Stokes theorem, $\varDelta u \ge 0$ in D in the distribution sense iff $\varDelta u \ge 0$ in D in the usual sense.

$\Longrightarrow$. Let u_ε denote the regularization of the function u (as in Theorem 8.2.25). By Theorems 8.2.25 and 8.2.27, $\varDelta u_\varepsilon \ge 0$ in D_ε in the distribution sense, i.e.

$$\int_{D_\varepsilon} u_\varepsilon \cdot (\varDelta\varphi)\, d\mathcal{L}^2 \ge 0$$

for every test function $\varphi \in \mathcal{C}_0^\infty(D_\varepsilon, \mathbb{R}_+)$. Since $u_\varepsilon \searrow u$ (Theorem 8.2.25), we get

$$\int_D u \cdot (\varDelta\varphi)\, d\mathcal{L}^2 \ge 0, \quad \varphi \in \mathcal{C}_0^\infty(D, \mathbb{R}_+).$$

$\Longleftarrow$. For every function $\varphi \in \mathcal{C}_0^\infty(D_\varepsilon, \mathbb{R}_+)$ we have

$$\begin{aligned}\int_{D_\varepsilon} u_\varepsilon \cdot (\varDelta\varphi)\, d\mathcal{L}^2 = \int_{D_\varepsilon} (\varDelta u_\varepsilon)\varphi\, d\mathcal{L}^2 &= \int_{D_\varepsilon} \Big(\int_D u(w)(\varDelta\Psi_\varepsilon)(z-w)\, d\mathcal{L}^2(w)\Big)\varphi(z)\, d\mathcal{L}^2(z)\\ &= \int_{D_\varepsilon} \Big(\int_D u(w)(\varDelta(\Psi_\varepsilon(z-\cdot)))(w)\, d\mathcal{L}^2(w)\Big)\varphi(z)\, d\mathcal{L}^2(z) \ge 0.\end{aligned}$$

This proves that $u_\varepsilon \in \mathcal{SH}(D_\varepsilon)$.

We show now that $u_\varepsilon \searrow$ when $\varepsilon \searrow 0$. Let $0 < \varepsilon_1 < \varepsilon_2$. By Theorem 8.2.25 applied for $z \in D_{\varepsilon_2}$ we have

$$\begin{aligned}u_{\varepsilon_2}(z) = \lim_{\varepsilon\to 0}(u_{\varepsilon_2})_\varepsilon(z) &= \lim_{\varepsilon\to 0}\int_{\mathbb{D}}\int_{\mathbb{D}} u(z+\varepsilon w+\varepsilon_2\xi)\Psi(\xi)\, d\mathcal{L}^2(\xi)\Psi(w)\, d\mathcal{L}^2(w)\\ &= \lim_{\varepsilon\to 0}\int_{\mathbb{D}}\int_{\mathbb{D}} u(z+\varepsilon w+\varepsilon_2\xi)\Psi(w)\, d\mathcal{L}^2(w)\Psi(\xi)\, d\mathcal{L}^2(\xi)\\ &= \lim_{\varepsilon\to 0}(u_\varepsilon)_{\varepsilon_2}(z) \ge \lim_{\varepsilon\to 0}(u_\varepsilon)_{\varepsilon_1}(z) = \lim_{\varepsilon\to 0}(u_{\varepsilon_1})_\varepsilon(z) = u_{\varepsilon_1}(z).\end{aligned}$$

Let $v := \lim_{\varepsilon\to 0} u_\varepsilon$. Then $v \in \mathcal{SH}(D)$. On the other hand, it is well known (cf. [*Rudin*]) that $u_\varepsilon \longrightarrow u$ in $L^1(D, \mathrm{loc})$. In particular, $u_\varepsilon \longrightarrow u$ $\mathcal{L}^2$-almost everywhere in D. Hence $u = v$ $\mathcal{L}^2$-almost everywhere D.

□

Theorem 8.2.29. *For every $f \in \mathcal{O}(\Omega, G)$ (G is an open subset of $\mathbb{C}$) and $u \in \mathcal{SH}(G)$ we have $u \circ f \in \mathcal{SH}(\Omega)$.*

Proof. If $u \in \mathcal{C}^2(G)$ it is sufficient to note that

$$\varDelta(u \circ f) = ((\varDelta u) \circ f) \cdot |f'|^2,$$

and use Theorem 8.2.27. For the general case we use the regularizations $(u_\varepsilon)_{\varepsilon>0}$, cf. Theorem 8.2.25. Let $v_\varepsilon := u_\varepsilon \circ f$. Then $v_\varepsilon \in \mathcal{SH}(f^{-1}(G_\varepsilon))$, and $v_\varepsilon \searrow u \circ f$ in G, and so, by Theorem 8.2.9(a), $u \circ f \in \mathcal{SH}(\Omega)$. $\square$

Theorem 8.2.30 (Liouville type theorem)**.** *If $u \in \mathcal{SH}(\mathbb{C})$ is bounded from above, then $u \equiv \text{const}$.*

Proof. Let $v(z) := u(1/z)$, $z \in \mathbb{C}_*$. Then, by Theorem 8.2.29, $v \in \mathcal{SH}(\mathbb{C}_*)$ and v is bounded from above. Hence, by Theorem 8.2.17, v extends to a function $\widetilde{v} \in \mathcal{SH}(\mathbb{C})$. Now, by the maximum principle, for arbitrary $z \in \mathbb{C}$, we have

$$u(z) \le \max\{\max_{\mathbb{T}} u, \max_{\mathbb{T}} v\} = u(z_0)$$

for some $z_0 \in \mathbb{T}$. Using once again the maximum principle we conclude that $u \equiv \text{const}$. $\square$

Theorem 8.2.31 (Oka theorem)**.** *For every function $u \in \mathcal{SH}(\Omega)$, and for every $\mathbb{R}$-analytic curve $\gamma : [0,1] \longrightarrow \Omega$ it holds*

$$u(\gamma(0)) = \limsup_{t\to 0+} u(\gamma(t)).$$

Proof. Since the curve γ is $\mathbb{R}$-analytic, there exists a function $f \in \mathcal{O}(G)$, where $G \subset \mathbb{C}$ is an open neighborhood of the interval $[0,1]$, such that $f = \gamma$ on $[0,1]$ and $f(G) \subset \Omega$. Put $u_1 := u \circ f$. To prove the assertion, it is sufficient to show that $\limsup_{x\to 0+} u_1(x) = u_1(0)$. Moreover, we may assume that $u_1 \le 0$.

Suppose that $\limsup_{x\to 0+} u_1(x) < C < u_1(0)$. Let

$$u_2 := -\frac{1}{C}\max\{u_1, C\} + 1.$$

Then $u_2 \in \mathcal{SH}(G)$, $0 \le u_2 \le 1$, $u_2(0) > 0$, and $u_2 = 0$ on $(0,\delta]$ for some $0 < \delta \ll 1$. We may assume that $\delta\overline{\mathbb{D}} \subset G$. Define $v(z) := u_2(\delta z)$, $z \in \overline{\mathbb{D}}$. Then $v \in \mathcal{SH}$, $0 \le v \le 1$, $v(0) > 0$, and $v = 0$ on $(0,1]$. Let

$$S_\nu := \{re^{i\vartheta} : 0 < r < 1,\ 0 < \vartheta < \frac{2\pi}{\nu}\},$$

$$v_\nu(z) := \begin{cases} v(z^\nu), & \text{for } z \in S_\nu \\ 0, & \text{for } z \in \mathbb{D}_* \setminus S_\nu \end{cases}, \quad \nu \in \mathbb{N}.$$

It is not difficult to check that $v_\nu \in \mathcal{SH}(\mathbb{D}_*)$ (cf. Theorem 8.2.11). Since $v_\nu \le 1$, the function v_ν extends to a subharmonic function on $\mathbb{D}$; denote the extension also by v_ν. Observe that

$$v_\nu(0) = \limsup_{\mathbb{D}_*\ni z\to 0} v_\nu(z) = \limsup_{S_\nu\ni z\to 0} v(z^\nu) = \limsup_{\mathbb{D}_*\ni z\to 0} v(z) = v(0).$$

Finally, for any $0 < r < 1$, we have

$$v(0) = v_\nu(0) \le \mathbb{J}(v_\nu; 0, r) = \frac{1}{2\pi}\int_0^{2\pi/\nu} v(r^\nu e^{i\nu\vartheta})\, d\vartheta = \frac{1}{2\pi}\int_0^{2\pi} v(r^\nu e^{i\vartheta})\frac{1}{\nu}\, d\vartheta \le \frac{1}{\nu}.$$

Letting $\nu \longrightarrow +\infty$ we obtain $v(0) = 0$; contradiction. $\square$

The above result can be generalized as follows:

Theorem 8.2.32 (Oka theorem). *For any $u \in \mathcal{SH}(\Omega)$ and a curve $\gamma : [0,1] \longrightarrow \Omega$ we have*

$$u(\gamma(0)) = \limsup_{t\to 0+} u(\gamma(t)).$$

Proof. Cf. [*Vla*]. We may assume that $\gamma(0) = 0 \in \Omega$. Suppose that

$$u(0) > A > \limsup_{t\to 0+} u(\gamma(t)).$$

Take $r > 0$ and $0 < t_0 \le 1$ such that:
- $K(r) \subset\subset \Omega$,
- $|\gamma(t)| < r$ for $0 \le t < t_0$,
- $|\gamma(t_0)| = r$,
- $u(\gamma(t)) < A$ for $0 < t \le t_0$.

We may assume that $t_0 = 1$. Let $\Omega_0 := \{z \in \Omega : u(z) < A\}$. Observe that Ω_0 is open and $\gamma((0,1]) \subset \Omega_0$. Let G denote the connected component of Ω_0 that contains $\gamma((0,1])$. For $0 < \rho < r$ let $0 < t_\rho < 1$ be such that $|\gamma(t_\rho)| = \rho$. Take a Jordan arc $\sigma_\rho : [0,1] \longrightarrow G$ such that $\sigma_\rho(0) = \gamma(t_\rho)$, $\sigma_\rho(1) = \gamma(1)$. There exist $0 \le \tau_0 < \tau_1 \le 1$ such that
- $|\sigma_\rho(\tau_0)| = \rho$,
- $\rho < |\sigma_\rho(t)| < r$ for $\tau_0 < t < \tau_1$,
- $|\sigma_\rho(\tau_1)| = r$.

We may assume that $\tau_0 = 0$, $\tau_1 = 1$. Let $L_\rho := \sigma_\rho([0,1])$, $D_\rho := K(r) \setminus L_\rho$. One can prove that D_ρ is simply connected (Exercise). Let $\varphi_\rho : \mathbb{D} \longrightarrow D_\rho$ be a biholomorphic mapping (from the Riemann theorem) with $\varphi_\rho(0) = 0$ and $\varphi'_\rho(0) \in \mathbb{R}_{>0}$. By the Carathéodory theorem (cf. [*Vla*]), the mapping φ_ρ extends continuously to $\overline{\mathbb{D}}$ (we denote this extension also by φ_ρ) and $\varphi_\rho(\mathbb{T}) \subset \partial D_\rho$. Let

$$T_\rho := \{\vartheta \in [0, 2\pi) : \varphi_\rho(e^{i\vartheta}) \in L_\rho\}$$

(observe that T_ρ is relatively closed in $[0, 2\pi)$) and let $m_\rho := \mathcal{L}^1(T_\rho)/(2\pi)$. Notice that $|\varphi_\rho(e^{i\vartheta})| = r$ for $\vartheta \in T'_\rho := [0, 2\pi) \setminus T_\rho$. The function

$$\psi_\rho(z) := \begin{cases} \varphi_\rho(z)/z, & z \neq 0 \\ \varphi'_\rho(0), & z = 0 \end{cases}$$

is holomorphic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. Moreover, $\psi_\rho(z) \neq 0$, $z \in \overline{\mathbb{D}}$. In particular, $\log|\psi_\rho|$ is harmonic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$ and, therefore,

$$\begin{aligned} \log \varphi'_\rho(0) &= \log|\psi_\rho(0)| = \mathbb{J}(\log|\psi_\rho|; 0, 1) = \mathbb{J}(\log|\varphi_\rho|; 0, 1) \\ &= \frac{1}{2\pi}\Big(\int_{T_\rho} \log|\varphi_\rho(e^{i\vartheta})|\, d\vartheta + \int_{T'_\rho} \log|\varphi_\rho(e^{i\vartheta})|\, d\vartheta\Big) \ge m_\rho \log\rho + (1 - m_\rho)\log r. \end{aligned}$$

On the other hand, by the Koebe theorem (cf. [*Vla*]), since $\overline{K}(\rho) \not\subset \varphi_\rho(\mathbb{D})$, we get $\varphi'_\rho(0) \le 4\rho$. Hence

$$4\rho^{1-m_\rho} \ge r^{1-m_\rho},$$

and, consequently, $\lim_{\rho\to 0} m_\rho = 1$.

Since $u \circ \varphi_\rho$ is subharmonic in $\mathbb{D}$ and upper semicontinuous in $\overline{\mathbb{D}}$, we get

$$u(0) \le \mathbb{J}(u \circ \varphi_\rho; 0, 1) = \frac{1}{2\pi}\Big(\int_{T_\rho} u(\varphi_\rho(e^{i\vartheta}))\, d\vartheta + \int_{T'_\rho} u(\varphi_\rho(e^{i\vartheta}))\, d\vartheta\Big) \le m_\rho A + (1 - m_\rho)c,$$

where $c := \sup_{\overline{K}(r)} u$. Letting $\rho \longrightarrow 0$ gives $u(0) \le A$; contradiction □

Theorem 8.2.33. *Let $u \in \mathcal{C}^{\uparrow}(\Omega, \mathbb{R}_+)$. Then $\log u \in \mathcal{SH}(\Omega)$ ([14]) iff for every polynomial $p \in \mathcal{P}(\mathbb{C})$ the function $|e^p|u$ is subharmonic. In particular, if $\log u_1$, $\log u_2 \in \mathcal{SH}(\Omega)$, then $\log(u_1 + u_2) \in \mathcal{SH}(\Omega)$.*

Proof. $\Longrightarrow$. Let $v(z) := |e^{p(z)}|u(z)$, $z \in \Omega$. Then $\log v = \operatorname{Re} p + \log u$ and hence $\log v \in \mathcal{SH}(\Omega)$; therefore also $v \in \mathcal{SH}(\Omega)$.

$\Longleftarrow$. We use Theorem 8.2.7. Let $a \in \Omega$, $0 < r < d_\Omega(a)$ and let $p \in \mathcal{P}(\mathbb{C})$ be such that $\log u \le \operatorname{Re} p$ on $C(a, r)$. Then $v := |e^{-p}|u \le 1$ on $C(a, r)$. Since the function v is subharmonic, it follows from the maximum principle that $v \le 1$ in $K(a, r)$, which means that $\log u \le \operatorname{Re} p$ in $K(a, r)$. $\square$

Theorem 8.2.33 can be generalized in the following way:

Theorem 8.2.34. *Let $u \in \mathcal{C}^{\uparrow}(\Omega, \mathbb{R}_+)$. Then $\log u \in \mathcal{SH}(\Omega)$ iff for every $a \in \mathbb{C}$ the function $|e^{az}|u(z)$ is subharmonic.*

Proof. It is clear that the problem is to prove $\Longleftarrow$. Suppose first that $u \in \mathcal{C}^2(\Omega, \mathbb{R}_{>0})$. It is sufficient to check that $\Delta \log u \ge 0$ in Ω. Note that

$$\Delta \log u = \frac{1}{u}\Big(\Delta u - \frac{(\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2}{u}\Big).$$

Let $a = \alpha + i\beta$ and put $v_a := |e^{az}|u$. Then

$$0 \le \Delta v_a = |e^{az}|\Big(\Delta u + |a|^2 u + 2(\alpha\frac{\partial u}{\partial x} - \beta\frac{\partial u}{\partial y})\Big).$$

Fix a $z_0 \in \Omega$ and put

$$\alpha := -\frac{\frac{\partial u}{\partial x}(z_0)}{u(z_0)}, \quad \beta := \frac{\frac{\partial u}{\partial y}(z_0)}{u(z_0)}.$$

Then

$$(\Delta \log u)(z_0) = \frac{|e^{-az_0}|}{u(z_0)}\Delta v_a(z_0) \ge 0.$$

Now consider the general case. Note that the function u is subharmonic (because $u = |e^{0z}|u$). Let $(u_\varepsilon)_{\varepsilon>0}$ denote the regularizations of the function u. Since $u_\varepsilon + \varepsilon \searrow u$, it suffices to show that $\log(u_\varepsilon + \varepsilon) \in \mathcal{SH}(\Omega_\varepsilon)$, $\varepsilon > 0$. Fix an $\varepsilon > 0$. In virtue of the first part of the proof it remains to show that $|e^{az}|u_\varepsilon \in \mathcal{SH}(\Omega_\varepsilon)$ for every $a \in \mathbb{C}$. Fix an $a \in \mathbb{C}$. Then

$$|e^{az}|u_\varepsilon(z) = \int_{\mathbb{D}} |e^{a(z+\varepsilon w)}|u(z + \varepsilon w)\Psi(w)|e^{-a\varepsilon w}|\, d\mathcal{L}^2(w), \quad z \in \Omega_\varepsilon.$$

Now we apply Corollary 8.2.26. $\square$

Theorem 8.2.35 (Schwarz type lemma). *Let $u : \mathbb{D} \longrightarrow [0, 1]$ be such that $\log u \in \mathcal{SH}(\mathbb{D})$, $u(0) = 0$, and*

$$\limsup_{\mathbb{D}_* \ni z \to 0} \frac{u(z)}{|z|} < +\infty.$$

([14]) That is u is *logarithmically subharmonic.*

Then

$$u(z) \le |z|,\ z \in \mathbb{D}, \quad \text{and} \quad \limsup_{\mathbb{D}_* \ni z \to 0} \frac{u(z)}{|z|} \le 1.$$

Moreover, if

$$\exists_{z_0 \in \mathbb{D}_*} :\ u(z_0) = |z_0| \quad \text{or} \quad \limsup_{\mathbb{D}_* \ni z \to 0} \frac{u(z)}{|z|} = 1,$$

then $u(z) = |z|$ for all $z \in \mathbb{D}$.

Proof. Let $v(z) := u(z)/|z|$, $z \in \mathbb{D}_*$. Since $\log v = \log u - \log |z|$, it follows that $\log v \in \mathcal{SH}(\mathbb{D}_*)$, and hence $v \in \mathcal{SH}(\mathbb{D}_*)$. By the assumption we conclude that the function v is locally bounded in $\mathbb{D}$. Hence, putting $v(0) := \limsup_{\mathbb{D}_* \ni z \to 0} v(z)$, and using Theorem 8.2.17, we obtain a function subharmonic in $\mathbb{D}$. By the maximum principle we get $v \le 1$, which gives the required inequalities.

Moreover, if $v(z_0) = 1$ for some $z_0 \in \mathbb{D}$, then $v \equiv 1$. □

Theorem 8.2.36. *Let $D \subset \mathbb{C}$ be a convex domain and let $u : D \longrightarrow \mathbb{R}$ be a convex function Then $u \in \mathcal{SH}(D)$.*

Proof. Since u is convex, it is also continuous (cf. [*Schwartz:Analiza*]). Fix an $a \in D$ and $0 < r < d_D(a)$. Then we have

$$\mathbb{J}(u; a, r) = \lim_{N \to +\infty} \sum_{j=1}^{N} \frac{1}{N} u(a + re^{i\frac{2\pi j}{N}}) \ge \lim_{N \to +\infty} u\Big(\sum_{j=1}^{N} \frac{1}{N}(a + re^{i\frac{2\pi j}{N}})\Big) = u(a).$$

It remains to apply Theorem 8.2.5. □

Theorem 8.2.37 (Hadamard's three circles theorem)**.** *Let*

$$A := \{z \in \mathbb{C} : r_1 < |z| < r_2\}$$

($0 < r_1 < r_2 < +\infty$) and let $\log u \in \mathcal{SH}(A)$. Assume that

$$\limsup_{|z| \to r_j} u(z) \le M_j, \quad j = 1, 2.$$

Then

$$u(z) \le M_1^{\frac{\log \frac{r_2}{|z|}}{\log \frac{r_2}{r_1}}} M_2^{\frac{\log \frac{|z|}{r_1}}{\log \frac{r_2}{r_1}}}, \quad z \in A.$$

Proof. For $\alpha \in \mathbb{R}$ put $u_\alpha(z) := |z|^\alpha u(z)$, $z \in A$. Observe that u_α is logarithmically subharmonic on A. Now, by the maximum principle (Corollary 8.2.4), we get

$$|z|^\alpha u(z) = u_\alpha(z) \le \max\{r_1^\alpha M_1, r_2^\alpha M_2\}, \quad z \in A.$$

Taking $\alpha \in \mathbb{R}$ so that $r_1^\alpha M_1 = r_2^\alpha M_2$, we get the required estimate. □