Jagiellonian University

Faculty of Mathematics and Computer Science

Institute of Mathematics

LECTURES

ON HOLOMORPHIC FUNCTIONS

OF SEVERAL COMPLEX

VARIABLES

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Preface

This book is based on lectures on several complex variables given by the authors at the Jagiellonian University in Kraków during the period of 1991–1999. The material contains two-semestral course for graduate students of III and IV year.

The text contains the background theory of several complex variables. Chapter I is of preparatory nature. In Chapters II-VI we discuss the extension of holomorphic functions, automorphisms, domains of holomorphy, subharmonic and plurisubharmonic functions, pseudoconvexity, the solution of the $\overline{\partial}$-problem by means of Hörmander's L^2-methods, and Cousin problems. The aim has not been to obtain completeness in any direction; in particular, several classical topics (like sheaf theory, Cartan's A and B theorems, local theory, or theory of analytic subsets) are postponed to the second part of the book, which is planned to be written in the future.

The treatment of the subject is rather classical and mostly oriented on $\overline{\partial}$-problem techniques developed by Hörmander in [17]. The reader is however encouraged to consult also other monographs on the theory of several complex variables, e.g.[15], [25] (where the approach is based on the theory of integral formulas), [13], [12] (where similar results are obtained by means of local theory); cf. also [10], [19], [23], [24], [33], [35]. We would like to stress out that the choice of bibliography reflects only personal preferences of the authors, and should be by no means treated as the try to valuate the textbooks on complex analysis.

Every chapter begins with a short summary which contains a rough outline of the material. The exercises which follow some chapters are based on problems proposed to the students during tutorials.

The reader will note that the contents of Chapter III, devoted to the theory of subharmonic and plurisubharmonic functions, is much larger than the amount of the material on this subject presented in most of textbooks on several complex variables. This is due to the traditionally strong position of the theory of plurisubharmonic functions in the Institute of Mathematics of the Jagiellonian University, and by the increasing importance of this subject to several complex variables, e.g. the recent development of the theory of the complex Monge-Ampère operator.

The reader is required to be familiar with elements of classical real analysis and complex analysis of one variable.

The references concerning one-variable theory are directed onto the textbook [4]. Of course, a similar material on one complex variable can be found in many other classical textbooks, e.g. [2], [26].

It is our great pleasure to record a debt of gratitude to our teacher, Professor Józef Siciak, who introduced us into complex analysis.

We thank Professor Peter Pflug for numerous helpful discussions and suggestions during writing this book. We are greatly indebted to our colleagues Armen Edigarian, Sławomir Kołodziej, and Włodzimierz Zwonek who have been conducting tutorials to our lectures and who helped us in corrections of the text.

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Contents

CHAPTER 1

Holomorphic functions

1.1. Formal derivatives

Let $\Omega \subset \mathbb{C}^n$ be open, let $a \in \Omega$, $j \in \{1, \dots, n\}$, and let

$$f = (f_1, \dots, f_m) : \Omega \longrightarrow \mathbb{C}^m$$

be a mapping such that

$$\frac{\partial f}{\partial x_j}(a), \quad \frac{\partial f}{\partial y_j}(a)$$

exist [1]. Define

$$\frac{\partial f}{\partial z_j}(a) := \frac{1}{2}\Big(\frac{\partial f}{\partial x_j}(a) - i\frac{\partial f}{\partial y_j}(a)\Big), \quad \frac{\partial f}{\partial \overline{z}_j}(a) := \frac{1}{2}\Big(\frac{\partial f}{\partial x_j}(a) + i\frac{\partial f}{\partial y_j}(a)\Big);$$

$\frac{\partial f}{\partial z_j}(a)$ and $\frac{\partial f}{\partial \overline{z}_j}(a)$ are called the j*-th formal partial derivatives* (or *Wirtinger derivatives*) of f at a.

Remark 1.1.1. (a)

$$\overline{\frac{\partial f}{\partial z_j}(a)} = \frac{\partial \overline{f}}{\partial \overline{z}_j}(a). \quad (^2)$$

(b) Assume that the j*-th complex partial derivative* $\frac{\partial f}{\partial z_j}(a)$ of f at a exists, i.e. the limit

$$\frac{\partial f}{\partial z_j}(a) := \lim_{\mathbb{C} \ni \lambda \to 0} \frac{1}{\lambda}(f(a + \lambda e_j) - f(a))$$

exists and is finite [3] (observe the difference between the complex partial derivative $\frac{\partial f}{\partial z_j}(a)$ and the formal partial derivative $\frac{\partial f}{\partial z_j}(a)$). Then

$$\frac{\partial f}{\partial x_j}(a) = \frac{\partial f}{\partial z_j}(a), \quad \frac{\partial f}{\partial y_j}(a) = i\frac{\partial f}{\partial z_j}(a)$$

and hence

$$\frac{\partial f}{\partial z_j}(a) = \frac{\partial f}{\partial z_j}(a), \quad \frac{\partial f}{\partial \overline{z}_j}(a) = 0.$$

Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Recall that the $\mathbb{K}$*-(Fréchet) differential* $f'_{\mathbb{K}}(a)$ of f at a is the $\mathbb{K}$-linear mapping $f'_{\mathbb{K}}(a) : \mathbb{C}^n \longrightarrow \mathbb{C}^m$ such that

$$f(a + h) = f(a) + f'_{\mathbb{K}}(a)(h) + o(\|h\|) \text{ when } \mathbb{C}^n \ni h \longrightarrow 0.$$

Obviously, if $f'_{\mathbb{C}}(a)$ exists, then $f'_{\mathbb{R}}(a)$ exists and they coincide.

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(1) We always use the following identification of $\mathbb{C}^k$ and $\mathbb{R}^{2k}$

$$\mathbb{C}^k \ni (x_1 + iy_1, \dots, x_k + iy_k) \longmapsto (x_1, y_1, \dots, x_k, y_k) \in \mathbb{R}^{2k}.$$

(2) For $w = (w_1, \dots, w_m) \in \mathbb{C}^m$ we put $\overline{w} := (\overline{w}_1, \dots, \overline{w}_m)$.
(3) $e_1, \dots, e_n$ are vectors of the canonical basis in $\mathbb{C}^n$; $e_j := (e_{j,1}, \dots, e_{j,n})$, $e_{j,k} = 0$ for $j \neq k$ and $e_{j,j} := 1$, $j = 1, \dots, n$.

If $f'_{\mathbb{R}}(a)$ exists, then $\frac{\partial f}{\partial x_j}(a)$, $\frac{\partial f}{\partial y_j}(a)$, $j = 1, \dots, n$, exist and

$$\begin{aligned} f'_{\mathbb{R}}(a)(h) &= \sum_{j=1}^{n} \Big(\frac{\partial f}{\partial x_j}(a) \operatorname{Re} h_j + \frac{\partial f}{\partial y_j}(a) \operatorname{Im} h_j \Big) \\ &= \sum_{j=1}^{n} \frac{\partial f}{\partial z_j}(a) h_j + \sum_{j=1}^{n} \frac{\partial f}{\partial \overline{z}_j}(a) \overline{h}_j, \quad h = (h_1, \dots, h_n) \in \mathbb{C}^n. \end{aligned}$$

If $f'_{\mathbb{C}}(a)$ exists, then $\frac{\partial f}{\partial z_j}(a)$, $j = 1, \dots, n$, exist and

$$f'_{\mathbb{C}}(a)(h) = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j}(a) h_j, \quad h = (h_1, \dots, h_n) \in \mathbb{C}^n.$$

Remark 1.1.2. Assume that $f'_{\mathbb{R}}(a)$ exists. Then the following conditions are equivalent:
(i) $f'_{\mathbb{R}}(a)$ is $\mathbb{C}$-linear;
(ii) $f'_{\mathbb{C}}(a)$ exists;
(iii) $\frac{\partial f}{\partial \overline{z}_j}(a) = 0$, $j = 1, \dots, n$ (i.e. f satisfies the *Cauchy–Riemann equations* at a).

Assume that $m = n$ and let

$$u_j := \operatorname{Re} f_j, \quad v_j := \operatorname{Im} f_j, \quad j = 1, \dots, n.$$

If $\frac{\partial f}{\partial x_k}(a)$ and $\frac{\partial f}{\partial y_k}(a)$, $k = 1, \dots, n$, exist, then we put

$$J_{\mathbb{R}} f(a) := \det \begin{bmatrix} \frac{\partial u_1}{\partial x_1}(a), \frac{\partial u_1}{\partial y_1}(a), \dots, \frac{\partial u_1}{\partial x_n}(a), \frac{\partial u_1}{\partial y_n}(a) \\ \frac{\partial v_1}{\partial x_1}(a), \frac{\partial v_1}{\partial y_1}(a), \dots, \frac{\partial v_1}{\partial x_n}(a), \frac{\partial v_1}{\partial y_n}(a) \\ \dots \\ \frac{\partial u_n}{\partial x_1}(a), \frac{\partial u_n}{\partial y_1}(a), \dots, \frac{\partial u_n}{\partial x_n}(a), \frac{\partial u_n}{\partial y_n}(a) \\ \frac{\partial v_n}{\partial x_1}(a), \frac{\partial v_n}{\partial y_1}(a), \dots, \frac{\partial v_n}{\partial x_n}(a), \frac{\partial v_n}{\partial y_n}(a) \end{bmatrix} \det \left[\begin{bmatrix} \frac{\partial u_j}{\partial x_k}(a), \frac{\partial u_j}{\partial y_k}(a) \\ \frac{\partial v_j}{\partial x_k}(a), \frac{\partial v_j}{\partial y_k}(a) \end{bmatrix}_{j,k=1,\dots,n} \right].$$

Observe that

$$J_{\mathbb{R}} f(a) = \det \begin{bmatrix} \frac{\partial u_1}{\partial x_1}(a), \dots, \frac{\partial u_1}{\partial x_n}(a), \frac{\partial u_1}{\partial y_1}(a), \dots, \frac{\partial u_1}{\partial y_n}(a) \\ \dots \\ \frac{\partial u_n}{\partial x_1}(a), \dots, \frac{\partial u_n}{\partial x_n}(a), \frac{\partial u_n}{\partial x_1}(a), \dots, \frac{\partial u_n}{\partial y_n}(a) \\ \frac{\partial v_1}{\partial x_1}(a), \dots, \frac{\partial v_1}{\partial x_n}(a), \frac{\partial v_1}{\partial x_1}(a), \dots, \frac{\partial v_1}{\partial y_n}(a) \\ \dots \\ \frac{\partial v_n}{\partial x_1}(a), \dots, \frac{\partial v_n}{\partial x_n}(a), \frac{\partial v_n}{\partial y_1}(a), \dots, \frac{\partial v_n}{\partial y_n}(a) \end{bmatrix} = \det \begin{bmatrix} \Big[\frac{\partial u_j}{\partial x_k}(a)\Big]_{j,k=1,\dots,n}, \Big[\frac{\partial u_j}{\partial y_k}(a)\Big]_{j,k=1,\dots,n} \\ \Big[\frac{\partial v_j}{\partial x_k}(a)\Big]_{j,k=1,\dots,n}, \Big[\frac{\partial v_j}{\partial y_k}(a)\Big]_{j,k=1,\dots,n} \end{bmatrix}.$$

Define

$$J_{\mathbb{R},\mathcal{W}} f(a) := \det \begin{bmatrix} \frac{\partial f_1}{\partial z_1}(a), \dots, \frac{\partial f_1}{\partial z_n}(a), \frac{\partial f_1}{\partial \overline{z}_1}(a), \dots, \frac{\partial f_1}{\partial \overline{z}_n}(a) \\ \dots \\ \frac{\partial f_n}{\partial z_1}(a), \dots, \frac{\partial f_n}{\partial z_n}(a), \frac{\partial f_n}{\partial \overline{z}_1}(a), \dots, \frac{\partial f_n}{\partial \overline{z}_n}(a) \\ \frac{\partial \overline{f}_1}{\partial z_1}(a), \dots, \frac{\partial \overline{f}_1}{\partial z_n}(a), \frac{\partial \overline{f}_1}{\partial \overline{z}_1}(a), \dots, \frac{\partial \overline{f}_1}{\partial \overline{z}_n}(a) \\ \dots \\ \frac{\partial \overline{f}_n}{\partial z_1}(a), \dots, \frac{\partial \overline{f}_n}{\partial z_n}(a), \frac{\partial \overline{f}_n}{\partial \overline{z}_1}(a), \dots, \frac{\partial \overline{f}_n}{\partial \overline{z}_n}(a) \end{bmatrix} \det \begin{bmatrix} \Big[\frac{\partial f_j}{\partial z_k}(a)\Big]_{j,k=1,\dots,n}, \Big[\frac{\partial f_j}{\partial \overline{z}_k}(a)\Big]_{j,k=1,\dots,n} \\ \Big[\frac{\partial \overline{f}_j}{\partial z_k}(a)\Big]_{j,k=1,\dots,n}, \Big[\frac{\partial \overline{f}_j}{\partial \overline{z}_k}(a)\Big]_{j,k=1,\dots,n} \end{bmatrix}.$$

It is clear that if $\frac{\partial f}{\partial \overline{z}_k}(a) = 0$, $k = 1, \dots, n$, then

$$J_{\mathbb{R},\mathcal{W}} f(a) = \left| \det \Big[\frac{\partial f_j}{\partial z_k}(a) \Big]_{j,k=1,\dots,n} \right|^2.$$

If $\frac{\partial f_j}{\partial z_k}(a)$, $j,k=1,\dots,n$, exist, then we define

$$J_{\mathbb{C}}f(a) := \det\begin{bmatrix} \frac{\partial f_1}{\partial z_1}(a),\dots,\frac{\partial f_1}{\partial z_n}(a) \\ \dots \\ \frac{\partial f_n}{\partial z_1}(a),\dots,\frac{\partial f_n}{\partial z_n}(a)\end{bmatrix} = \det\left[\frac{\partial f_j}{\partial z_k}(a)\right]_{j,k=1,\dots,n}.$$

Proposition 1.1.3.

$$J_{\mathbb{R}}f(a) = J_{\mathbb{R},\mathcal{W}}f(a).$$

In particular, if $\partial f_k(a)$, $k=1,\dots,n$, *exist, then*

$$J_{\mathbb{R}}f(a) = |J_{\mathbb{C}}f(a)|^2.$$

Proof. $J_{\mathbb{R},\mathcal{W}}f(a)$

$$=\det\begin{bmatrix} \left[\frac12\Big(\frac{\partial f_j}{\partial x_k}(a) - i\frac{\partial f_j}{\partial y_k}(a)\Big)\right]_{j,k=1,\dots,n}, & \left[\frac12\Big(\frac{\partial f_j}{\partial x_k}(a) + i\frac{\partial f_j}{\partial y_k}(a)\Big)\right]_{j,k=1,\dots,n} \\ \left[\frac12\Big(\frac{\partial \overline{f}_j}{\partial x_k}(a) - i\frac{\partial \overline{f}_j}{\partial y_k}(a)\Big)\right]_{j,k=1,\dots,n}, & \left[\frac12\Big(\frac{\partial \overline{f}_j}{\partial x_k}(a) + i\frac{\partial \overline{f}_j}{\partial y_k}(a)\Big)\right]_{j,k=1,\dots,n}\end{bmatrix}$$

$$=\frac{1}{4^n}\det\begin{bmatrix} \left[\frac{\partial f_j}{\partial x_k}(a) - i\frac{\partial f_j}{\partial y_k}(a)\right]_{j,k=1,\dots,n}, & \left[\frac{\partial f_j}{\partial x_k}(a) + i\frac{\partial f_j}{\partial y_k}(a)\right]_{j,k=1,\dots,n} \\ \left[\frac{\partial \overline{f}_j}{\partial x_k}(a) - i\frac{\partial \overline{f}_j}{\partial y_k}(a)\right]_{j,k=1,\dots,n}, & \left[\frac{\partial \overline{f}_j}{\partial x_k}(a) + i\frac{\partial \overline{f}_j}{\partial y_k}(a)\right]_{j,k=1,\dots,n}\end{bmatrix}$$

we add the $(n+k)$-th column to the k-th column

$$=\frac{1}{2^n}\det\begin{bmatrix} \left[\frac{\partial f_j}{\partial x_k}(a)\right]_{j,k=1,\dots,n}, & \left[\frac{\partial f_j}{\partial x_k}(a) + i\frac{\partial f_j}{\partial y_k}(a)\right]_{j,k=1,\dots,n} \\ \left[\frac{\partial \overline{f}_j}{\partial x_k}(a)\right]_{j,k=1,\dots,n}, & \left[\frac{\partial \overline{f}_j}{\partial x_k}(a) + i\frac{\partial \overline{f}_j}{\partial y_k}(a)\right]_{j,k=1,\dots,n}\end{bmatrix}$$

we subtract the k-th column from the $(n+k)$-th column

$$=\Big(\frac{i}{2}\Big)^n\det\begin{bmatrix} \left[\frac{\partial f_j}{\partial x_k}(a)\right]_{j,k=1,\dots,n}, & \left[\frac{\partial f_j}{\partial y_k}(a)\right]_{j,k=1,\dots,n} \\ \left[\frac{\partial \overline{f}_j}{\partial x_k}(a)\right]_{j,k=1,\dots,n}, & \left[\frac{\partial \overline{f}_j}{\partial y_k}(a)\right]_{j,k=1,\dots,n}\end{bmatrix}$$

$$=\Big(\frac{i}{2}\Big)^n\det\begin{bmatrix} \left[\frac{\partial u_j}{\partial x_k}(a) + i\frac{\partial v_j}{\partial x_k}(a)\right]_{j,k=1,\dots,n}, & \left[\frac{\partial u_j}{\partial y_k}(a) + i\frac{\partial v_j}{\partial y_k}(a)\right]_{j,k=1,\dots,n} \\ \left[\frac{\partial u_j}{\partial x_k}(a) - i\frac{\partial v_j}{\partial x_k}(a)\right]_{j,k=1,\dots,n}, & \left[\frac{\partial u_j}{\partial y_k}(a) - i\frac{\partial v_j}{\partial y_k}(a)\right]_{j,k=1,\dots,n}\end{bmatrix}$$

we add the $(n+k)$-th row to the k-th row

$$=i^n\det\begin{bmatrix} \left[\frac{\partial u_j}{\partial x_k}(a)\right]_{j,k=1,\dots,n}, & \left[\frac{\partial u_j}{\partial y_k}(a)\right]_{j,k=1,\dots,n} \\ \left[\frac{\partial u_j}{\partial x_k}(a) - i\frac{\partial v_j}{\partial x_k}(a)\right]_{j,k=1,\dots,n}, & \left[\frac{\partial u_j}{\partial y_k}(a) - i\frac{\partial v_j}{\partial y_k}(a)\right]_{j,k=1,\dots,n}\end{bmatrix}$$

we subtract the k-th row from the $(n+k)$-th row

$$=\det\begin{bmatrix} \left[\frac{\partial u_j}{\partial x_k}(a)\right]_{j,k=1,\dots,n}, & \left[\frac{\partial u_j}{\partial y_k}(a)\right]_{j,k=1,\dots,n} \\ \left[\frac{\partial v_j}{\partial x_k}(a)\right]_{j,k=1,\dots,n}, & \left[\frac{\partial v_j}{\partial y_k}(a)\right]_{j,k=1,\dots,n}\end{bmatrix} = J_{\mathbb{R}}f(a).$$ □

In the sequel we will use also the following differential operators

$$D^{\alpha,\beta}:\mathcal{C}^k(\Omega,\mathbb{C}^m)\longrightarrow\mathcal{C}^{k-|\alpha|-|\beta|}(\Omega,\mathbb{C}^m),\qquad D^{\alpha,\beta}:=(\frac{\partial}{\partial z_1})^{\alpha_1}\circ\dots\circ(\frac{\partial}{\partial z_n})^{\alpha_n}\circ(\frac{\partial}{\partial \overline{z}_1})^{\beta_1}\circ\dots\circ(\frac{\partial}{\partial \overline{z}_n})^{\beta_n},$$

where $\alpha, \beta \in \mathbb{N}_0^n$ (4) , $|\alpha| + |\beta| \le k$. Moreover, we put

$$D^\alpha := D^{\alpha,0} = (\frac{\partial}{\partial z_1})^{\alpha_1} \circ \cdots \circ (\frac{\partial}{\partial z_n})^{\alpha_n}, \quad \alpha \in \mathbb{N}_0^n,\ |\alpha| \le k.$$

1.2. Separately holomorphic functions

Let $\Omega \subset \mathbb{C}^n$ be open and let $f : \Omega \longrightarrow \mathbb{C}^m$. Given $a \in \mathbb{C}^n$ and $X \in \mathbb{C}^n$, define

$$\Omega_{a,X} := \{\lambda \in \mathbb{C} : a + \lambda X \in \Omega\}, \qquad f_{a,X}(\lambda) := f(a + \lambda X), \quad \lambda \in \Omega_{a,X}.$$

Note that $\Omega_{a,X}$ is an open subset of $\mathbb{C}$.

Definition 1.2.1. A function $f : \Omega \longrightarrow \mathbb{C}$ is *separately holomorphic* on Ω ($f \in \mathcal{O}_s(\Omega)$) if

$$f_{a,e_j} \in \mathcal{O}(\Omega_{a,e_j}), \quad a \in \Omega,\ j = 1, \dots, n. \quad (5)$$

Observe that $\mathcal{O}_s(\Omega)$ is a ring. Obviously, if $n = 1$, then $\mathcal{O}_s(\Omega) = \mathcal{O}(\Omega)$.

Remark 1.2.2. (a) $f \in \mathcal{O}_s(\Omega)$ iff $\frac{\partial f}{\partial z_j}(a)$ exists for any $a \in \Omega$ and $j = 1, \dots, n$.
(b) A function $f : \Omega \longrightarrow \mathbb{C}$ is separately holomorphic in Ω iff every point $a \in \Omega$ admits an open neighborhood $U_a \subset \Omega$ such that $f|_{U_a} \in \mathcal{O}_s(U_a)$ (6) .

Proposition 1.2.3 (Osgood). *For $f \in \mathcal{O}_s(\Omega)$ the following conditions are equivalent:*
(i) *$f \in \mathcal{C}(\Omega)$;*
(ii) *f is locally bounded.*

Proof. The implication (i) $\Longrightarrow$ (ii) is trivial.

To prove the implication (ii) $\Longrightarrow$ (i) fix an $a = (a_1, \dots, a_n) \in \Omega$ and $r > 0$ such that $\mathbb{P}(a, r) \subset\subset \Omega$ (7) . Put $C := \sup_{\mathbb{P}(a,r)} |f|$. Observe that for any $(b_1, \dots, b_n) \in \mathbb{P}(a, r)$ the functions

$$K(a_j, r) \ni \lambda \longmapsto f(b_1, \dots, b_{j-1}, \lambda, b_{j+1}, \dots, b_n), \quad j = 1, \dots, n,$$

are holomorphic. Hence, by the classical Schwarz lemma (cf. [4], Th. VI.2.1), we get

$$\begin{aligned}|f(z) - f(a)| &\le |f(z_1, a_2, \dots, a_n) - f(a_1, a_2, \dots, a_n)| + \cdots + |f(z_1, \dots, z_{n-1}, z_n) - f(z_1, \dots, z_{n-1}, a_n)| \\ &\le \frac{2C}{r}(|z_1 - a_1| + \cdots + |z_n - a_n|), \quad z = (z_1, \dots, z_n) \in \mathbb{P}(a, r),\end{aligned}$$

which implies that f is continuous at a. $\square$

Definition 1.2.4. We say that a bounded domain $D \subset \mathbb{C}$ is *regular* if ∂D is the finite union of images of pairwise disjoint Jordan piecewise $\mathcal{C}^1$ curves having positive orientation with respect to D.

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(4) $A_+ := \{a \in A : a \ge 0\}$. To simplify notation we write $\mathbb{N}_0^n$ instead of $(\mathbb{N}_0)^n$.

(5) If $U \subset \mathbb{C}$ is open, then $\mathcal{O}(U)$ denotes the space of all holomorphic functions on U in the sense of the one-variable theory.

(6) That is, f is separately holomorphic iff f is locally separately holomorphic.

(7) $\mathbb{P}(a, \boldsymbol{r}) = \mathbb{P}_n(a, \boldsymbol{r}) := K(a_1, r_1) \times \cdots \times K(a_n, r_n)$, where $K(a, r) := \{z \in \mathbb{C} : |z - a| < r\}$; $\mathbb{P}(a, r) = \mathbb{P}_n(a, r) := \mathbb{P}(a, (r, \dots, r))$.

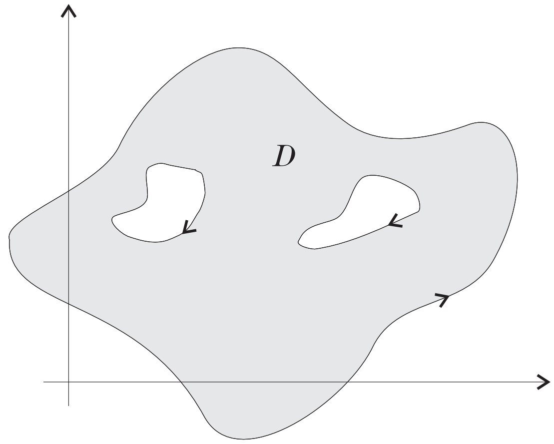

Figure 1.2.1

Proposition 1.2.5 (Cauchy's integral formula)**.** *Let $D_1, \dots, D_n \subset \mathbb{C}$ be regular domains. Put $D := D_1 \times \dots \times D_n$, $\partial_0 D := \partial D_1 \times \dots \times \partial D_n$ and let $f \in \mathcal{O}_s(D) \cap \mathcal{C}(\overline{D})$. Then*

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\partial D_1} \cdots \int_{\partial D_n} \frac{f(\zeta_1, \dots, \zeta_n)}{(\zeta_1 - z_1) \cdot \dots \cdot (\zeta_n - z_n)} d\zeta_1 \dots d\zeta_n =: \frac{1}{(2\pi i)^n} \int_{\partial_0 D} \frac{f(\zeta)}{\zeta - z} d\zeta,$$
$$z = (z_1, \dots, z_n) \in D. \quad (1.2.1)$$

The set $\partial_0 D$ is called the *distinguished boundary* of D. Observe that the integral is well defined and independent of the order of integration.

Proof. We apply induction with respect to n. For $n = 1$ the result reduces to the classical Cauchy integral formula (cf. [4], IV.5).

$n - 1 \rightsquigarrow n$. Fix an $a = (a', a_n) \in D' \times D_n := (D_1 \times \dots \times D_{n-1}) \times D_n$. We have

$$f(a) = f(a', a_n) = \frac{1}{(2\pi i)^{n-1}} \int_{\partial_0 (D')} \frac{f(\zeta', a_n)}{\zeta' - a'} d\zeta'. \quad (1.2.2)$$

Observe that $f(\zeta', \cdot) \in \mathcal{O}(D_n) \cap \mathcal{C}(\overline{D}_n)$ for any $\zeta' \in \partial_0 D'$.

Indeed, fix a $\zeta' \in \partial_0 D'$ and let $D' \ni \zeta'_\nu \longrightarrow \zeta'$. Since f is separately holomorphic, $f(\zeta'_\nu, \cdot) \in \mathcal{O}(D_n)$ for any ν. Obviously, $f(\zeta'_\nu, \cdot) \longrightarrow f(\zeta', \cdot)$ uniformly on D_n. Hence, by Weierstrass' theorem (cf. [4], Th. VII.2.1), $f(\zeta', \cdot) \in \mathcal{O}(D_n)$.

Consequently, by the classical Cauchy formula,

$$f(\zeta', a_n) = \frac{1}{2\pi i} \int_{\partial D_n} \frac{f(\zeta', \zeta_n)}{\zeta_n - a_n} d\zeta_n,$$

which together with (1.2.2) gives (1.2.1). $\square$

1.3. Domains of convergence of power series

Definition 1.3.1. Any series

$$\sum_{\alpha \in \mathbb{N}_0^n} a_\alpha (z - z_0)^\alpha, \quad z \in \mathbb{C}^n,$$

where $(a_\alpha)_{\alpha \in \mathbb{N}_0^n} \subset \mathbb{C}$ and $z_0 \in \mathbb{C}^n$ (8) , is called a *power series* with center at z_0.

In other words, a power series with center at z_0 is the series generated by a family $(\mathbb{C}^n \ni z \longmapsto a_\alpha (z - z_0)^\alpha)_{\alpha \in \mathbb{N}_0^n}$.

(8) $w^\alpha := w_1^{\alpha_1} \cdot \dots \cdot w_n^{\alpha_n}$, $w = (w_1, \dots, w_n) \in \mathbb{C}^n$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$; $0^0 := 1$.

Example 1.3.2. (a) The *geometric series*

$$\sum_{\alpha\in\mathbb{N}_0^n}\frac{z^\alpha}{\boldsymbol{r}^\alpha},$$

where $\boldsymbol{r}\in\mathbb{R}^n_{>0}$ [9], is locally normally convergent in $\mathbb{P}(\boldsymbol{r})$ [10] to the function

$$\mathbb{P}(\boldsymbol{r})\ni(z_1,\dots,z_n)\longmapsto\prod_{j=1}^n\frac{1}{1-z_j/r_j}.$$

(b) The series

$$\sum_{\alpha\in\mathbb{Z}^n}\Big(\frac{z_1}{r_1}\Big)^{|\alpha_1|}\dots\Big(\frac{z_n}{r_n}\Big)^{|\alpha_n|},$$

where $\boldsymbol{r}\in\mathbb{R}^n_{>0}$, is locally normally convergent in $\mathbb{P}(\boldsymbol{r})$ to the function

$$\mathbb{P}(\boldsymbol{r})\ni(z_1,\dots,z_n)\longmapsto\prod_{j=1}^n\frac{r_j+z_j}{r_j-z_j}.$$

Proposition 1.3.3 (Abel's lemma)**.** *If*

$$|a_\alpha|\boldsymbol{r}^\alpha\le C,\quad \alpha\in\mathbb{N}_0^n,$$

where $\boldsymbol{r}\in\mathbb{R}^n_{>0}$, then the series $\sum_{\alpha\in\mathbb{N}_0^n}a_\alpha z^\alpha$ is locally normally convergent in $\mathbb{P}(\boldsymbol{r})$.

Proof. Since $|a_\alpha z^\alpha|\le C|z^\alpha|/\boldsymbol{r}^\alpha$, the result follows immediately from Example 1.3.2(a). □

Definition 1.3.4. A set $A\subset\mathbb{C}^n$ is called:
- *circular* if $\lambda a\in A$ for arbitrary $\lambda\in\mathbb{T}$, $a\in A$ [11];
- *n-circled* if $(\lambda_1a_1,\dots,\lambda_na_n)\in A$ for arbitrary $\lambda_1,\dots,\lambda_n\in\mathbb{T}$, $(a_1,\dots,a_n)\in A$;
- *balanced* if $\lambda a\in A$ for arbitrary $\lambda\in\overline{\mathbb{D}}$, $a\in A$;
- *complete n-circled* if $(\lambda_1a_1,\dots,\lambda_na_n)\in A$ for arbitrary $\lambda_1,\dots,\lambda_n\in\overline{\mathbb{D}}$, $(a_1,\dots,a_n)\in A$ [12].

Observe that

$$\begin{tikzcd}
A \text{ is complete } n\text{-circled} \arrow[r] \arrow[dr] & A \text{ is } n\text{-circled} \arrow[r] & A \text{ is circular} \\
& A \text{ is balanced} \arrow[ur] &
\end{tikzcd}$$

Let

$$\mathbb{C}^n\ni(z_1,\dots,z_n)\overset{R}{\longmapsto}(|z_1|,\dots,|z_n|)\in\mathbb{R}^n_+.$$

Observe that a set $A\subset\mathbb{C}^n$ is n-circled iff $A=R^{-1}(R(A))$. Consequently, any n-circled set $A\subset\mathbb{C}^n$ is completely determined by the set $R(A)\subset\mathbb{R}^n_+$. Obviously, if $A\subset\mathbb{C}^n$ is n-circled, then $R(A)=A\cap\mathbb{R}^n_+$.

---

[9] $A_{>0}:=\{a\in A: a>0\}$. To simplify notation we write $\mathbb{R}^n_{>0}$ instead of $(\mathbb{R}_{>0})^n$.
[10] $\mathbb{P}(\boldsymbol{r})=\mathbb{P}_n(\boldsymbol{r}):=\mathbb{P}(0,\boldsymbol{r})$, $\mathbb{P}(r)=\mathbb{P}_n(r):=\mathbb{P}(0,r)$.
[11] $\mathbb{D}$ denotes the unit disc.
[12] More generally: if $a_0\in\mathbb{C}^n$ is fixed, then the set A is called *circular with respect to* a_0 if $a_0+\lambda(a-a_0)\in A$ for arbitrary $\lambda\in\mathbb{T}$, $a\in A$. The other definitions may be generalized similarly.

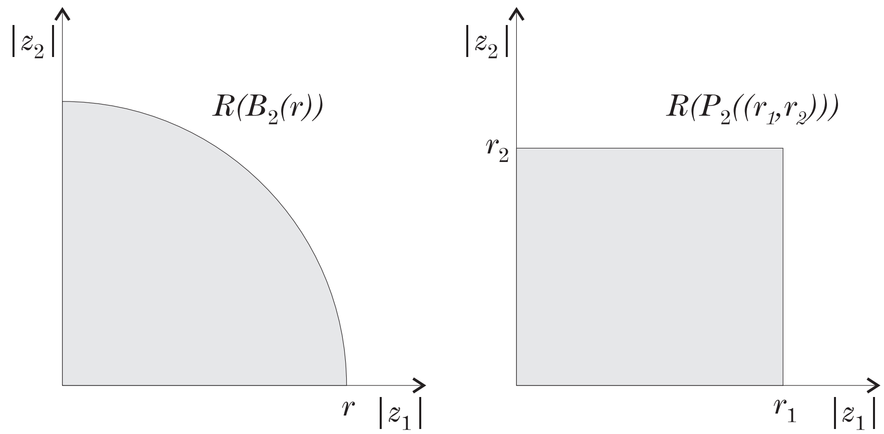

Figure 1.3.1

The mapping R is open. Consequently, if $A \subset \mathbb{C}^n$ is n-circled, then A is open in $\mathbb{C}^n$ iff $R(A)$ is open in $\mathbb{R}^n_+$.

Moreover, if $B \subset \mathbb{R}^n_+$ is arcwise connected, then so is $R^{-1}(B)$. In particular, if $A \subset \mathbb{C}^n$ is n-circled, then A is a domain in $\mathbb{C}^n$ iff $R(A)$ is a domain in $\mathbb{R}^n_+$ (cf. Exercise 1.3).

For every n-circled set $A \subset \mathbb{C}^n$ put

$$\log A := \{(x_1, \dots, x_n) \in \mathbb{R}^n : (e^{x_1}, \dots, e^{x_n}) \in A\}.$$

The set $\log A$ is called the *logarithmic image* of A. Note that $\log A = \log(A \cap \mathbb{R}^n_{>0})$.

We say that A is *logarithmically convex* (*log-convex*) if $\log A$ is convex.

Notice that A is logarithmically convex iff for any $(x_1, \dots, x_n)$, $(y_1, \dots, y_n) \in A \cap \mathbb{R}^n_{>0}$ and for any $t \in [0, 1]$ the point $(x_1^{1-t} y_1^t, \dots, x_n^{1-t} y_n^t)$ belongs to A.

We will see (cf. the Riemann removable singularities theorem 2.1.6) that if $D \subset \mathbb{C}^n$ is a domain, then $D \setminus \{(z_1, \dots, z_n) \in \mathbb{C}^n : z_1 \cdot \dots \cdot z_n = 0\}$ is connected. In particular, if $D \subset \mathbb{C}^n$ is an n-circled domain, then $\log D$ is a domain in $\mathbb{R}^n$ (cf. Exercise 1.15).

Example 1.3.5. Let

$$D := \{(z_1, \dots, z_n) \in \mathit{\Omega}(\alpha) : |z_1|^{\alpha_1} \dots |z_n|^{\alpha_n} < C\},$$

where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$, $C > 0$, and $\mathit{\Omega}(\alpha) := U(\alpha_1) \times \dots \times U(\alpha_n)$ with

$$U(x) := \begin{cases} \mathbb{C} & \text{if } x \geq 0 \\ \mathbb{C}_* & \text{if } x < 0 \end{cases}, \quad x \in \mathbb{R}.$$

Then

$$\log D = \{x \in \mathbb{R}^n : \langle x, \alpha \rangle < \log C\},$$

where $\langle\ ,\ \rangle$ denotes the standard scalar product in $\mathbb{R}^n$.

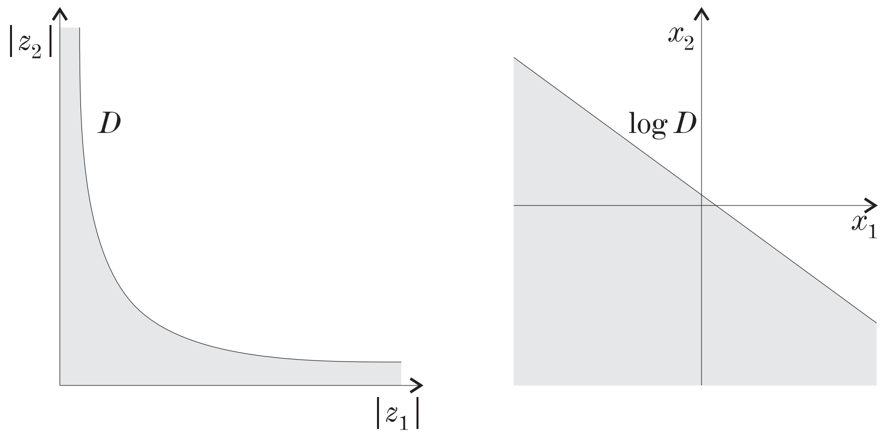

Figure 1.3.2

Fix a power series

$$\Sigma = \sum_{\alpha \in \mathbb{N}_0^n} a_\alpha z^\alpha$$

and let

$$\begin{aligned}\mathfrak{B} = \mathfrak{B}(\Sigma) &:= \{z \in \mathbb{C}^n : \exists_{C>0} : \forall_{\alpha \in \mathbb{N}_0^n} :\ |a_\alpha z^\alpha| \leq C\},\\ \mathfrak{C} = \mathfrak{C}(\Sigma) &:= \{z \in \mathbb{C}^n : \Sigma \text{ is summable at } z\},\\ \mathfrak{D} = \mathfrak{D}(\Sigma) &:= \operatorname{int} \mathfrak{C}.\end{aligned}$$

Clearly $\mathfrak{D} \subset \mathfrak{C} \subset \mathfrak{B}$. The set $\mathfrak{D}$ is called the *domain of convergence* of Σ. We will see (Proposition 1.3.6) that $\mathfrak{D}$ is connected and, therefore, is indeed a domain in $\mathbb{C}^n$.

Recall that for $n = 1$, if $\varnothing \neq \mathfrak{D} \neq \mathbb{C}$, then $\overline{\mathfrak{B}} = \overline{\mathfrak{C}} = \overline{\mathfrak{D}} = \overline{K}(R)$ (13), where R is the radius of convergence of Σ. For $n > 1$ the situation is more complicated, for instance if $\Sigma := \sum_{\nu=0}^\infty z_1^\nu z_2$, then $\mathbb{C} \times \{0\} \subset \mathfrak{C}$, but $\mathfrak{D} = \mathbb{D} \times \mathbb{C}$.

Proposition 1.3.6. (a) *The set $\mathfrak{B}$ is complete n-circled and log-convex.*
(b) $\mathfrak{D} = \operatorname{int} \mathfrak{B}$. *In particular, $\mathfrak{D}$ is a complete n-circled and log-convex domain* (14).
(c) *The series Σ is locally normally convergent in $\mathfrak{D}$.*

All the above properties (after formal changes) remain true for power series with an arbitrary center.

Proof. (a) is obvious. Notice that

$$\log \mathfrak{B} = \{x \in \mathbb{R}^n : \exists_{C>0} :\ \forall_{\alpha \in \mathbb{N}_0^n} : \langle x, \alpha\rangle \leq \log C - \log |a_\alpha|\}.$$

(b) Since $\mathfrak{D} = \operatorname{int} \mathfrak{C} \subset \operatorname{int} \mathfrak{B}$, it remains to prove that $\operatorname{int} \mathfrak{B} \subset \mathfrak{D}$. Let $a \in \operatorname{int} \mathfrak{B}$. Since $\mathfrak{B}$ is complete n-circled, there exists an $\boldsymbol{r} \in \mathbb{R}^n_{>0} \cap \mathfrak{B}$ such that $a \in \mathbb{P}(\boldsymbol{r})$. Now, by Abel's lemma, we have $\mathbb{P}(\boldsymbol{r}) \subset \mathfrak{C}$. Hence $a \in \mathfrak{D}$.

(c) Take a point $a \in \mathfrak{D}$ and let $\boldsymbol{r} \in \mathbb{R}^n_{>0} \cap \mathfrak{B}$ and $0 < \theta < 1$ be such that $a \in \mathbb{P}(\theta\boldsymbol{r})$. By Abel's lemma, the series is convergent normally in $\mathbb{P}(\theta\boldsymbol{r})$ and, therefore, it is convergent normally in a neighborhood of a. □

(13) $K(r) := K(0, r)$.

(14) Observe that if $A \subset \mathbb{C}^n$ is n-circled (resp. complete n-circled), then $\operatorname{int} A$ is n-circled (resp. complete n-circled). Moreover, if $A \subset \mathbb{C}^n$ is n-circled, then $\operatorname{int} \log A = \log \operatorname{int} A$.

Let $f : \Omega \longrightarrow \mathbb{C}$ be a function having all complex derivatives at any point of Ω. Put

$$T_a f(z) := \sum_{\alpha\in\mathbb{N}_0^n} \frac{D^\alpha f(a)}{\alpha!}(z-a)^\alpha, \quad a \in \Omega. \quad (^{15})$$

The series $T_a f$ is called the *Taylor series* of f at a. The number

$$d(T_a f) := \sup\{r > 0 : \mathbb{P}(a,r) \subset \boldsymbol{\mathfrak{D}}(T_a f)\}$$

is called the *radius of convergence* of $T_a f$.

Proposition 1.3.7. *Assume that* $D = \boldsymbol{\mathfrak{D}}(\Sigma) \neq \varnothing$ *and let*

$$f(z) := \sum_{\alpha\in\mathbb{N}_0^n} a_\alpha z^\alpha, \quad z \in D.$$

For $\beta \in \mathbb{N}_0^n$ *let* $D^\beta\Sigma$ *denote the series*

$$\sum_{\alpha\in\mathbb{N}_0^n:\ \alpha\geq\beta} \binom{\alpha}{\beta} \beta!\, a_\alpha\, z^{\alpha-\beta}. \quad (^{16})$$

Then f *has all complex derivatives in* D, $D \subset \boldsymbol{\mathfrak{D}}(D^\beta\Sigma)$, *and*

$$D^\beta f(z) = \sum_{\alpha\in\mathbb{N}_0^n:\ \alpha\geq\beta} \binom{\alpha}{\beta} \beta!\, a_\alpha\, z^{\alpha-\beta}, \quad z \in D,\ \beta \in \mathbb{N}_0^n.$$

In particular, $\Sigma = T_0 f$ *and* $f(z) = T_0 f(z)$ *for* $z \in D$.

All the aforementioned properties of the series Σ remain valid (with obvious changes) for power series with arbitrary center.

Notice the following difference between one and several variables. For $n = 1$ the radius of convergence of Σ is equal to the radius of convergence of the series of derivatives. This is no longer true for $n > 1$, for instance if

$$\Sigma := \sum_{\nu=0}^\infty z_1^\nu + \sum_{\nu=0}^\infty z_2^\nu,$$

then $\boldsymbol{\mathfrak{D}}(\Sigma) = \mathbb{D}\times\mathbb{D}$, but $\boldsymbol{\mathfrak{D}}(\frac{\partial\Sigma}{\partial z_1}) = \mathbb{D}\times\mathbb{C}$ $(^{17})$.

Proof. It is sufficient to consider the case $\beta = e_j$ for some $j \in \{1,\dots,n\}$. We show first that the series

$$\frac{\partial\Sigma}{\partial z_j} = \sum_{\alpha\in\mathbb{N}_0^n:\ \alpha\geq e_j} \alpha_j a_\alpha\, z^{\alpha-e_j}$$

is locally normally convergent in D. It is sufficient to prove that if $\boldsymbol{r} \in \mathbb{R}_{>0}^n \cap \boldsymbol{\mathfrak{B}}(\Sigma)$, then the series $\partial_j\Sigma$ is locally normally convergent in $\mathbb{P}(\boldsymbol{r})$. Let $C > 0$ be such that $|a_\alpha|\boldsymbol{r}^\alpha \leq C$, $\alpha \in \mathbb{N}_0^n$. Then for any $0 < \theta < 1$ we have

$$\sum_{\alpha\in\mathbb{N}_0^n:\ \alpha\geq e_j} \sup_{\mathbb{P}(\theta\boldsymbol{r})}\{|\alpha_j a_\alpha\, z^{\alpha-e_j}|\} \leq \frac{C}{\theta r_j} \sum_{\alpha\in\mathbb{N}_0^n:\ \alpha\geq e_j} \alpha_j \theta^{|\alpha|},$$

which gives the normal convergence in $\mathbb{P}(\theta\boldsymbol{r})$.

Now fix a $z_0 \in D$. The series

$$\sum_{\alpha\in\mathbb{N}_0^n} (a_\alpha z^\alpha)_{z_0,e_j}(\lambda) = \sum_{\alpha\in\mathbb{N}_0^n} a_\alpha (z_0 + \lambda e_j)^\alpha$$

$(^{15})$ $\alpha! := \alpha_1! \cdot \dots \cdot \alpha_n!$, $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{N}_0^n$.

$(^{16})$ $\binom{\alpha}{\beta} := \binom{\alpha_1}{\beta_1} \cdot \dots \cdot \binom{\alpha_n}{\beta_n}$, $\alpha = (\alpha_1,\dots,\alpha_n)$, $\beta = (\beta_1,\dots,\beta_n) \in \mathbb{N}_0^n$, $\alpha \leq \beta$.

$(^{17})$ $\partial_1\Sigma = \sum_{\nu=1}^\infty \nu z_1^{\nu-1}$.

converges locally normally in D_{z_0,e_j} to the function f_{z_0,e_j}. Hence, by the classical one-dimensional Weierstrass theorem (cf. [4], Th. VII.2.1), the function f_{z_0,e_j} is holomorphic (in particular the derivative $\frac{\partial f}{\partial z_j}(z_0) = (f_{z_0,e_j})'(0)$ exists) and

$$\frac{\partial f}{\partial z_j}(z_0) = (f_{z_0,e_j})'(0) = \sum_{\alpha\in\mathbb{N}_0^n}((a_\alpha z^\alpha)_{z_0,e_j})'(0) = \sum_{\alpha\in\mathbb{N}_0^n:\ \alpha\geq e_j}\alpha_j a_\alpha\, z_0^{\alpha-e_j}. \qquad \square$$

1.4. Holomorphic functions

Definition 1.4.1. Let $\Omega\subset\mathbb{C}^n$ be open. Put

$$d_\Omega(a) := \sup\{r>0 : \mathbb{P}(a,r)\subset\Omega\},\quad a\in\Omega.$$

We say that a function $f:\Omega\longrightarrow\mathbb{C}$ is *holomorphic* in Ω ($f\in\mathcal{O}(\Omega)$) if for any $a\in\Omega$ there exists a power series

$$\sum_{\alpha\in\mathbb{N}_0^n} a_\alpha(z-a)^\alpha$$

and $0<r\leq d_\Omega(a)$ such that

$$f(z) = \sum_{\alpha\in\mathbb{N}_0^n} a_\alpha(z-a)^\alpha,\quad z\in\mathbb{P}(a,r). \quad ([18])$$

A mapping $f=(f_1,\dots,f_m):\Omega\longrightarrow\mathbb{C}^m$ is called *holomorphic* ($f\in\mathcal{O}(\Omega,\mathbb{C}^m)$) if $f_1,\dots,f_m\in\mathcal{O}(\Omega)$.
The functions from $\mathcal{O}(\mathbb{C}^n)$ are called *entire* functions.

Remark 1.4.2. (a) $\mathcal{O}(\Omega)$ is a ring.
(b) A function $f:\Omega\longrightarrow\mathbb{C}$ is holomorphic on Ω iff for any point $a\in\Omega$ there exists a neighborhood U_a such that $f|_{U_a}\in\mathcal{O}(U_a)$.
(c) Every polynomial of n complex variables is an entire function, i.e. $\mathcal{P}(\mathbb{C}^n)\subset\mathcal{O}(\mathbb{C}^n)$.
(d) Holomorphic functions are infinitely differentiable in the complex sense (by Proposition 1.3.7).
(e) If $f\in\mathcal{O}(\Omega)$, then $D^\alpha f\in\mathcal{O}(\Omega)$ for arbitrary $\alpha\in\mathbb{N}_0^n$.

Proposition 1.4.3 (Identity principle)**.** *Let $f,g\in\mathcal{O}(D)$, where $D\subset\mathbb{C}^n$ is a domain. Then the following conditions are equivalent:*
(i) *$f\equiv g$;*
(ii) *there exists an $a\in D$ such that $T_af=T_ag$;*
(iii) *$\operatorname{int}(\{z\in D: f(z)=g(z)\})\neq\varnothing$.*

Proof. Clearly (i) $\Longrightarrow$ (ii) $\Longleftrightarrow$ (iii). To prove the implication (ii) $\Longrightarrow$ (i) it is sufficient to note that the set $D_0:=\{z\in D: T_zf=T_zg\}$ is non-empty open and closed in D. $\square$

Lemma 1.4.4. *Let $\gamma_j:[0,1]\longrightarrow\mathbb{C}$ be a piecewise $\mathcal{C}^1$ curve. Put $\gamma_j^*:=\gamma_j([0,1])$, $j=1,\dots,n$, and let $\varphi:\gamma_1^*\times\dots\times\gamma_n^*\longrightarrow\mathbb{C}$ be a continuous function. Define*

$$\Phi(z) := \frac{1}{(2\pi i)^n}\int_{\gamma_1}\cdots\int_{\gamma_n}\frac{\varphi(\zeta_1,\dots,\zeta_n)}{(\zeta_1-z_1)\cdot\dots\cdot(\zeta_n-z_n)}d\zeta_1\dots d\zeta_n =: \frac{1}{(2\pi i)^n}\int_{\gamma_1\times\dots\times\gamma_n}\frac{\varphi(\zeta)}{\zeta-z}d\zeta,$$
$$z\in\mathbb{C}^n\setminus(\gamma_1^*\times\dots\times\gamma_n^*) =: \Omega.$$

Then
(a) *$\Phi\in\mathcal{O}(\Omega)$;*

([18]) Here and in the sequel, if we write $f(z)=\sum_{\alpha\in\mathbb{N}_0^n}a_\alpha(z-a)^\alpha$, $z\in A$, then we assume that A is contained in the domain of convergence of the series.

(b)

$$D^\alpha\Phi(z)=\frac{\alpha!}{(2\pi i)^n}\int_{\gamma_1}\cdots\int_{\gamma_n}\frac{\varphi(\zeta_1,\dots,\zeta_n)}{(\zeta_1-z_1)^{\alpha_1+1}\cdot\dots\cdot(\zeta_n-z_n)^{\alpha_n+1}}d\zeta_1\dots d\zeta_n$$
$$=\frac{\alpha!}{(2\pi i)^n}\int_{\gamma_1\times\dots\times\gamma_n}\frac{\varphi(\zeta)}{(\zeta-z)^{\alpha+\mathbf{1}}}d\zeta,\quad z\in\Omega,\ \alpha\in\mathbb{N}_0^n,$$

where $\mathbf{1}:=(1,\dots,1)\in\mathbb{N}^n$*;*
(c) *for any polydisc* $\mathbb{P}(a,\boldsymbol{r})\subset\Omega$ *we have*

$$\Phi(z)=T_a\Phi(z),\quad z\in\mathbb{P}(a,\boldsymbol{r}).$$

Proof. Fix a $\mathbb{P}(a,\boldsymbol{r})\subset\Omega$ and observe that for $(\zeta,z)\in(\gamma_1^*\times\dots\times\gamma_n^*)\times\mathbb{P}(a,\boldsymbol{r})$

$$\frac{1}{\zeta-z}=\sum_{\alpha\in\mathbb{N}_0^n}\frac{(z-a)^\alpha}{(\zeta-a)^{\alpha+\mathbf{1}}},$$

and the series is locally normally convergent. Hence

$$\Phi(z)=\sum_{\alpha\in\mathbb{N}_0^n}\Big[\frac{1}{(2\pi i)^n}\int_{\gamma_1\times\dots\times\gamma_n}\frac{\varphi(\zeta)}{(\zeta-a)^{\alpha+\mathbf{1}}}d\zeta\Big](z-a)^\alpha,\quad z\in\mathbb{P}(a,\boldsymbol{r}).$$

It remains to apply Proposition 1.3.7. □

The above lemma and the Cauchy integral formula (Proposition 1.2.5) imply the following important corollaries.

Corollary 1.4.5. *Let* $f:\Omega\longrightarrow\mathbb{C}$*. The following conditions are equivalent:*
(i) $f\in\mathcal{O}(\Omega)$*;*
(ii) $f\in\mathcal{O}_s(\Omega)\cap\mathcal{C}(\Omega)$*;*
(iii) f *is differentiable in the complex sense at an arbitrary point of* Ω*.*

Corollary 1.4.6. *If* $f\in\mathcal{O}(\Omega)$*, then for every polydisc* $\mathbb{P}(a,\boldsymbol{r})\subset\Omega$ *we have*

$$f(z)=T_af(z),\quad z\in\mathbb{P}(a,\boldsymbol{r}).$$

In particular,

$$f(z)=T_af(z),\quad z\in\mathbb{P}(a,d_\Omega(a)),\ a\in\Omega.$$

Corollary 1.4.7. *Let* $D\subset\mathbb{C}^n$ *be a complete* n*-circled domain. Then*

$$f(z)=T_0f(z),\quad z\in D.$$

Corollary 1.4.8 (Cauchy's inequalities)**.** *If* $f\in\mathcal{O}(\mathbb{P}(a,\boldsymbol{r}))\cap\mathcal{C}(\overline{\mathbb{P}}(a,\boldsymbol{r}))$*, then*

$$|D^\alpha f(a)|\le\frac{\alpha!}{\boldsymbol{r}^\alpha}\|f\|_{\partial_0\mathbb{P}(a,\boldsymbol{r})},\quad\alpha\in\mathbb{N}_0^n.\quad(^{19})$$

Similarly as in the case of one complex variable, the following corollary is an easy consequence of the Cauchy inequalities.

Corollary 1.4.9 (Liouville theorem)**.** *Let* $f\in\mathcal{O}(\mathbb{C}^n)$*,* $k\in\mathbb{N}_0$*. Then the following conditions are equivalent:*
(i) f *is a polynomial of degree* $\le k$*;*
(ii) $\exists_{C,R_0}>0:\ |f(z)|\le C\|z\|^k$ *for* $\|z\|\ge R_0$*.*

For $A\subset\mathbb{C}^n$, $\boldsymbol{r}\in\mathbb{R}^n_{>0}$, and $r>0$ let

$$A^{(\boldsymbol{r})}:=\bigcup_{a\in A}\overline{\mathbb{P}}(a,\boldsymbol{r}),\qquad A^{(r)}:=A^{(r,\dots,r)}.$$

Note that if A is compact, then $A^{(\boldsymbol{r})}$ is also compact.

(19) $\|f\|_A:=\sup_A|f|$.

Corollary 1.4.10. *For arbitrary compact $K \subset \Omega$ and polyradius $\boldsymbol{r}$ such that $K^{(\boldsymbol{r})} \subset \Omega$ we have*

$$\|D^\alpha f\|_K \le \frac{\alpha!}{\boldsymbol{r}^\alpha}\|f\|_{K^{(\boldsymbol{r})}}, \quad f \in \mathcal{O}(\Omega),\ \alpha \in \mathbb{N}_0^n.$$

Hence, using Corollary 1.4.5, we get

Corollary 1.4.11 (Weierstrass theorem)**.** *If $\mathcal{O}(\Omega) \ni f_\nu \longrightarrow f$ locally uniformly on Ω, then $f \in \mathcal{O}(\Omega)$ and $D^\alpha f_\nu \longrightarrow D^\alpha f$ locally uniformly on Ω for any $\alpha \in \mathbb{N}_0^n$.*

In other words, we have

Corollary 1.4.12. *The space $\mathcal{O}(\Omega)$ endowed with the topology defined by seminorms*

$$\mathcal{O}(\Omega) \ni f \longmapsto \|f\|_K, \quad K \subset\subset \Omega,$$

is a Fréchet space such that for arbitrary $\alpha \in \mathbb{N}_0^n$ the mapping

$$\mathcal{O}(\Omega) \ni f \longmapsto D^\alpha f \in \mathcal{O}(\Omega)$$

is continuous.

Corollary 1.4.13. (a) *Let*

$$\mathcal{H}^\infty(\Omega) := \{f \in \mathcal{O}(\Omega) : \|f\|_\Omega < +\infty\}.$$

Then $(\mathcal{H}^\infty(\Omega), \|\ \|_\Omega)$ is a Banach algebra.
(b) *Assume that Ω is bounded, and let*

$$\mathcal{A}^k(\Omega) := \{f \in \mathcal{O}(\Omega) : \forall_{\alpha\in\mathbb{N}_0^n,\ |\alpha|\le k} : \exists_{\varphi_\alpha\in\mathcal{C}(\overline{\Omega})} : \varphi_\alpha = D^\alpha f \text{ in } \Omega\}, \quad k \in \mathbb{N}_0 \cup \{\infty\}.$$

Then $\mathcal{A}^k(\Omega)$ endowed with the topology defined by seminorms

$$\mathcal{A}^k(\Omega) \ni f \longmapsto \|D^\alpha f\|_\Omega, \quad |\alpha| \le k,$$

is a Fréchet space. If $k < \infty$, then the space $\mathcal{A}^k(\Omega)$ endowed with the norm

$$f \longmapsto \sum_{\alpha\in\mathbb{N}_0^n:\ |\alpha|\le k} \|D^\alpha f\|_\Omega$$

is a Banach space.

Corollary 1.4.5(iii) implies also

Corollary 1.4.14. *The composition of holomorphic mappings is holomorphic.*

A bijective holomorphic mapping $f : \Omega \longrightarrow \Omega'$ (where Ω and Ω' are open in $\mathbb{C}^n$) is called *biholomorphic* if f^{-1} is also holomorphic (cf. Chapter VII).

Corollary 1.4.15 (Inverse mapping theorem)**.** *Let $f : \Omega \longrightarrow \mathbb{C}^n$ be a holomorphic mapping with $J_{\mathbb{C}}f(a) \neq 0$ for some $a \in \Omega$. Then there exists an open neighborhood U of a ($U \subset \Omega$) such that $f(U)$ is an open set and $f|_U : U \longrightarrow f(U)$ is biholomorphic.*

Corollary 1.4.16 (Implicit mapping theorem)**.** *Let Ω be an open subset of $\mathbb{C}^n \times \mathbb{C}^m$ and let $f : \Omega \longrightarrow \mathbb{C}^m$ be a holomorphic mapping. Assume that*

$$\det\Big(\Big[\frac{\partial f_j}{\partial w_k}(a,b)\Big]_{j,k=1,\dots,m}\Big) \neq 0$$

for some $(a,b) \in \Omega$, where $(z,w) = (z_1,\dots,z_n,w_1,\dots,w_m) \in \mathbb{C}^n \times \mathbb{C}^m$. Then there exist
an open neighborhood U of a,
an open neighborhood V of b, $U \times V \subset \Omega$,
a holomorphic mapping $\varphi : U \longrightarrow V$,
such that

$$\{(z,w) \in U \times V : f(z,w) = f(a,b)\} = \{(z,\varphi(z)) : z \in U\}.$$

Corollary 1.4.17 (Rank theorem)**.** *Let $f : \Omega \longrightarrow \mathbb{C}^m$ be holomorphic and such that* $\operatorname{rank} f'(z) = r$ *for any $z \in \Omega$. Then for arbitrary $a \in \Omega$ there exist*
an open neighborhood U of a, $U \subset \Omega$,
an open neighborhood V of $f(a)$, $V \subset \mathbb{C}^m$,
biholomorphic mappings $\Phi : \mathbb{D}^n \longrightarrow U$, $\Psi : V \longrightarrow E^m$,
such that $\Phi(0) = a$, $\Psi(f(a)) = 0$, $f(U) \subset V$, and

$$\Psi \circ f \circ \Phi(z_1, \dots, z_n) = (z_1, \dots, z_r, 0, \dots, 0), \quad (z_1, \dots, z_n) \in \mathbb{D}^n.$$

Proposition 1.4.18. *Let $D \subset \mathbb{C}^n$ be a domain and let $f \in \mathcal{O}(D)$, $f \not\equiv \text{const}$. Then f is an open mapping.*

Proof. Fix an $a \in D$. By the identity principle, there exists an $X \in \mathbb{C}^n$ such that $f_{a,X} \not\equiv \text{const}$ in the connected component $S_{a,X}$ of $D_{a,X}$ with $0 \in S_{a,X}$. Consequently, the function $f_{a,X} : S_{a,X} \longrightarrow \mathbb{C}$ is open. Hence $f(a) \in \operatorname{int} f(U)$ for any neighborhood U of a. □

Corollary 1.4.19 (Maximum principle)**.** *Let $D \subset \mathbb{C}^n$ be a domain and let $f \in \mathcal{O}(D)$, $f \not\equiv \text{const}$. Then*
(a) *$|f|$ does not attain local maxima in D;*
(b) *if, moreover, D is bounded, then*

$$|f(z)| < \sup_{\zeta \in \partial D} \{\limsup_{D \ni z \to \zeta} |f(z)|\}, \quad z \in D.$$

Lemma 1.4.20. *For any compact $K \subset \Omega$ and $\boldsymbol{r} = (r_1, \dots, r_n)$ such that $K^{(\boldsymbol{r})} \subset \Omega$ we have*

$$\|f\|_K \le \frac{1}{(\pi r_1^2) \dots (\pi r_n^2)} \int_{K^{(\boldsymbol{r})}} |f| \, d\mathcal{L}^{2n}, \quad f \in \mathcal{O}(\Omega),$$

where $\mathcal{L}^{2n}$ denotes Lebesgue measure in $\mathbb{C}^n$. In particular, for arbitrary $1 \le p < \infty$ and for arbitrary compact $K \subset \Omega$ there exists a constant $C > 0$ such that

$$\|f\|_K \le C \|f\|_{L^p}, \quad f \in L_h^p(\Omega) := \mathcal{O}(\Omega) \cap L^p(\Omega, \mathcal{L}^{2n}),$$

where

$$\|f\|_{L^p} := \Big(\int_\Omega |f|^p d\mathcal{L}^{2n} \Big)^{1/p}.$$

Proof. By Cauchy's integral formula, for every $a = (a_1, \dots, a_n) \in K$ we have

$$\begin{aligned}(\frac{1}{2} r_1^2) \dots (\frac{1}{2} r_n^2) |f(a)| &\le \int_0^{r_1} \tau_1 \, d\tau_1 \cdots \int_0^{r_n} \tau_n \, d\tau_n \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} |f(a_1 + \tau_1 e^{i\theta_1}, \dots, a_n + \tau_n e^{i\theta_n})| d\theta_1 \dots d\theta_n \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{P}(a, \boldsymbol{r})} |f| \, d\mathcal{L}^{2n} \le \frac{1}{(2\pi)^n} \int_{K^{(\boldsymbol{r})}} |f| \, d\mathcal{L}^{2n}.\end{aligned}$$

□

Corollary 1.4.21. *For arbitrary $1 \le p < \infty$ $(L_h^p(\Omega), \| \; \|_{L^p})$ is a Banach space. The space $L_h^2(\Omega)$ (with the scalar product induced from $L^2(\Omega, \mathcal{L}^{2n})$ (20)) is a Hilbert space.*

Lemma 1.4.22. *Assume that a family $\mathcal{F} \subset \mathcal{O}(\Omega)$ is locally uniformly bounded in Ω. Then $\mathcal{F}$ is equicontinuous.*

Proof. Fix a $\mathbb{P}(a, r) \subset\subset \Omega$. Set $C := \sup_{f \in \mathcal{F}} \{\|f\|_{\mathbb{P}(a,r)}\}$. Now, using the Schwarz lemma, similarly as in the proof of Proposition 1.2.3, we obtain

$$|f(z) - f(a)| \le \frac{2C}{r} (|z_1 - a_1| + \dots + |z_n - a_n|), \quad f \in \mathcal{F}, \; z \in \mathbb{P}(a, r).$$ □

Having Lemma 1.4.22, we can repeat the proof of the classical (one-dimensional) Montel theorem (cf. [4], Th. VII.2.9) and we obtain

(20) $\langle f, g \rangle_{L^2} := \int_\Omega f \bar{g} \, d\mathcal{L}^{2n}$, $f, g \in L^2(\Omega, \mathcal{L}^{2n})$.

Corollary 1.4.23 (Montel theorem). *Let $\mathcal{F} \subset \mathcal{O}(\Omega)$ be a family locally uniformly bounded in Ω. Then for arbitrary sequence $(f_\nu)_{\nu=1}^\infty \subset \mathcal{F}$ there exists a subsequence which converges locally uniformly to a holomorphic function on Ω.*

Proposition 1.4.24 (Vitali theorem). *Let $D \subset \mathbb{C}^n$ be a domain and let a sequence $(f_\nu)_{\nu=1}^\infty \subset \mathcal{O}(D)$ be locally uniformly bounded and pointwise convergent on a non-empty open subset $U \subset D$. Then the sequence $(f_\nu)_{\nu=1}^\infty$ is convergent locally uniformly in D.*

Proof. Similarly as in the case of one complex variable, the main difficulty is to show that the sequence $(f_\nu)_{\nu=1}^\infty$ is pointwise convergent in all of D. Let

$$D_0 := \{a \in D : \text{ the sequence } (f_\nu)_{\nu=1}^\infty \text{ is pointwise convergent a neighborhood of } a\}.$$

The set D_0 is non-empty and open. It is sufficient to show that it is closed in D. Fix an accumulation point $b \in D$ of D_0. Let $\mathbb{P}(b, r) \subset D$ and $a \in D_0 \cap \mathbb{P}(b, r)$. For every $X \in \mathbb{C}^n$, $X \neq 0$, the sequence $((f_\nu)_{a,X})_{\nu=1}^\infty$, considered on the connected component $S_{a,X}$ of $D_{a,X}$ with $0 \in S_{a,X}$, is locally uniformly bounded and pointwise convergent in $(D_0)_{a,X} \cap S_{a,X}$. It is easy to see that this last set is non-empty (because $a \in D_0$) and open. Hence, by the classical one-dimensional Vitali theorem, the sequence $(f_\nu)_{\nu=1}^\infty$ is pointwise convergent in $\widetilde{S}_{a,X} := \{a + \lambda X : \lambda \in S_{a,X}\}$. Therefore the sequence $(f_\nu)_{\nu=1}^\infty$ is pointwise convergent in the set $\bigcup_{X \in \mathbb{C}^n} \widetilde{S}_{a,X}$ which is a neighborhood of b. □

Recall that, by the Riemann Mapping Theorem (cf. [4], Th. VII.4.2), any simply connected domain $D \subsetneq \mathbb{C}$ is biholomorphic to the unit disc $\mathbb{D}$. In other words, if $D \subsetneq \mathbb{C}$ is a domain, then D and $\mathbb{D}$ are biholomorphically equivalent iff they are topologically equivalent.

It is surprising, but for $n \geq 2$ the above theorem is not true even in the category of bounded convex domains.

Theorem 1.4.25 (Poincaré theorem). *For $n \geq 2$ the unit Euclidean ball $\mathbb{B}_n$ is not biholomorphic to the unit polydisc $\mathbb{D}^n$.*

The proof will be based on the following version of the Schwarz lemma (cf. Exercise 4.1).

Lemma 1.4.26. *Let $\| \ \|_1$, $\| \ \|_2 : \mathbb{C}^n \longrightarrow \mathbb{R}_+$ be arbitrary $\mathbb{C}$–norms. Put*

$$B_j := \{z \in \mathbb{C}^n : \|z\|_j < 1\}, \quad j = 1, 2,$$

and let $F : B_1 \longrightarrow B_2$ be a holomorphic mapping with $F(0) = 0$. Then

$$\|F(z)\|_2 \leq \|z\|_1, \quad z \in B_1.$$

Proof. Fix a $z_0 \in B_1 \setminus \{0\}$. Let $L : \mathbb{C}^n \longrightarrow \mathbb{C}$ be a $\mathbb{C}$–linear functional with $|L| \leq \| \ \|_2$ and $|L(F(z_0))| = \|F(z_0)\|_2$ (use the Hahn–Banach theorem). Consider the holomorphic mapping

$$\varphi(\lambda) := L(F(\lambda z_0)), \quad |\lambda| < 1/\|z_0\|_1.$$

Then by the classical Schwarz lemma, we obtain

$$|\varphi(\lambda)| \leq |\lambda| \|z_0\|_1, \quad |\lambda| < 1/\|z_0\|_1.$$

In particular, if $\lambda = 1$, then we get $\|F(z_0)\|_2 \leq \|z_0\|_1$. □

Proof of the Poincaré theorem. Suppose that $f : \mathbb{B}_n \longrightarrow \mathbb{D}^n$ is biholomorphic. Let $a = (a_1, \dots, a_n) := f(0)$. Define

$$\mathbb{D}^n \ni (z_1, \dots, z_n) \overset{\Phi}{\longmapsto} \Big(\frac{z_1 - a_1}{1 - \overline{a}_1 z_1}, \dots, \frac{z_n - a_n}{1 - \overline{a}_n z_n}\Big) \in \mathbb{D}^n.$$

It is easy to see that Φ maps biholomorphically $\mathbb{D}^n$ onto $\mathbb{D}^n$. Replacing f by $\Phi \circ f$ we may assume that $f(0) = 0$. Put $g = (g_1, \dots, g_n) := f^{-1} : \mathbb{D}^n \longrightarrow \mathbb{B}_n$. By Lemma 1.4.26, we conclude that $\|g(w)\| = |w|$ for any $w \in \mathbb{D}^n$. Thus

$$|g_1(w)|^2 + \dots + |g_n(w)|^2 = \max\{|w_1|^2, \dots, |w_n|^2\}, \quad w = (w_1, \dots, w_n) \in \mathbb{D}^n.$$

Observe that the left-hand side defines a function of class $\mathcal{C}^\infty$, but the right-hand side is even not differentiable (for $n \geq 2$); contradiction. □

1.5. Hartogs' theorem

Theorem 1.5.1 (Hartogs' theorem)**.** *$\mathcal{O}(\Omega) = \mathcal{O}_s(\Omega)$ for an arbitrary open subset $\Omega \subset \mathbb{C}^n$.*

Remark 1.5.2. Observe that there is no analogous theorem for separately $\mathbb{R}$-analytic functions [21]. For example, let

$$f(x_1, x_2) := \begin{cases} \frac{x_1 x_2}{x_1^2 + x_2^2} & \text{if } (x_1, x_2) \neq (0,0) \\ 0 & \text{if } (x_1, x_2) = (0,0) \end{cases}.$$

Then f is separately $\mathbb{R}$-analytic, but is not continuous at $(0,0)$.

The proof of Hartogs' theorem will be based on the following

Lemma 1.5.3 (Hartogs' lemma)**.** *Let $f : \mathbb{P}_n(r) \longrightarrow \mathbb{C}$ be such that*
$f(z, \cdot) \in \mathcal{O}(\mathbb{P}_{n-1}(r))$ for arbitrary $z \in K(r)$,
$f \in \mathcal{O}(K(r) \times \mathbb{P}_{n-1}(\delta))$ for some $0 < \delta < r$.
Then $f \in \mathcal{O}(\mathbb{P}_n(r))$.

Remark 1.5.4. The lemma is not true without the assumption that $f \in \mathcal{O}(K(r) \times \mathbb{P}_{n-1}(\delta))$ for some $0 < \delta < r$ (even if f satisfies some additional regularity conditions).

For, let us consider the following counterexample due to Leja (cf. [20]; we are going to construct a function $f \in \mathcal{O}((\mathbb{C} \setminus \mathbb{R}_-) \times \mathbb{C})$ such that $f(z, \cdot) \in \mathcal{O}(\mathbb{C})$ for any $z \in \mathbb{C}$, but f is not holomorphic in any neighborhood of $(0,0)$).

Let

$$L_k := \bigcup_{x \in \mathbb{R}:\, x \leq 0} K(x, 1/k) \subset \mathbb{C},$$

$$A_k := \overline{K}(k) \setminus L_k, \quad B_k := \overline{K}(k) \cap (\overline{L}_{k+1} \setminus L_{k+2}), \quad C_k := \overline{K}(k) \cap \overline{L}_{k+3}, \quad k \in \mathbb{N}.$$

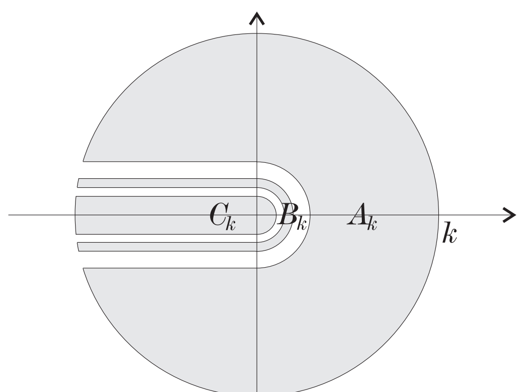

Figure 1.5.1

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[21] If $G \subset \mathbb{R}^n$ is open and $f : G \longrightarrow \mathbb{C}$, then f is $\mathbb{R}$-*analytic* if for any $x_0 \in G$ there exist $(a_\alpha)_{\alpha \in \mathbb{N}_0^n} \subset \mathbb{C}$ and an open neighborhood $U_{x_0} \subset G$ of x_0 such that for any $x \in U_{x_0}$ the family $(a_\alpha (x - x_0)^\alpha)_{\alpha \in \mathbb{N}_0^n}$ is summable and $f(x) = \sum_{\alpha \in \mathbb{N}_0^n} a_\alpha (x - x_0)^\alpha$. By Abel's lemma 1.3.3, the series $\sum_{\alpha \in \mathbb{N}_0^n} a_\alpha (z - x_0)^\alpha$ is normally convergent in a $\mathbb{C}^n$-neighborhood $\widetilde{U}_{x_0} \subset \mathbb{C}^n$ of x_0 with $\widetilde{U}_{x_0} \cap \mathbb{R}^n \subset U_{x_0}$. Put $\widetilde{f}_{x_0}(z) := \sum_{\alpha \in \mathbb{N}_0^n} a_\alpha (z - x_0)^\alpha$, $z \in \widetilde{U}_{x_0}$. Then $\widetilde{f}_{x_0} \in \mathcal{O}(\widetilde{U}_{x_0})$ and $\widetilde{f}_{x_0} = f$ in $\widetilde{U}_{x_0} \cap \mathbb{R}^n$.

Thus we have the following equivalent definition: f is $\mathbb{R}$-analytic if for any point $x_0 \in G$ there exist a $\mathbb{C}^n$-neighborhood $\widetilde{U}_{x_0} \subset \mathbb{C}^n$ of x_0 with $\widetilde{U}_{x_0} \cap \mathbb{R}^n \subset G$ and $\widetilde{f} \in \mathcal{O}(\widetilde{U}_{x_0})$ such that $\widetilde{f}_{x_0} = f$ in $\widetilde{U}_{x_0} \cap \mathbb{R}^n$.

Obviously, $\mathbb{R}$-analytic functions are of class $\mathcal{C}^\infty$.

By the Runge theorem (cf. [4], Th. VIII.1.7) for each $k \in \mathbb{N}$ there exists a polynomial $P_k \in \mathcal{P}(\mathbb{C})$ such that

$$|P_k(z)| \le 1/k^k, \quad z \in A_k \cup C_k, \qquad |P_k(z)| \ge k^k, \quad z \in B_k.$$

Let

$$f(z,w) := \sum_{k=1}^{\infty} P_k(z) w^k, \quad (z,w) \in \mathbb{C}^2.$$

Observe that f is well defined because for any $z \in \mathbb{C}$ there exists a $k_0(z) \in \mathbb{N}$ such that $z \in A_k \cup C_k$ for any $k \ge k_0(z)$ and therefore

$$|P_k(z) w^k| \le (|w|/k)^k, \quad k \ge k_0(z).$$

In particular, $f(z,\cdot) \in \mathcal{O}(\mathbb{C})$ for any $z \in \mathbb{C}$.

Moreover, for any $z_0 \in \mathbb{C} \setminus \mathbb{R}_-$ there exist $r_0 > 0$ and $k_0 \in \mathbb{N}$ such that $K(z_0, r_0) \subset A_k$ for $k \ge k_0$. Hence

$$|P_k(z) w^k| \le (|w|/k)^k, \quad (z,w) \in K(z_0, r_0) \times \mathbb{C}, \ k \ge k_0,$$

and consequently, by the Weierstrass theorem, $f \in \mathcal{O}((\mathbb{C} \setminus \mathbb{R}_-) \times \mathbb{C})$.

Suppose that f is bounded in a neighborhood of $(0,0)$. Let $|f(z,w)| \le C$ for $(z,w) \in P_2(r)$. Then, by the Cauchy inequalities, we get

$$|P_k(z)| \le C/r^k, \quad k \in \mathbb{N}, \ z \in K(r).$$

Consequently, taking $z \in B_k \cap K(r)$ with $k \gg 1$, we get

$$k^k \le |P_k(z)| \le C/r^k, \quad k \gg 1;$$

contradiction.

Proof that Lemma 1.5.3 implies Theorem 1.5.1. We use induction with respect to n. For $n = 1$ the theorem is trivial.

$n-1 \rightsquigarrow n$. Fix an $\Omega \subset \mathbb{C}^n = \mathbb{C} \times \mathbb{C}^{n-1}$ and $f \in \mathcal{O}_s(\Omega)$.

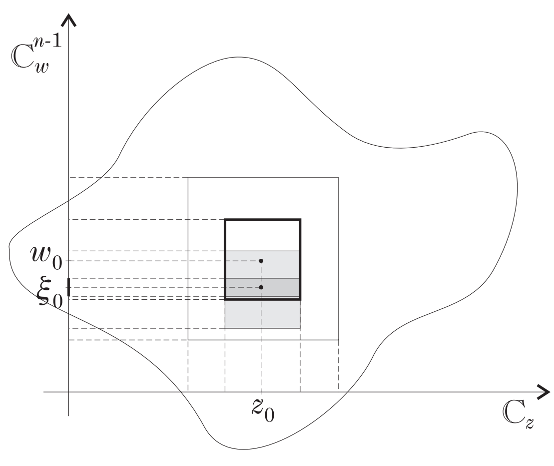

Figure 1.5.2

It is sufficient to show that f is holomorphic in a neighborhood of an arbitrary point $(z_0, w_0) \in \Omega$. Let $\mathbb{P}_n((z_0, w_0), 2r) \subset \Omega$, and let

$$A_k := \{w \in \overline{\mathbb{P}}_{n-1}(w_0, r) : \forall_{z \in K(z_0, r)} : \ |f(z,w)| \le k\}.$$

Clearly $A_k \subset A_{k+1}$. Since $f(z,\cdot) \in \mathcal{C}(\mathbb{P}_{n-1}(w_0, 2r))$ for arbitrary $z \in K(z_0, 2r)$ (the inductive assumption), the sets A_k are closed. Since $f(\cdot, w) \in \mathcal{C}(K(z_0, 2r))$ for any $w \in \overline{\mathbb{P}}_{n-1}(w_0, r)$, we get $\bigcup_{k \in \mathbb{N}} A_k = \overline{\mathbb{P}}_{n-1}(w_0, r)$. Using Baire's property we conclude that $\operatorname{int} A_{k_0} \ne \varnothing$ for some k_0. Let $\mathbb{P}_{n-1}(\xi_0, \delta) \subset A_{k_0}$. In particular, by

Osgood's theorem (Proposition 1.2.3), $f \in \mathcal{O}(K(z_0, r) \times \mathbb{P}_{n-1}(\xi_0, \delta))$. Now we apply Lemma 1.5.3 to the function

$$\mathbb{P}_n(r) \ni (z, w) \longmapsto f(z_0 + z, \xi_0 + w),$$

and we conclude that $f \in \mathcal{O}(\mathbb{P}_n((z_0, \xi_0), r))$. It remains to observe that $(z_0, w_0) \in \mathbb{P}_n((z_0, \xi_0), r)$. $\square$

Proof of Lemma 1.5.3. Observe that it is sufficient to show that $f \in \mathcal{O}(\mathbb{P}_n(r'))$ for arbitrary $0 < r' < r$. Thus we may assume that $|f| \le c < +\infty$ in $K(r) \times \mathbb{P}_{n-1}(\delta)$ and that $f(z, \cdot)$ is bounded for any $z \in \mathbb{P}_{n-1}(r)$. We have

$$f(z, w) = \sum_{\beta \in \mathbb{N}_0^{n-1}} f_\beta(z) w^\beta, \quad z \in K(r),\ w \in \mathbb{P}_{n-1}(r),$$

where

$$f_\beta(z) = \frac{1}{\beta!}(D^\beta f(z, \cdot))(0) = \frac{1}{\beta!}(D^{(0,\beta)} f)(z, 0), \quad \beta \in \mathbb{N}_0^{n-1},\ z \in K(r).$$

The last equality follows from the fact that $f \in \mathcal{O}(K(r) \times \mathbb{P}_{n-1}(\delta))$. In particular, $f_\beta \in \mathcal{O}(K(r))$ for arbitrary β. Moreover, by Cauchy's inequalities, we obtain

$$|f_\beta| \le \frac{1}{\delta^{|\beta|}} c, \quad \beta \in \mathbb{N}_0^{n-1}.$$

Applying once more Cauchy's inequalities (for the function $f(z, \cdot)$), we have

$$|f_\beta(z)| \le \frac{1}{r^{|\beta|}} \|f(z, \cdot)\|_{\mathbb{P}_{n-1}(r)}, \quad \beta \in \mathbb{N}_0^{n-1},\ z \in K(r),$$

and so

$$\limsup_{|\beta| \to +\infty} |f_\beta(z)|^{1/|\beta|} \le \frac{1}{r}, \quad z \in K(r).$$

We need now the following auxiliary

Lemma 1.5.5. *Let $\Omega \subset \mathbb{C}$ be open, $\varphi_\nu \in \mathcal{O}(\Omega)$, $p_\nu > 0$, $\nu \ge 1$. Assume that the sequence $(|\varphi_\nu|^{p_\nu})_{\nu=1}^\infty$ is locally uniformly bounded in Ω and*

$$\limsup_{\nu \to +\infty} |\varphi_\nu(z)|^{p_\nu} \le m, \quad z \in \Omega.$$

Then for any $K \subset\subset \Omega$ and $\varepsilon > 0$ there exists a ν_0 such that

$$|\varphi_\nu|^{p_\nu} \le m + \varepsilon \text{ on } K \text{ for } \nu \ge \nu_0. \quad (^{22})$$

Assume for the moment that the lemma is true, and let us finish the main proof.

Write $\mathbb{N}_0^{n-1} = \{\beta_1, \beta_2, \dots\}$ so that $|\beta_\nu| \le |\beta_{\nu+1}|$, $\nu = 1, 2, \dots$. Let $\Omega := K(r)$, $\varphi_\nu := f_{\beta_\nu}$, $p_\nu := 1/|\beta_\nu|$, $m := 1/r$. It is easy to see that all the assumptions of Lemma 1.5.5 are satisfied. Fix a $\theta \in (0, 1)$ and let $\varepsilon > 0$ be such that $(1 + r\varepsilon)\theta < 1$. Applying Lemma 1.5.5 to the compact $K := \overline{K}(\theta r)$ we obtain $|\varphi_\nu(z)|^{p_\nu} \le 1/r + \varepsilon$ for $z \in \overline{K}(\theta r)$ and $\nu \ge \nu_0$. This means that

$$|f_\beta(z)| \le \left(\frac{1}{r} + \varepsilon\right)^{|\beta|}, \quad z \in K(\theta r),\ |\beta| \gg 1.$$

Hence

$$|f_\beta(z) w^\beta| \le [(1 + r\varepsilon)\theta]^{|\beta|}, \quad z \in K(\theta r),\ w \in \mathbb{P}_{n-1}(\theta r),\ |\beta| \gg 1.$$

Consequently, the series

$$\sum_{\beta \in \mathbb{N}_0^{n-1}} f_\beta(z) w^\beta$$

is convergent normally in $\mathbb{P}_n(\theta r)$, which by the Weierstrass theorem (Corollary 1.4.11) implies that $f \in \mathcal{O}(\mathbb{P}_n(\theta r))$. Since θ was arbitrary, we get $f \in \mathcal{O}(\mathbb{P}_n(r))$. $\square$

$(^{22})$ Lemma 1.5.5 will be generalized in Proposition 3.2.20.

Proof of Lemma 1.5.5. The result is local — it is sufficient to show that for any $\varepsilon > 0$ and $a \in \Omega$ there exist a disc $K(a,\delta) \subset \Omega$ and ν_0 such that

$$\sup_{K(a,\delta)} \{|\varphi_\nu|^{p_\nu}\} \le m + \varepsilon, \quad \nu \ge \nu_0.$$

We may assume that $\Omega = K(2)$, $a = 0$. Let $c > 0$ be such that $|\varphi_\nu|^{p_\nu} \le c$ in $\mathbb{D}$ for arbitrary ν. We may also assume that $\varphi_\nu \not\equiv 0$, $\nu \ge 1$. Denote by $a_{\nu,1}, \dots, a_{\nu,\mu(\nu)}$ the zeros of φ_ν in the disc $\mathbb{D}$ counted with multiplicities (if φ_ν has zeros in $\mathbb{D}$). Define

$$B_\nu(z) := \begin{cases} \prod_{j=1}^{\mu(\nu)} \frac{z - a_{\nu,j}}{1 - \overline{a}_{\nu,j} z}, & \text{if } \varphi_\nu \text{ has zeros in } \mathbb{D} \\ 1, & \text{otherwise} \end{cases}$$

and let $\psi_\nu := \varphi_\nu / B_\nu$. Observe that $|B_\nu| \le 1$ in $\mathbb{D}$, and that $|B_\nu| = 1$ in $\mathbb{T}$. The function ψ_ν has no zeros in $\mathbb{D}$. In particular, it admits a branch χ_ν of the p_ν-th power in $\mathbb{D}$. Given arbitrary $\zeta \in \mathbb{T}$, we have

$$\limsup_{\mathbb{D} \ni z \to \zeta} |\chi_\nu(z)| = \limsup_{\mathbb{D} \ni z \to \zeta} |\psi_\nu(z)|^{p_\nu} = \limsup_{\mathbb{D} \ni z \to \zeta} |\varphi_\nu(z)|^{p_\nu} \le c,$$

and so $|\chi_\nu| \le c$ in $\mathbb{D}$ for arbitrary ν. This means in particular that the family $(\chi_\nu)_{\nu=1}^\infty$ is equicontinuous in $\mathbb{D}$. Fix an $\varepsilon > 0$ and let $0 < \delta < 1$ be such that $|\chi_\nu(z) - \chi_\nu(0)| \le \varepsilon/2$ for $z \in \overline{K}(\delta)$ and $\nu \ge 1$. Then

$$|\varphi_\nu(z)|^{p_\nu} = |B_\nu(z)\psi_\nu(z)|^{p_\nu} \le |\psi_\nu(z)|^{p_\nu} = |\chi_\nu(z)| \le \varepsilon/2 + |\chi_\nu(0)|, \quad z \in \overline{K}(\delta),\ \nu \ge 1.$$

It remains to estimate $\chi_\nu(0)$. Since

$$|\chi_\nu(0)| \le \frac{1}{2\pi} \int_0^{2\pi} |\chi_\nu(re^{i\theta})|\, d\theta = \frac{1}{2\pi} \int_0^{2\pi} \frac{|\varphi_\nu(re^{i\theta})|^{p_\nu}}{|B_\nu(re^{i\theta})|^{p_\nu}}\, d\theta, \quad 0 < r < 1,$$

the Lebesgue dominated convergence theorem (recall that $|\chi_\nu(re^{i\theta})| \le c$ and $|\chi_\nu(re^{i\theta})| \longrightarrow |\varphi_\nu(e^{i\theta})|^{p_\nu}$ as $r \longrightarrow 1$) gives

$$|\chi_\nu(0)| \le \frac{1}{2\pi} \int_0^{2\pi} |\varphi_\nu(e^{i\theta})|^{p_\nu}\, d\theta, \quad \nu \ge 1.$$

Let

$$A_k := \{\theta \in [0, 2\pi] : |\varphi_\nu(e^{i\theta})|^{p_\nu} \le m + \varepsilon/4,\ \nu \ge k\}.$$

The sets A_k are closed, $A_k \subset A_{k+1}$, and $\bigcup_{k \in \mathbb{N}} A_k = [0, 2\pi]$ (23) . In particular, $\mathcal{L}^1(A_k) \longrightarrow 2\pi$. For $\nu \ge k$ we have

$$|\chi_\nu(0)| \le \frac{1}{2\pi} \Big(\int_{A_k} + \int_{[0,2\pi] \setminus A_k} \Big) |\varphi_\nu(e^{i\theta})|^{p_\nu}\, d\theta \le \frac{1}{2\pi} [(m + \varepsilon/4)\mathcal{L}^1(A_k) + c(2\pi - \mathcal{L}^1(A_k))].$$

Hence $|\chi_\nu(0)| \le m + \varepsilon/2$ for ν large enough. □

Proposition 1.5.6 (Hartogs' lemma)**.** *Let G be a domain in $\mathbb{C}^{n-k}$ and let D be a domain in $G \times \mathbb{C}^k$ such that for each $z \in G$ the fiber*

$$D_z := \{w \in \mathbb{C}^k : (z, w) \in D\}$$

is connected. Assume that $f : D \longrightarrow \mathbb{C}$ is such that:
$f(z, \cdot) \in \mathcal{O}(D_z)$, $z \in G$, $(*)$
$f \in \mathcal{O}(U)$, where $U \subset D$ is an open set such that $U_z \neq \varnothing$ for any $z \in G$.
Then $f \in \mathcal{O}(D)$.

Proof. First, we consider the case where $G := \mathbb{D}^{n-k}$, $D := \mathbb{D}^n$, $U := \mathbb{D}^{n-k} \times (\delta\mathbb{D})^k$ $(0 < \delta < 1)$.

The case $n - k = 1$ reduces to Lemma 1.5.3.

Assume that $n - k \ge 2$. By virtue of Hartogs' theorem, it suffices to prove that $f \in \mathcal{O}_s(\mathbb{D}^n)$. In view of $(*)$ we only need to check that $f(\cdot, w) \in \mathcal{O}_s(\mathbb{D}^{n-k})$. Fix a $z_0 \in \mathbb{D}^{n-k}$ and $j \in \{1, \dots, n-k\}$. Define

$$g(\zeta, w) := f(z_{0,1}, \dots, z_{0,j-1}, \zeta, z_{0,j+1}, \dots, z_{0,n-k}, w), \quad (\zeta, w) \in \mathbb{D} \times \mathbb{D}^k.$$

(23) Because $\limsup_{\nu \to +\infty} |\varphi_\nu|^{p_\nu} \le m$.

Then g satisfies all the assumptions of the lemma with $n = k + 1$. Consequently, $g \in \mathcal{O}(\mathbb{D} \times \mathbb{D}^k)$, which shows that f is holomorphic as a function of z_j.

In the general case let $\widetilde{D}$ denote the maximal open subset of D such that $f \in \mathcal{O}(\widetilde{D})$. Obviously $U \subset \widetilde{D}$. Suppose that $\widetilde{D}_{z_0} \subsetneq D_{z_0}$ for some $z_0 \in G$. Since $\varnothing \neq U_{z_0} \subset \widetilde{D}_{z_0}$ and D_{z_0} is connected, there exists a $w_0 \in \widetilde{D}_{z_0}$ such that $d_{\widetilde{D}_{z_0}}(w_0) < d_{D_{z_0}}(w_0)$. Take an $r > 0$ such that $d_{\widetilde{D}_{z_0}}(w_0) < r < d_{D_{z_0}}(w_0)$ and let $0 < \varepsilon < r$ be such that

$$\mathbb{P}_{n-k}(z_0, \varepsilon) \times \mathbb{P}_k(w_0, r) \subset D, \quad \mathbb{P}_{n-k}(z_0, \varepsilon) \times \mathbb{P}_k(w_0, \varepsilon) \subset \widetilde{D}.$$

Define

$$g(z, w) := f(z_0 + \varepsilon z, w_0 + rw), \quad (z, w) \in \mathbb{D}^{n-k} \times \mathbb{D}^k.$$

Then, by the first part of the proof (with $\delta := \varepsilon/r$), $g \in \mathcal{O}(\mathbb{D}^n)$. Consequently, $f \in \mathcal{O}(P_{n-k}(z_0, \varepsilon) \times P_k(w_0, r))$; contradiction. $\square$

Corollary 1.5.7. *Let $G_j \subset \widetilde{G}_j \subset \mathbb{C}^{n_j}$ be domains such that $\mathcal{O}(G_j) = \mathcal{O}(\widetilde{G}_j)|_{G_j}$, $j = 1, 2$. Then $\mathcal{O}(\widetilde{G}_1 \times \widetilde{G}_2)|_{G_1 \times G_2} = \mathcal{O}(G_1 \times G_2)$.*

Proof. Let $f \in \mathcal{O}(G_1 \times G_2)$. For any $w \in G_2$ the function $f(\cdot, w)$ extends to a function $\widetilde{f(\cdot, w)} \in \mathcal{O}(\widetilde{G}_1)$. Define $g(z, w) := \widetilde{f(\cdot, w)}(z)$, $(z, w) \in \widetilde{G}_1 \times G_2$. Then, by Proposition 1.5.6, g is holomorphic.

Now, for any $z \in \widetilde{G}_1$ the function $g(z, \cdot)$ extends holomorphically to $\widetilde{g(z, \cdot)} \in \mathcal{O}(\widetilde{G}_2)$. The same argument as above shows that the function $\widetilde{f}(z, w) := \widetilde{g(z, \cdot)}(w)$, $(z, w) \in \widetilde{G}_1 \times \widetilde{G}_2$, is holomorphic on $\widetilde{G}_1 \times \widetilde{G}_2$. $\square$

1.6. Special domains

Recall (Corollary 1.4.7) that if $D \subset \mathbb{C}^n$ is a complete n-circled domain, then any function $f \in \mathcal{O}(D)$ can be represented by the power series

$$f(z) = \sum_{\alpha \in \mathbb{N}_0^n} \frac{1}{\alpha!} D^\alpha f(0) z^\alpha, \quad z \in D.$$

Moreover, the series is locally normally convergent in D.

Consider a more general case where D is balanced (i.e. $\overline{\mathbb{D}} \cdot D = D$). Let h_D denote the *Minkowski functional* of D, i.e.

$$h_D(z) := \inf\{t > 0 : \frac{z}{t} \in D\}, \quad z \in \mathbb{C}^n.$$

Lemma 1.6.1. (a) *If $D \subset \mathbb{C}^n$ is a balanced domain and $h := h_D$, then*

$$h(\lambda z) = |\lambda| h(z), \quad \lambda \in \mathbb{C},\ z \in \mathbb{C}^n, \tag{1.6.1}$$

$$D = \{z \in \mathbb{C}^n : h(z) < 1\}, \tag{1.6.2}$$

$$h \text{ is upper semicontinuous on } \mathbb{C}^n. \tag{1.6.3}$$

Conversely, if $h : \mathbb{C}^n \longrightarrow \mathbb{R}_+$ satisfies (1.6.1) *and* (1.6.3)*, then the set D given by* (1.6.2) *is a balanced domain.*
(b) *If D is a complete n-circled domain and $h := h_D$, then*

$$h(\lambda_1 z_1, \dots, \lambda_n z_n) \le h(z),\ \lambda_1, \dots, \lambda_n \in \overline{\mathbb{D}},\ z = (z_1, \dots, z_n) \in \mathbb{C}^n, \tag{1.6.4}$$

$$h(e^{i\theta_1} z_1, \dots, e^{i\theta_n} z_n) = h(z),\ \theta_1, \dots, \theta_n \in \mathbb{R},\ z = (z_1, \dots, z_n) \in \mathbb{C}^n, \tag{1.6.5}$$

$$h \text{ is continuous.}$$

Conversely, if h satisfies (1.6.1), (1.6.3)*, and* (1.6.4)*, then the set D given by* (1.6.2) *is a complete n-circled domain.*

In particular, if h satisfies (1.6.1), (1.6.3)*, and* (1.6.4)*, then h is continuous.*

Proof. (a) Property (1.6.1) is a direct consequence of the definition of the Minkowski functional.

It is clear that $\{h < 1\} \subset D$. Take an $a \in D \setminus \{0\}$ and let $0 < \theta < 1$ be such that $a/\theta \in D$ (D is open). Then $h(a/\theta) \leq 1$, which gives $h(a) = \theta h(a/\theta) \leq \theta < 1$. Thus (1.6.2) is proved.

Take an $a \in \mathbb{C}^n$. To prove that h is upper semicontinuous at a we have to prove that for any $C > h(a)$ there exists a neighborhood U of a such that $h < C$ in U. Observe that $b := a/C \in D$. Let V be a neighborhood of b with $V \subset D$. Put $U := CV$. Then, for $z \in U$ we have $h(z) = Ch(z/C) < C$.

(b) The proof of (1.6.4) is elementary. Property (1.6.5) follows directly from (1.6.4).

To prove that h is continuous it suffices to show that h is lower semicontinuous at any point $a \in \mathbb{C}^n$ such that $h(a) > 0$. Fix such an $a = (a_1, \dots, a_n)$. We may assume that $a_1 \cdot \dots \cdot a_s \neq 0$, $a_{s+1} = \dots = a_n = 0$ for some $1 \leq s \leq n$. Fix a $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, put

$$m := \min\{|\frac{z_\nu}{a_\nu}| : j = 1, \dots, s\},$$

and let $\lambda_j \in \overline{\mathbb{D}}$ be such that

$$\lambda_j \frac{z_j}{a_j} = m, \quad j = 1, \dots, s.$$

Then conditions (1.6.1) and (1.6.4) give

$$mh(a) = h(ma_1, \dots, ma_s, 0, \dots, 0) = h(\lambda_1 z_1, \dots, \lambda_s z_s, 0z_{s+1}, \dots, 0z_n) \leq h(z).$$

Consequently,

$$\min\{|\frac{z_\nu}{a_\nu}| : j = 1, \dots, s\}h(a) \leq h(z), \quad z = (z_1, \dots, z_n) \in \mathbb{C}^n,$$

which implies the lower semicontinuity of h at a. $\square$

Proposition 1.6.2. *Let $D \subset \mathbb{C}^n$ be a balanced domain and let $f \in \mathcal{O}(D)$. Then*

$$f(z) = \sum_{\nu=0}^{\infty} Q_\nu(z), \quad z \in D, \tag{1.6.6}$$

where

$$Q_\nu(z) := \sum_{\alpha \in \mathbb{N}_0^n :\, |\alpha| = \nu} \frac{1}{\alpha!} D^\alpha f(0) z^\alpha. \quad (^{24})$$

Moreover, for any compact $K \subset D$ there exist $C > 0$ and $\theta \in (0, 1)$ such that

$$|Q_\nu(z)| \leq C\theta^\nu, \quad z \in K,\ \nu \in \mathbb{N}_0.$$

In particular, the series converges locally normally in D.

Proof. Let $h := h_D$. Take an $a \in D \setminus \{0\}$. The function

$$K(1/h(a)) \ni \lambda \overset{\varphi_a}{\longmapsto} f(\lambda a) \quad (^{25})$$

is holomorphic. Hence

$$f(a) = \varphi_a(1) = \sum_{\nu=0}^{\infty} \frac{1}{\nu!} \varphi_a^{(\nu)}(0) = \sum_{\nu=0}^{\infty} Q_\nu(a).$$

Thus the formula (1.6.6) is true (and the series is pointwise convergent in D). It remains to prove the estimate.

Take a compact $K \subset D$. Let $\theta \in (0, 1)$ be such that

$$L := \{\lambda z : |\lambda| \leq 1/\theta,\ z \in K\} \subset D.$$

Then, for any $a \in K$, by Cauchy's inequalities, we get

$$|Q_\nu(a)| = \frac{1}{\nu!} |\varphi_a^{(\nu)}(0)| \leq \|\varphi_a\|_{K(1/\theta)} \theta^\nu \leq \|f\|_L \theta^\nu, \quad \nu \in \mathbb{N}_0.$$

(^{24}) Observe that $Q_\nu : \mathbb{C}^n \longrightarrow \mathbb{C}$ is a homogeneous polynomial of degree ν.
(^{25}) $K(1/0) := \mathbb{C}$.

□

Definition 1.6.3. Let D be a domain in $\mathbb{C}^n$, let $1 \le k \le n-1$, and let G denote the projection of D onto $\mathbb{C}^{n-k}$, $G := \pi(D)$, where

$$\mathbb{C}^{n-k} \times \mathbb{C}^k \ni (z, w) \overset{\pi}{\longmapsto} z \in \mathbb{C}^{n-k}.$$

We say that D is a *Hartogs domain over G with k-dimensional balanced fibers* if for any $z \in G$ the fiber

$$D_z := \{w \in \mathbb{C}^k : (z, w) \in D\}$$

is balanced.

If D_z is complete k-circled for any $z \in G$, then we say that D is a *Hartogs domain over G with complete k-circled fibers.* Of course, if $k = 1$, then there is no difference between Hartogs domains with 1-dimensional balanced fibers and Hartogs domains with complete 1-circled fibers (26) ; in this case we simply say that D is a *complete Hartogs domain over G*.

If D_z is only k-circled for any $z \in G$, then D is called a *Hartogs domain over G with k-circled fibers* (we point out that we do not assume that D_z is connected). If $k = 1$, then we shortly say D is a *Hartogs domain over G*.

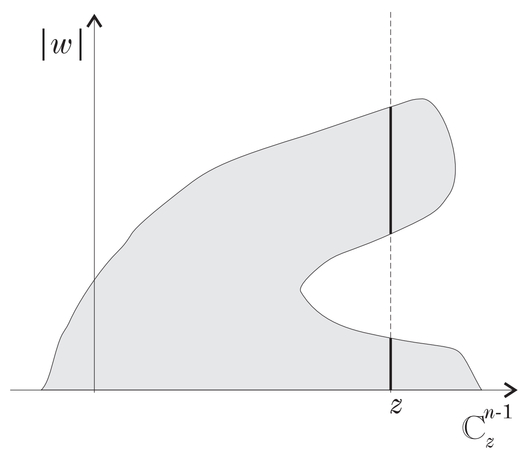

Figure 1.6.1

Remark 1.6.4. (a) Let D be a Hartogs domain over G with k-dimensional balanced fibers. Define

$$H(z, w) = H_D(z, w) := h_{D_z}(w), \quad (z, w) \in G \times \mathbb{C}^k,$$

where h_{D_z} is the Minkowski functional of D_z, $z \in G$. Observe that

$$D = \{(z, w) \in G \times \mathbb{C}^k : H(z, w) < 1\}, \tag{1.6.7}$$

$$H(z, \lambda w) = |\lambda| H(z, w), \quad (z, w) \in G \times \mathbb{C}^k, \tag{1.6.8}$$

$$H \text{ is upper semicontinuous on } G \times \mathbb{C}^k. \tag{1.6.9}$$

To prove that H is upper semicontinuous we can argue as in the proof of Lemma 1.6.1(a): if $H(z_0, w_0) < C$, then $(z_0, w_0/C) \in D$. Hence there exists a neighborhood V of $(z_0, w_0/C)$ such that $V \subset D$. Put $U := \{(z, Cw) : (z, w) \in V\}$. Then U is a neighborhood of (z_0, w_0) and $H < C$ in U.

Conversely, if a function $H : G \times \mathbb{C}^k \longrightarrow \mathbb{R}_+$ satisfies (1.6.8) and (1.6.9), then the set D given by (1.6.7) is a Hartogs domain over G with k-dimensional balanced fibers.

(26) If $k = 1$, then $D_z = \mathbb{C}$ or $D_z = K(R(z))$ for some $R(z) > 0$.

(b) D is a Hartogs domain over G with complete k-circled fibers iff the function $H = H_D$ from (a) satisfies additionally the following condition

$$H(z, \lambda_1 w_1, \dots, \lambda_k w_k) \le H(z, w), \quad \lambda_1, \dots, \lambda_k \in \overline{\mathbb{D}},\ z \in G,\ w = (w_1, \dots, w_k) \in \mathbb{C}^k.$$

(c) In particular, if $k = 1$, then D is a complete Hartogs domain over G iff

$$D = \{(z, w) \in G \times \mathbb{C} : |w| < e^{-u(z)}\},$$

where $u : G \longrightarrow [-\infty, +\infty)$ is upper semicontinuous (27) .

(d) If $k = 1$, then D is a Hartogs domain over G such that D_z is an annulus (28) for any $z \in G$ iff

$$D = \{(z, w) \in G \times \mathbb{C} : e^{v(z)} < |w| < e^{-u(z)}\},$$

where $v, u : G \longrightarrow [-\infty, +\infty)$ are upper semicontinuous and such that $v(z) + u(z) < 0$ for any $z \in G$. Hartogs domains of the above type are sometimes called *Hartogs-Laurent domains over* G.

Proposition 1.6.5. (a) *Let D be a Hartogs domain over G with complete k-circled fibers. Then any $f \in \mathcal{O}(D)$ can be represented by the* Hartogs series

$$f(z, w) = \sum_{\beta \in \mathbb{N}_0^k} f_\beta(z) w^\beta, \quad (z, w) \in D,$$

where

$$f_\beta(z) := \frac{1}{\beta!} D^{(0,\beta)} f(z, 0), \quad z \in G,\ \beta \in \mathbb{N}_0^k.$$

Moreover, for any compact $K \subset D$ there exist $C > 0$ and $\theta \in (0, 1)$ such that

$$|f_\beta(z) w^\beta| \le C\theta^{|\beta|}, \quad (z, w) \in K,\ \beta \in \mathbb{N}_0^k.$$

In particular, the series converges locally normally in D.

(b) *If D is a Hartogs domain over G with k-dimensional balanced fibers, then any $f \in \mathcal{O}(D)$ can be represented by the* Hartogs series

$$f(z, w) = \sum_{\nu=0}^{\infty} Q_\nu(z, w), \quad (z, w) \in D,$$

where

$$Q_\nu(z, w) := \sum_{|\beta| = \nu} \frac{1}{\beta!} D^{(0,\beta)} f(z, 0) w^\beta, \quad (z, w) \in G \times \mathbb{C}^k,\ \nu \in \mathbb{N}_0. \quad (29)$$

Moreover, for any compact $K \subset D$ there exist $C > 0$ and $\theta \in (0, 1)$ such that

$$|Q_\nu(z, w)| \le C\theta^\nu, \quad (z, w) \in K,\ \nu \in \mathbb{N}_0.$$

In particular, the series converges locally normally in D.

The case of Hartogs domains with k-circled fibers will be considered in Proposition 2.6.3.

Proof. In virtue of Corollary 1.4.7 and Proposition 1.6.2, we only need to check the estimates. Take a compact $K \subset D$.

(a) Let $\theta \in (0, 1)$ be such that

$$L := \{(z, \lambda_1 w_1, \dots, \lambda_k w_k) : (z, w_1, \dots, w_k) \in K,\ |\lambda_j| \le 1/\theta,\ j = 1, \dots, k\} \subset D.$$

Now, if $(z, w) \in K$, then by Cauchy's inequalities we get

$$|f_\beta(z) w^\beta| \le \|f\|_L \theta^{|\beta|}, \quad \beta \in \mathbb{N}_0^k.$$

(b) We argue as in the proof of Proposition 1.6.2. Let $\theta \in (0, 1)$ be such that

$$L := \{(z, \lambda w) : (z, w) \in K,\ |\lambda| \le 1/\theta\} \subset D.$$

---

(27) $u = \log H(\cdot, 1)$.
(28) That is, $D_z = \{w \in \mathbb{C} : r(z) < |w| < R(z)\}$ for some $0 \le r(z) < R(z) \le +\infty$.
(29) Notice that $Q_\nu \in \mathcal{O}(G \times \mathbb{C}^k)$ and $Q_\nu(z, \cdot)$ is a homogeneous polynomial of degree ν.

Then, by Cauchy's inequalities, we get

$$|Q_\nu(z,w)| \le \|f\|_L \theta^\nu, \quad (z,w) \in K,\ \nu \in \mathbb{N}_0.$$

□

1.7. Weierstrass Preparation and Division Theorems

If f is a function holomorphic in a connected neighborhood of $0 \in \mathbb{C}$ with $f(0)=0$, $f \not\equiv 0$, then f has a unique decomposition $f = z^p g$, where g is holomorphic and $g(0) \neq 0$. The aim of this section is to generalize the above result to the case of several variables.

Definition 1.7.1. Any function W of the form

$$W(z', z_n) = z_n^p + \sum_{j=1}^{p} W_j(z') z_n^{p-j},$$

where W_j is holomorphic in a neighborhood of $0' \in \mathbb{C}^{n-1}$ and $W_j(0') = 0$, $j = 1, \dots, p$, is called a *Weierstrass polynomial of degree p with center at $0 \in \mathbb{C}^n$.* [30]

Note that if W is a Weierstrass polynomial of degree p, then $W(0', z_n) = z_n^p$, and so $W(0', \cdot)$ has a zero of order p at $z_n = 0$.

Theorem 1.7.2 (Weierstrass Preparation Theorem)**.** *Let U_0 be a neighborhood of $0 \in \mathbb{C}^n$ and let $f \in \mathcal{O}(U_0)$ be such that the function $f(0', \cdot)$ has a zero of order p at $z_n = 0$. Then there exists a polydisc P, $0 \in P \subset U_0$, such that the function f has in P a unique decomposition $f = h \cdot W$, where $h, W \in \mathcal{O}(P)$, $h(z) \neq 0$, $z \in P$, and W is a Weierstrass polynomial of degree p.*

The Weierstrass Preparation Theorem will be a consequence of the following theorem.

Theorem 1.7.3 (Weierstrass Division Theorem)**.** *Let U_0 be a neighborhood of $0 \in \mathbb{C}^n$ and let $f \in \mathcal{O}(U_0)$ be such that the function $f(0', \cdot)$ has a zero of order p at $z_n = 0$. Then there exist a polydisc $P = \mathbb{P}(\boldsymbol{\varrho}) \subset U_0$ and a constant $c > 0$ such that any function $g \in \mathcal{H}^\infty(P)$ has in P a unique decomposition $g = q \cdot f + r$, where $q \in \mathcal{O}(P)$, $r \in \mathcal{O}(\mathbb{P}_{n-1}(\boldsymbol{\varrho}'))[Z_n]$* [31] *, $\deg_{Z_n} r < p$, and $\|q\|_P \le c\|g\|_P$.*

Proof that the Weierstrass Division Theorem implies the Weierstrass Preparation Theorem. Let $P = \mathbb{P}(\boldsymbol{\varrho}) \subset U_0$ be as in the Weierstrass Division Theorem. Then, taking $g := z_n^p$, we obtain the decomposition

$$z_n^p = q \cdot f + r \ \text{ in } P$$

with

$$q \in \mathcal{O}(P), \quad r = \sum_{j=0}^{p-1} r_j Z_n^j \in \mathcal{O}(\mathbb{P}_{n-1}(\boldsymbol{\varrho}'))[Z_n], \quad \deg_{Z_n} r < p.$$

In particular,

$$z_n^p = q(0', z_n) f(0', z_n) + \sum_{j=0}^{p-1} r_j(0') z_n^j, \quad z_n \in K(\varrho_n).$$

Hence $q(0) \neq 0$ and $r_j(0') = 0$, $j = 0, \dots, p-1$. Shrinking $\boldsymbol{\varrho}$, we may assume that $q(z) \neq 0$, $z \in P$. Setting $h(z) := 1/q(z)$ and $W(z', z_n) := z_n^p - r(z', z_n)$, $z = (z', z_n) \in P$, we obtain the required decomposition. The uniqueness follows from the uniqueness of the decomposition in the Weierstrass Division Theorem.

In fact, if $f = \widetilde{h} \cdot \widetilde{W}$ in P, then setting $\widetilde{q} := 1/\widetilde{h}$ and $\widetilde{r} := z_n^p - \widetilde{W}$, we obtain the decomposition $z_n^p = \widetilde{q} \cdot f + \widetilde{r}$ in P. Hence $\widetilde{h} = h$ and $\widetilde{r} = r$ in a neighborhood of 0 and so, by the identity principle, $\widetilde{h} = h$ and $\widetilde{r} = r$ on P. □

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[30] More generally, a Weierstrass polynomial with center at $a = (a', a_n) \in \mathbb{C}^n$ of degree p is a function W of the form $W(z', z_n) = (z_n - a_n)^p + \sum_{j=1}^{p} W_j(z')(z_n - a_n)^{p-j}$, where the W_j are holomorphic in a neighborhood of $a' \in \mathbb{C}^{n-1}$ and $W_j(a') = 0$, $j = 1, \dots, p$.

[31] $A[Z]$ denotes the ring of polynomials with coefficients in A.

Lemma 1.7.4. *For arbitrary $P = \mathbb{P}(\boldsymbol{\varrho}) \subset \mathbb{C}^n$ and $p \in \mathbb{N}$, every function $\varphi \in \mathcal{O}(P)$ admits a unique decomposition*

$$\varphi(z', z_n) = z_n^p \varphi_1(z', z_n) + \varphi_2(z', z_n), \quad (z', z_n) \in P,$$

with $\varphi_1 \in \mathcal{O}(P)$, $\varphi_2 \in \mathcal{O}(\mathbb{P}_{n-1}(\boldsymbol{\varrho}'))[Z_n]$, $\deg_{Z_n} \varphi_2 < p$. Moreover

$$\|\varphi_1\|_P \le \frac{p+1}{\varrho_n^p} \|\varphi\|_P.$$

Proof. The existence and uniqueness of the decomposition are obvious. Since

$$\varphi_2(z', z_n) = \sum_{j=0}^{p-1} \frac{1}{j!} \frac{\partial^j \varphi}{\partial z_n^j}(z', 0) z_n^j,$$

Cauchy's inequalities imply that $\|\varphi_2\|_P \le p\|\varphi\|_P$, and hence

$$\sup_P |z_n^p \varphi_1| \le (p+1)\|\varphi\|_P.$$

Consequently,

$$\|\varphi_1\|_P \le \frac{p+1}{\varrho_n^p} \|\varphi\|_P.$$

$\square$

Proof of the Weierstrass Division Theorem. Let $P = \mathbb{P}(\boldsymbol{\varrho}) \subset\subset U_0$ be an arbitrary polydisc. The polyradius $\boldsymbol{\varrho}$ will be modified in the sequel. By Lemma 1.7.4, we may assume that $f = z_n^p f_1 + f_2$, where f_1, f_2 are holomorphic in a neighborhood of $\overline{P}$, f_2 is a polynomial of degree $< p$ with respect to z_n. In particular,

$$f(0', z_n) = z_n^p f_1(0', z_n) + f_2(0', z_n), \quad z_n \in K(\varrho_n).$$

Since $\lim_{z_n \to 0} f(0', z_n)/z_n^p \in \mathbb{C}_*$, we get $f_1(0) \ne 0$ and $f_2(0', \cdot) \equiv 0$. Shrinking $\boldsymbol{\varrho}$, we may assume that $f_1(z) \ne 0$ in a neighborhood of $\overline{P}$. Let $h := f_2/f_1$. Since $h(0', \cdot) \equiv 0$, we may shrink $\boldsymbol{\varrho}'$ (with fixed ϱ_n) so that

$$\|h\|_P \le \frac{\varrho_n^p}{2(p+1)}. \tag{1.7.1}$$

From now on $\boldsymbol{\varrho}$ is assumed to be fixed.

Take a $g \in \mathcal{H}^\infty(P)$. Setting $f = z_n^p f_1 + f_2$, we see that the required decomposition $g = qf + r$ is equivalent to the decomposition $g = s(z_n^p + h) + r$, where

$$s \in \mathcal{O}(P), \quad r \in \mathcal{O}(\mathbb{P}_{n-1}(\boldsymbol{\varrho}'))[Z_n], \quad \deg_{Z_n} r < p, \quad \|s\|_P \le \widetilde{c}\|g\|_P,$$

with a constant $\widetilde{c}$ independent of g. Moreover, the decomposition $g = qf + r$ is unique iff the decomposition $g = s(z_n^p + h) + r$ is unique.

We proceed by recurrence. Let $s_0 := 0$ and let

$$g - h \cdot s_{k-1} = z_n^p s_k + r_k \tag{1.7.2}$$

be the decomposition obtained from Lemma 1.7.4, $k \ge 1$. Since

$$h(s_{k-1} - s_k) = z_n^p(s_{k+1} - s_k) + r_{k+1} - r_k,$$

the second part of Lemma 1.7.4 (applied to $\varphi := h(s_{k-1} - s_k)$) and (1.7.1) imply that

$$\|s_{k+1} - s_k\|_P \le \frac{p+1}{\varrho_n^p} \|h(s_k - s_{k-1})\|_P \le \frac{1}{2}\|s_k - s_{k-1}\|_P.$$

This means that the sequence $(s_k)_{k=0}^\infty$ is convergent uniformly in P to a holomorphic function s. Applying (1.7.2) we conclude that $r_k \longrightarrow r$ uniformly in P (r must be a polynomial of degree $< p$ with respect to z_n) and $g = s(z_n^p + h) + r$. Applying once more Lemma 1.7.4 (with $\varphi := g - sh$) and (1.7.1), we obtain

$$\|s\|_P \le \frac{p+1}{\varrho_n^p} \|g - sh\|_P \le \frac{p+1}{\varrho_n^p} \|g\|_P + \frac{1}{2}\|s\|_P,$$

and hence
$$\|s\|_P \le \frac{2(p+1)}{\varrho_n^p}\|g\|_P = \widetilde{c}\|g\|_P.$$
Suppose that there exists another decomposition $g = \widetilde{s}(z_n^p + h) + \widetilde{r}$. Then $(s - \widetilde{s})(z_n^p + h) + r - \widetilde{r} \equiv 0$, and therefore, by Lemma 1.7.4 (with $\varphi := h(\widetilde{s} - s)$) and (1.7.1), we would have
$$\|s - \widetilde{s}\|_P \le \frac{p+1}{\varrho_n^p}\|h(s - \widetilde{s})\|_P \le \frac{1}{2}\|s - \widetilde{s}\|_P.$$
Hence $\widetilde{s} \equiv s$, and consequently $\widetilde{r} \equiv r$. □

1.8. Elementary properties of the ring of germs of holomorphic functions

Let $a \in \mathbb{C}^n$. Define
$$\widetilde{\mathcal{O}}_a := \{(U, f) : U \in \mathbb{B}(a),\ f \in \mathcal{O}(U)\},$$
where $\mathbb{B}(a)$ denotes the family of all open neighborhoods of a. For (U, f), $(V, g) \in \widetilde{\mathcal{O}}_a$ we put
$$(U, f) \overset{a}{\sim} (V, g) \overset{\text{df}}{\Longleftrightarrow} \exists_{W \in \mathbb{B}(a)} : W \subset U \cap V,\ f|_W = g|_W.$$
It is clear that $\overset{a}{\sim}$ is an equivalence relation. Put
$$\mathcal{O}_a = \mathcal{O}_a^{(n)} := \widetilde{\mathcal{O}}_a / \overset{a}{\sim}\ .$$
The class $[(U, f)]_{\overset{a}{\sim}}$ is called the *germ* of f at a. We write $\widehat{f}_a := [(U, f)]_{\overset{a}{\sim}}$. Define
$$[(U, f)]_{\overset{a}{\sim}} + [(V, g)]_{\overset{a}{\sim}} := [(U \cap V, f + g)]_{\overset{a}{\sim}}, \quad [(U, f)]_{\overset{a}{\sim}} \cdot [(V, g)]_{\overset{a}{\sim}} := [(U \cap V, f \cdot g)]_{\overset{a}{\sim}}.$$
One can easily check that the operations $+, \cdot : \mathcal{O}_a \times \mathcal{O}_a \longrightarrow \mathcal{O}_a$ are well defined and that $(\mathcal{O}_a, +, \cdot)$ is a commutative ring with the unit element (the *ring of germs of holomorphic functions at* a).

Let $\mathfrak{f} = \widehat{f}_a \in \mathcal{O}_a$. Observe that the series $T\mathfrak{f}_a := T_a f$ is well defined (it is independent of the representant f). The mapping
$$\mathcal{O}_a \ni \mathfrak{f} \longmapsto T_a\mathfrak{f} \in \text{the ring of all power series with center at } a \text{ which are convergent in a neighborhood of } a$$
is an isomorphism.

Let $\mathcal{O}^{(n)} := \mathcal{O}_0^{(n)}$.

Let $a = (a', a_n) \in \mathbb{C}^n$. We say that a germ $F \in \mathcal{O}_a^{(n)}$ is z_n-*normalized* if there exist $r > 0$ and a representation $(\mathbb{P}_n(a, r), f)$ of F such that $f(a', \cdot) \not\equiv 0$ in $K(a_n, r)$. Then we will denote by $\operatorname{ord}_{a,Z_n} F$ the order of zero of $f(a', \cdot)$ at a_n.

Note that the germ $F \in \mathcal{O}_a^{(n)}$ is an invertible element of the ring $\mathcal{O}_a^{(n)}$ (i.e. it is a unit in the ring $\mathcal{O}_a^{(n)}$) iff $F(a) \neq 0$ (32) .

Let $\mathcal{W}_a^{(n)} \subset \mathcal{O}_a^{(n)}$ denote the set of all germs of Weierstrass polynomials with center at a. Note that $\mathcal{W}_a^{(n)}$ can be considered as a subset of $\mathcal{O}_{a'}^{(n-1)}[Z_n]$. According to the general rule, let $\mathcal{W}^{(n)} := \mathcal{W}_0^{(n)}$.

We have $\operatorname{ord}_{a,Z_n} W = \deg_{Z_n} W$ for any $W \in \mathcal{W}_a^{(n)}$.

Lemma 1.8.1. (a) *Let F, G, $W \in \mathcal{O}^{(n)}$ be such that $F = G \cdot W$. Assume that $F \in \mathcal{O}^{(n-1)}[Z_n]$, $W \in \mathcal{W}^{(n)}$. Then $G \in \mathcal{O}^{(n-1)}[Z_n]$.* (33)
(b) *Assume that a germ $F \in \mathcal{O}^{(n-1)}[Z_n]$ is z_n-normalized and irreducible in $\mathcal{O}^{(n-1)}[Z_n]$. Then F is also irreducible in $\mathcal{O}^{(n)}$.*

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(32) The value of the germ F at a point a is well defined.
(33) The theorem is not true if W is not a germ of the Weierstrass polynomial; for instance $1 = (1/W) \cdot W$, where W is a germ of a polynomial such that $W(0) \neq 0$.

Proof. (a) By the division algorithm, we obtain $F = G_1 \cdot W + R$, where $G_1, R \in \mathcal{O}^{(n-1)}[Z_n]$, $\deg_{Z_n} R < \deg_{Z_n} W$. Therefore we have two factorizations: $F = G \cdot W$ and $F = G_1 \cdot W + R$. The uniqueness of the factorization in the Weierstrass Division Theorem implies that $G = G_1 \in \mathcal{O}^{(n-1)}[Z_n]$ (and $R = 0$).

(b) Suppose that $F = G \cdot H$, where $G, H \in \mathcal{O}^{(n)}$, and H is not a unit in $\mathcal{O}^{(n)}$, i.e. $H(0) = 0$. Clearly H is z_n-normalized and $\operatorname{ord}_{0,Z_n} H > 0$. By the Weierstrass Preparation Theorem we have $H = K \cdot W$, where $W \in \mathcal{W}^{(n)}$, $\deg_{Z_n} W > 0$. Therefore $F = (G \cdot K) \cdot W$, and so, by (a), $G \cdot K \in \mathcal{O}^{(n-1)}[Z_n]$. We have obtained a factorization in $\mathcal{O}^{(n-1)}[Z_n]$. Since W is not a unit in $\mathcal{O}^{(n-1)}[Z_n]$, $G \cdot K$ must be a unit. In particular, G is a unit in $\mathcal{O}^{(n)}$. $\square$

Lemma 1.8.2. *Let $F, G, W \in \mathcal{O}^{(n-1)}[Z_n]$ be such that $W = F \cdot G$. Assume that $W \in \mathcal{W}^{(n)}$. Then there exists a unit $H \in \mathcal{O}^{(n-1)}$ such that $H \cdot F$, $(1/H) \cdot G \in \mathcal{W}^{(n)}$.*

Proof. Let $r := \deg_{Z_n} F$, $s := \deg_{Z_n} G$,

$$F = \sum_{j=0}^{r} F_j Z_n^{r-j}, \quad G = \sum_{j=0}^{s} G_j Z_n^{s-j}, \quad W = \sum_{j=0}^{p} W_j Z_n^{p-j},$$

where $W_0 = 1$, $W_j(0') = 0$, $j = 1, \dots, p$. In particular,

$$z_n^p = W(0', z_n) = F(0', z_n) \cdot G(0', z_n) = \sum_{j=0}^{r} F_j(0') z_n^{r-j} \times \sum_{j=0}^{s} G_j(0') z_n^{s-j},$$

and hence $1 = F_0(0') \cdot G_0(0')$, $F_j(0') = 0$, $j = 1, \dots, r$, $G_j(0') = 0$, $j = 1, \dots, s$. $\square$

Lemma 1.8.3. *Let $f_j \in \mathcal{O}(\mathbb{P}_n(r))$, $f_j(0) = 0$, $f_j \not\equiv 0$, $j \in \mathbb{N}$. Then there exists a unitary transformation $L : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ such that $f_j \circ L$ is z_n-normalized, $j \in \mathbb{N}$.*

Proof. Let

$$f_j(z) = \sum_{k=k(j)}^{\infty} Q_{j,k}, \quad z \in \mathbb{P}_n(r),$$

where

$Q_{j,k}$ is a homogeneous polynomial of degree k,
$Q_{j,k} \equiv 0$, $k = 0, \dots, k(j) - 1$,
$Q_{j,k(j)} \not\equiv 0$, $k(j) \geq 1$.

Let $V_j := Q_{j,k(j)}^{-1}(0)$; V_j is a closed cone with $\operatorname{int} V_j = \varnothing$. Hence, by the Baire property, there exists an $X \in \mathbb{C}^n$, $\|X\| = 1$, such that $X \notin V_j$ for any $j \in \mathbb{N}$. Let $L : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be a unitary mapping with $L(e_n) = X$. Then

$$(f_j \circ L)(0', z_n) = f_j(L(z_n e_n)) = f_j(z_n X) = \sum_{k=k(j)}^{\infty} Q_{j,k}(X) z_n^k,$$

and so $(f_j \circ L)(0', \cdot) \not\equiv 0$, $j \in \mathbb{N}$. $\square$

Proposition 1.8.4. *$\mathcal{O}_a^{(n)}$ is a unique factorization domain.*

Proof. It is sufficient to consider the case $a = 0$. We apply induction with respect to n. The case $n = 1$ is clear: every germ $F \in \mathcal{O}^{(1)}$, $F \neq 0$, has a unique factorization $F = z^p G$, where G is a unit in $\mathcal{O}^{(1)}$.

Suppose that the theorem is true for $\mathcal{O}^{(n-1)}$. Consequently, $\mathcal{O}^{(n-1)}[Z_n]$ is also a unique factorization domain.

Fix a germ $F \in \mathcal{O}^{(n)}$ which is not a unit. We may assume that it is z_n-normalized. By the Weierstrass Preparation Theorem we have $F = G \cdot W$, where G is a unit, and $W \in \mathcal{W}^{(n)}$. By the inductive hypothesis W admits a factorization $W = W_1 \cdot \dots \cdot W_k$, where $W_1, \dots, W_k \in \mathcal{O}^{(n-1)}[Z_n]$ are irreducible. By Lemma 1.8.2 we may assume that $W_1, \dots, W_k \in \mathcal{W}^{(n)}$. By Lemma 1.8.1(b) the elements $W_1, \dots, W_k$ are also irreducible in $\mathcal{O}^{(n)}$. Therefore we have obtained a factorization $F = H \cdot W_1 \cdot \dots \cdot W_k$, where H is a unit and $W_1, \dots, W_k \in \mathcal{W}^{(n)}$ are irreducible Weierstrass polynomials.

Suppose that there exists another factorization

$$F = V_1 \cdot \cdots \cdot V_\ell,$$

where $V_1, \dots, V_\ell \in \mathcal{O}^{(n)}$ are irreducible. Then, by the Weierstrass Preparation Theorem, we get $F = \widehat{H} \cdot \widehat{W}_1 \cdot \cdots \cdot \widehat{W}_\ell$, where $\widehat{H}$ is a unit in $\mathcal{O}^{(n)}$ and $\widehat{W}_1, \dots, \widehat{W}_\ell$ are irreducible Weierstrass polynomials. Recall that the decomposition in the Weierstrass Preparation Theorem is unique. Hence $W_1 \cdot \cdots \cdot W_k = \widehat{W}_1 \cdot \cdots \cdot \widehat{W}_\ell$. Now, since $\mathcal{O}^{(n-1)}[Z_n]$ is a unique factorization domain, we get $k = \ell$ and $W_j = \widehat{W}_j$, $j = 1, \dots, k$ (up to a permutation). $\square$

Proposition 1.8.5. *Let U be a neighborhood of $0 \in \mathbb{C}^n$ and let $f, g \in \mathcal{O}(U)$ be such that $f(0) = g(0) = 0$. Assume that the germs $F := \widehat{f}_0$ and $G := \widehat{g}_0$ are relatively prime (in $\mathcal{O}^{(n)}$). Then:*
(a) *There exists a number $r > 0$ such that the germs $\widehat{f}_z$, $\widehat{g}_z$ are relatively prime in $\mathcal{O}_z^{(n)}$ for any $z \in \mathbb{P}_n(r) \subset U$.*
(b) *If $n \geq 2$, then for any neighborhood of zero $V \subset U$ and for any $w \in \mathbb{C}$ there exists a $z \in V$ such that $g(z) \neq 0$ and $f(z)/g(z) = w$.*

Proof. (a) We may assume that F and G are z_n-normalized. By Lemma 1.8.3 and the Weierstrass Preparation Theorem we may assume that $F, G \in \mathcal{W}^{(n)}$. The germs F and G are relatively prime in the ring $\mathcal{O}^{(n-1)}[Z_n]$. Hence, by the Gauss lemma, they are relatively prime in $\Bbbk[Z_n]$, where $\Bbbk$ denotes the quotient field of $\mathcal{O}^{(n-1)}$. Consequently, there exists an $r > 0$ and $f_1, g_1 \in \mathcal{O}(\mathbb{P}_{n-1}(r))[Z_n]$, $h \in \mathcal{O}(\mathbb{P}_{n-1}(r))$, $h \not\equiv 0$, such that

$$h(z') = f_1(z)f(z) + g_1(z)g(z), \quad z = (z', z_n) \in \mathbb{P}_n(r). \tag{1.8.1}$$

Suppose that for some $\zeta = (\zeta', \zeta_n) \in \mathbb{P}_n(r)$ the germs $\widehat{f}_\zeta$, $\widehat{g}_\zeta$ are not relatively prime, and let $C \in \mathcal{O}_\zeta^{(n)}$ be their nontrivial divisor. Then clearly C is z_n-normalized (in $\mathcal{O}_\zeta^{(n)}$). Consequently, by the Weierstrass Preparation Theorem, we may assume that $C \in \mathcal{W}_\zeta^{(n)}$. On the other hand, equality (1.8.1) shows that C must divide $\widehat{h}_\zeta$. Recall that h depends only on z'. Hence, using Lemma 1.8.1(a), we conclude that $\deg_{Z_n} C = 0$; contradiction.

(b) Fix a w. Replacing f by $f - wg$ we may assume that $w = 0$. We may also assume that f and g are z_n-normalized. By the proof of (a) there exist $r > 0$, $f_1, g_1 \in \mathcal{O}(\mathbb{P}_n(r))$, $h \in \mathcal{O}(\mathbb{P}_{n-1}(r))$, $h \not\equiv 0$, such that equality (1.8.1) is true.

Suppose that (b) does not hold, i.e. there exists a neighborhood of zero $V \subset \mathbb{P}_n(r)$ such that $\{z \in V : f(z) = 0\} \subset \{z \in V : g(z) = 0\}$. Let $\mathbb{P}(\boldsymbol{\tau}) \subset\subset V$ be such that $f(0', z_n) \neq 0$ for $0 < |z_n| \leq \tau_n$. Let $\varepsilon := \min\{|f(0', z_n)| : |z_n| = \tau_n\}$. Shrinking $\boldsymbol{\tau}'$ (with fixed τ_n) we may assume that $|f(z', z_n) - f(0', z_n)| < \varepsilon$ for $z' \in \mathbb{P}_{n-1}(\boldsymbol{\tau}')$, $|z_n| = \tau_n$. Now, by Rouché's theorem (cf. [4], Th. V.3.8), for every $z' \in \mathbb{P}_{n-1}(\boldsymbol{\tau}')$ the function $f(z', \cdot)$ has a zero in the disc $K(\tau_n)$. In particular, for any $z' \in \mathbb{P}_{n-1}(\boldsymbol{\tau}')$ there exists a $z_n \in K(\tau_n)$ such that $f(z', z_n) = g(z', z_n) = 0$. Hence, by (1.8.1), $h = 0$ on $\mathbb{P}_n(\boldsymbol{\tau})$; contradiction. $\square$

Proposition 1.8.6. *$\mathcal{O}_a^{(n)}$ is Noetherian.*

Proof. We may assume that $a = 0$. We apply induction on n. In the case $n = 1$ every ideal is principal. Assume that $\mathcal{O}^{(n-1)}$ is Noetherian.

Let $\mathcal{I} \subset \mathcal{O}^{(n)}$ be a nontrivial ideal. We may assume that there exists an $F_0 \in \mathcal{I}$ such that F_0 is z_n-normalized. Let $p := \operatorname{ord}_{0, Z_n} F_0$. Using the Weierstrass Division Theorem, we see that $\mathcal{I}$ is generated over $\mathcal{O}^{(n)}$ by $\{F_0\} \cup \mathcal{M}$, where

$$\mathcal{M} := \{F \in \mathcal{I} \cap \mathcal{O}^{(n-1)}[Z_n] : \deg_{Z_n} F < p\}.$$

Observe that $\mathcal{M}$ is an $\mathcal{O}^{(n-1)}$-module. Hence, by the Hilbert theorem, $\mathcal{M}$ is finitely generated over $\mathcal{O}^{(n-1)}$. Let $F_1, \dots, F_N$ be generators of $\mathcal{M}$. Then $F_0, F_1, \dots, F_N$ generate $\mathcal{I}$ over $\mathcal{O}^{(n)}$. $\square$

Exercises

1.1. Prove the following slight generalization of the Cauchy integral formula. Let $P := \mathbb{P}(a, \boldsymbol{r})$ and let $f \in \mathcal{O}(P) \cap \mathcal{C}(P \cup \partial_0 P)$. Then
$$f(z) = \frac{1}{(2\pi i)^n} \int_{\partial_0 P} \frac{f(\zeta)}{\zeta - z} d\zeta, \quad z \in P.$$
Verify whether, in general, the Cauchy integral formula from Proposition 1.2.5 remains true for $f \in \mathcal{O}(D) \cap \mathcal{C}(D \cup \partial_0 D)$.

1.2. Let $g_k : [0,1] \longrightarrow \mathbb{R}$,
$$g_k(x) := \begin{cases} 1 - \frac{1}{k} & \text{if } 0 \le x \le \frac{1}{k+1} \\ \frac{1}{2} - \frac{1}{k} + \frac{k+1}{2}x & \text{if } \frac{1}{k+1} \le x \le \frac{1}{k} \\ 1 - \frac{1}{2k} & \text{if } \frac{1}{k} \le x \le 1 \end{cases}.$$
Put $f_k := g_k - g_{k-1}$, $k \in \mathbb{N}$ ($g_0 := 0$). Prove that the series $\sum_{k\in\mathbb{N}} f_k$ is uniformly summable on $[0,1]$, but is not normally summable on $[0,1]$.

1.3. Let $A \subset \mathbb{C}^n$ be n-circled. Prove that A is connected iff $R(A)$ is connected.

1.4. Find the domains of convergence of the following series:
$$\sum_{n,m=0}^{\infty} z^n w^m, \quad \sum_{n,m=0}^{\infty} n! z^n w^m, \quad \sum_{n=0}^{\infty} (zw)^n, \quad \sum_{n,m=1}^{\infty} (n/m!) z^n w^m.$$

1.5. Check whether the set $(\mathbb{D} \times (2\mathbb{D})) \cup ((2\mathbb{D}) \times \mathbb{D})$ can be the domain of convergence of a power series.

1.6. Find a power series whose domain of convergence is the ball $\mathbb{B}_2 \subset \mathbb{C}^2$.

1.7. Prove the following version of the Cauchy inequalities. If $f \in \mathcal{O}(\mathbb{B}(a,r)) \cap \mathcal{C}(\overline{\mathbb{B}}(a,r))$, then
$$\|f^{(k)}(a)\| \le \frac{k!}{r^k} \|f\|_{\mathbb{B}(a,r)}, \quad k \in \mathbb{N}_0$$
(recall that for $L := f^{(k)}(a)$ we have $\|L\| := \max\{|L(X)| : \|X\| = 1\}$).

1.8. Let $D \subset \mathbb{C}^n$ be a domain such that $D \cap \mathbb{R}^n \neq \varnothing$. Show that if $f \in \mathcal{O}(D)$ is such that $f = 0$ in $D \cap \mathbb{R}^n$, then $f \equiv 0$.

1.9. Let $D \subset \mathbb{C}^n$ be a domain and let $D^* := \{\overline{z} : z \in D\}$. Assume that $f \in \mathcal{O}(D \times D^*)$ is such that $f(z, \overline{z}) = 0$ for z in a neighborhood of a point $p_0 \in D$. Prove that $f \equiv 0$.

1.10. A set $A \subset \Omega$ is called a *determining set* for $\mathcal{O}(\Omega)$ if for any function $f \in \mathcal{O}(\Omega)$ the following implication is true $f|_A = 0 \Longrightarrow f \equiv 0$.

Construct a countable set $A \subset \mathbb{C}^n$ such that $A \cap \mathbb{B}(r)$ is determining for $\mathcal{O}(\mathbb{B}(r))$ for any $r > 0$.

1.11. Suppose that $f \in \mathcal{O}(\mathbb{C}^n)$ is a complex polynomial with respect to each variable separately. Prove that f is a polynomial.

1.12. Let $D \subset \mathbb{C}^n$ be a bounded domain. The smallest closed subset $A \subset \overline{D}$ such that
$$\forall_{f \in \mathcal{O}(D) \cap \mathcal{C}(\overline{D})} \exists_{a \in A} : |f(a)| = \max\{|f(z)| : z \in \overline{D}\}.$$
is called the *Shilov boundary* of D and is denoted by $\partial_S D$. Notice that $\partial_S D \subset \partial D$.

Prove that:

(a) $\partial_S \mathbb{B}_n = \partial \mathbb{B}_n$.

(b) If $D_1, \dots, D_n$ are fat [34] bounded planar domains, then $\partial_S(D_1 \times \dots \times D_n) = \partial_0(D_1 \times \dots \times D_n) = (\partial D_1) \times \dots \times (\partial D_n)$. In particular, $\partial_S(\mathbb{D}^n) = \partial_0(\mathbb{D}^n) = \mathbb{T}^n$.

1.13. Let $D \subset \mathbb{C}^n$ be a bounded domain. The smallest closed subset $A \subset \overline{D}$ such that
$$\forall_{f \in \mathcal{O}(\Omega):\ \overline{D} \subset \Omega} \exists_{a \in A} : |f(a)| = \max\{|f(z)| : z \in \overline{D}\}$$
is called the *Bergman boundary* of D and is denoted by $\partial_B D$. Prove that $\partial_B D$ is well defined. Observe that $\partial_B D \subset \partial_S D$.

Calculate $\partial_B \mathbb{B}_n$ and $\partial_B \mathbb{D}^n$.

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[34] Ω is *fat* if $\Omega = \operatorname{int} \overline{\Omega}$.

Prove that for the domain

$$D := \{(z_1, z_2) \in \mathbb{C}^2 : 0 < |z_1| < 1,\ |z_2| < |z_1|^{-\log |z_1|}\}$$

we have

$$\partial_0 \mathbb{D}^2 = \partial_B D \subsetneq \partial_S D = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| \le 1,\ |z_2| = |z_1|^{-\log |z_1|}\}.$$

1.14. Show that a complete Hartogs domain $D \subset \mathbb{C}^n$ is convex iff the set $\{(z', |z_n|) \in \mathbb{C}^{n-1} \times \mathbb{R} : (z', z_n) \in D\}$ is convex in $\mathbb{R}^{2n-1}$.

1.15. Let $D \subset \mathbb{C}^n$ be an n-circled domain. Prove that $D \setminus \{(z_1, \dots, z_n) \in \mathbb{C}^n : z_1 \cdot \dots \cdot z_n = 0\}$ is connected (without using the Riemann removable singularities theorem).

CHAPTER 2

Extension of holomorphic functions

2.1. Hartogs and Riemann theorems

Theorem 2.1.1. *Let D be a domain in $\mathbb{C}^n = \mathbb{C}^{n-1} \times \mathbb{C}$, $n \geq 2$, and let M be a relatively closed subset of D such that $D \setminus M$ is connected. Put*

$$\mathbb{C}^{n-1} \times \mathbb{C} \ni (z, w) \overset{p}{\longmapsto} z \in \mathbb{C}^{n-1}.$$

Assume that:
1^o *for every point $a \in p(D)$ there exist an open neighborhood $U_a \subset p(D)$ of a and a compact set $K_a \subset \mathbb{C}$ such that*

$$p^{-1}(U_a) \cap M \subset U_a \times K_a \subset D.$$

2^o *there exists a point $a_0 \in p(D)$ such that*

$$p^{-1}(a_0) \cap M = \varnothing.$$

Then $\mathcal{O}(D \setminus M) = \mathcal{O}(D)|_{D\setminus M}$, i.e. any function $f \in \mathcal{O}(D \setminus M)$ extends holomorphically to D.

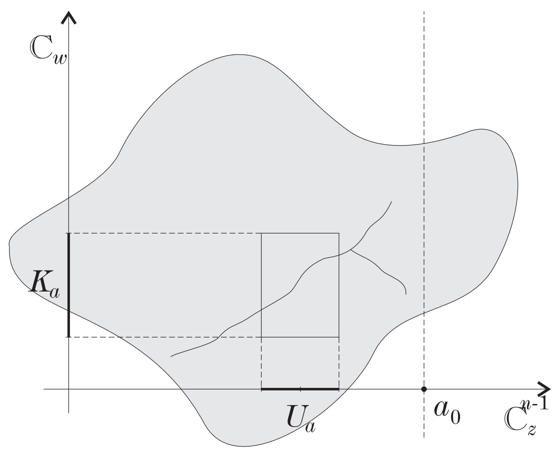

Figure 2.1.1

Proof. Fix a function $f \in \mathcal{O}(D \setminus M)$. Consider the family $\mathfrak{F}$ of all pairs (P, Ω), where

P is an open convex subset of $p(D)$,

Ω is an open subset of $\mathbb{C}$ being a finite union of regular domains (cf. Definition 1.2.4) $\Omega = G_1 \cup \dots \cup G_N$ with $\overline{G}_j \cap \overline{G}_k = \varnothing$ for $j \neq k$,

such that

$$p^{-1}(\overline{P}) \cap M \subset\subset \overline{P} \times \Omega \subset\subset D.$$

Define

$$\widetilde{f}_{P,\Omega}(z, w) := \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(z, \zeta)}{\zeta - w}\, d\zeta, \quad (z, w) \in P \times \Omega.$$

Then $\widetilde{f}_{P,\Omega}(z,\cdot) \in \mathcal{O}(\Omega)$ for every $z \in P$, and $\widetilde{f}_{P,\Omega}(\cdot,w) \in \mathcal{O}_s(P)$ for every $w \in \Omega$. Moreover, $\widetilde{f}_{P,\Omega}$ is continuous. Hence, by Osgood's theorem 1.2.3, $\widetilde{f}_{P,\Omega} \in \mathcal{O}(P \times \Omega)$. Using the Cauchy theorem (cf. [4], IV.5) we easily conclude that $\widetilde{f}_{P',\Omega'} = \widetilde{f}_{P'',\Omega''}$ in $(P' \times \Omega') \cap (P'' \times \Omega'')$ for arbitrary two (P',Ω'), $(P'',\Omega'') \in \mathfrak{F}$ (1) .

Observe that, by 1^o, for each $(a,w_0) \in D$ there exists a pair $(P,\Omega) \in \mathfrak{F}$ such that $(a,w_0) \in P \times \Omega$ (2) . Thus the formula

$$\widetilde{f}(z,w) := \widetilde{f}_{P,\Omega}(z,w), \quad (z,w) \in P \times \Omega,$$

defines a function holomorphic on D.

By 2^o and the Cauchy integral formula we see that there exists an open neighborhood U_{a_0} such that $\widetilde{f} = f$ in $D \cap p^{-1}(U_{a_0}) \subset D \setminus M$. Since $D \setminus M$ is connected, the identity principle implies that $\widetilde{f} = f$ in $D \setminus M$. □

It is clear that if $M \subset D$ is compact then conditions 1^o and 2^o from Theorem 2.1.1 are always satisfied. Consequently, we get the following

Corollary 2.1.2 (Hartogs' extension theorem)**.** *Let D be a domain in $\mathbb{C}^n$, $n \geq 2$, and let K be a compact subset of D such that $D \setminus K$ is connected. Then $\mathcal{O}(D \setminus K) = \mathcal{O}(D)|_{D\setminus K}$* (3) .

Notice that the above result does not hold for $n = 1$ (4) .

Corollary 2.1.3. *For $n \geq 2$ the zeros of holomorphic functions are not isolated.*

Proof. Suppose that $f \in \mathcal{O}(\mathbb{P}_n(a,r))$, $n \geq 2$, $f(a) = 0$, and $f(z) \neq 0$ for $z \neq a$. Then, by Hartogs' extension theorem, the function $1/f$ would extend holomorphically onto $\mathbb{P}_n(a,r)$; contradiction. □

Definition 2.1.4. A subset M of an open set $\Omega \subset \mathbb{C}^n$ is called *thin* in Ω if for every point $a \in \Omega$ there exist a polydisc $P = \mathbb{P}(a,r) \subset \Omega$ and a function $\varphi \in \mathcal{O}(P)$, $\varphi \not\equiv 0$, such that $M \cap P \subset \varphi^{-1}(0)$ (5) .

Remark 2.1.5. (a) If M is thin, then $\operatorname{int} M = \varnothing$.
(b) If M is thin in Ω and $N \subset M$, then N is thin in Ω.
(c) If M_1, M_2 are thin in Ω, then $M_1 \cup M_2$ is thin in Ω.

Assume that $\varphi \in \mathcal{O}(D)$, $\varphi \not\equiv 0$, where $D \subset \mathbb{C}^n$ is a domain. Then $\varphi^{-1}(0)$ is thin in D.

Theorem 2.1.6 (Riemann removable singularities theorem)**.** *Let D be a domain in $\mathbb{C}^n$ and let $M \subset D$ be thin and relatively closed in D. Then every function $f \in \mathcal{O}(D \setminus M)$ which is locally bounded in D* (6) *extends holomorphically to D.*

Moreover, the set $D \setminus M$ is connected.

Proof. Fix a function $f \in \mathcal{O}(D \setminus M)$ such that f is locally bounded on D. Observe that the problem of continuation across M is local. In fact, if every point $a \in D$ admits a polydisc P_a and a function $\widetilde{f}_a \in \mathcal{O}(P_a)$ such that $\widetilde{f}_a = f$ in $P_a \setminus M$, then by identity principle and Remark 2.1.5(a), the function $\widetilde{f}$ defined as $\widetilde{f} := \widetilde{f}_a$ in P_a gives the required extension.

Fix an $a \in D$. We may assume that $a = 0 \in M$. Let $\varphi \in \mathcal{O}(\mathbb{P}(\boldsymbol{r}))$, $\varphi \not\equiv 0$, be such that $M \cap \mathbb{P}(\boldsymbol{r}) \subset \varphi^{-1}(0)$. Let $\boldsymbol{r} = (\boldsymbol{r}', r_n) = (r_1, \dots, r_n)$. Changing the coordinate system if necessary (cf. Lemma 1.8.3) we may assume that:

---

(1) Assume that $(P' \times \Omega') \cap (P'' \times \Omega'') \neq \varnothing$ and let $(z_0, w_0) \in (P' \times \Omega') \cap (P'' \times \Omega'')$ be fixed. Let Ω'_0 be the connected component of $\Omega' \cap \Omega''$ that contains w_0. There exists an $\Omega \subset \Omega' \cap \Omega''$ such that $(P' \cap P'', \Omega) \in \mathfrak{F}$ and $\Omega \cap \Omega'_0 \neq \varnothing$. By the Cauchy theorem, we conclude that $f_{P',\Omega'}(z_0,w) = f_{P' \cap P'',\Omega}(z_0,w) = f_{P'',\Omega''}(z_0,w)$ for any $w \in \Omega$. Hence, by the identity principle, $f_{P',\Omega'}(z_0,w_0) = f_{P'',\Omega''}(z_0,w_0)$.

(2) Let U_a, K_a be as in 1^o. It is clear that there exists an Ω as in the definition of $\mathfrak{F}$ such that $K_a \subset \Omega$, $w_0 \in \Omega$, and $\{a\} \times \Omega \subset\subset D$. Let $P \subset\subset U_a$ be an open convex neighborhood of a such that $\overline{P} \times \Omega \subset\subset D$. Then (P,Ω) satisfies all the required conditions.

(3) See the end of § 4.2 for another proof.

(4) For example, $f(z) := 1/z$, $z \in \mathbb{C} \setminus \{0\}$.

(5) Note that M need not be closed in Ω.

(6) That is, every point $a \in D$ has a neighborhood U_a such that f is bounded in $U_a \setminus M$.

φ is z_n-normalized, $\varphi(0', \cdot)$ has zero of order p at $z_n = 0$ ($p \in \mathbb{N}$),
φ is holomorphic in a neighborhood of $\overline{\mathbb{P}}(\boldsymbol{r})$,
$\varphi(0', z_n) \neq 0$ for $z \in \overline{K}(r_n) \setminus \{0\}$.

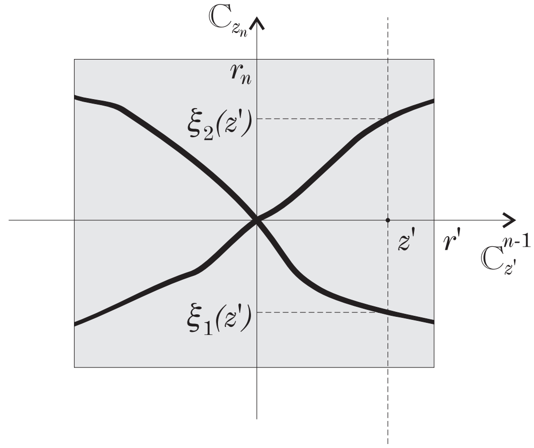

Figure 2.1.2

Let $\varepsilon := \min\{|\varphi(0', z_n)| : |z_n| = r_n\}$. Shrinking $\boldsymbol{r}'$ (with fixed r_n) we may assume that $|\varphi(z', z_n) - \varphi(0', z_n)| < \varepsilon$ for $z' \in \mathbb{P}_{n-1}(\boldsymbol{r}')$, $|z_n| = r_n$. Now, by Rouché's theorem (cf. [4], Th. V.3.8), for every $z' \in \mathbb{P}_{n-1}(\boldsymbol{r}')$ the function $\varphi(z', \cdot)$ has exactly p zeros (counted with multiplicities) in the disc $K(r_n)$, say $\xi_1(z'), \dots, \xi_p(z')$, and $\varphi(z', \cdot)$ does not vanish on the circle $C(r_n)$. In particular, for every $z' \in \mathbb{P}_{n-1}(\boldsymbol{r}')$ the function $f(z', \cdot)$ is holomorphic in $\overline{K}(r_n) \setminus \{\xi_1(z'), \dots, \xi_p(z')\}$ and locally bounded in $K(r_n)$. Hence, by the classical (one-dimensional) Riemann theorem on removable singularities (cf. [4], V.1), $f(z', \cdot)$ extends holomorphically to a function $\widetilde{f(z', \cdot)} \in \mathcal{O}(K(r_n))$. Let $\widetilde{f}(z', z_n) := \widetilde{f(z', \cdot)}(z_n)$, $(z', z_n) \in \mathbb{P}(\boldsymbol{r})$. By the Hartogs Lemma 1.5.6 $\widetilde{f} \in \mathcal{O}(\mathbb{P}(\boldsymbol{r}))$. It is clear that $\widetilde{f} = f$ in $\mathbb{P}(\boldsymbol{r}) \setminus M$.

It remains to prove that $D \setminus M$ is connected. Suppose that $D \setminus M = \Omega_0 \cup \Omega_1$, where Ω_0 and Ω_1 are non-empty, disjoint, and open. Then the function $f := j$ in Ω_j, $j = 0, 1$, would extend holomorphically onto D; contradiction. □

Proposition 2.1.7. *Let $D \subset \mathbb{C}$ be a regular domain (cf. Definition 1.2.4), let $G \subset \mathbb{C}^{n-1}$ be an arbitrary bounded domain, and let $w_0 \in G$ be fixed. Put*

$$H := (\overline{D} \times \{w_0\}) \cup (\partial D \times \overline{G}) \subset \mathbb{C}^n.$$

Then every function holomorphic in a neighborhood of H extends holomorphically to a neighborhood of $\overline{D} \times \overline{G}$.

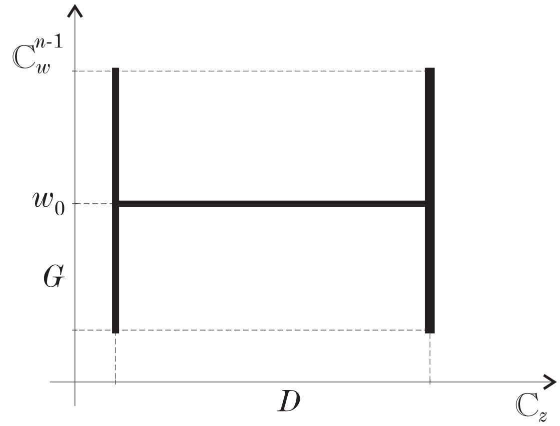

Figure 2.1.3

Proof. Note that H is connected. Let $f \in \mathcal{O}(U)$, where U is a neighborhood of H. Then there exist domains $D_0 \subset \mathbb{C}$, $G_0 \subset \mathbb{C}^{n-1}$, and a neighborhood V of w_0 such that:

- D_0 is regular, G_0 is bounded,
- $D \subset\subset D_0$, $G \subset\subset G_0$,
- $(\overline{D}_0 \times V) \cup (\partial D_0 \times G_0) \subset U$.

Define

$$\widetilde{f}(z,w) := \frac{1}{2\pi i}\int_{\partial D_0} \frac{f(\zeta,w)}{\zeta - z}\,d\zeta, \quad (z,w) \in D_0 \times G_0.$$

One can easily check that $\widetilde{f} \in \mathcal{O}(D_0 \times G_0)$ and $\widetilde{f} = f$ in $D_0 \times V$ (by the Cauchy integral formula). Finally, by the identity principle, $\widetilde{f} = f$ in a neighborhood of H. $\square$

Corollary 2.1.8. *Every function holomorphic in a neighborhood of the set*

$$H := (\overline{\mathbb{D}} \times \{0\}) \cup (\mathbb{T} \times \overline{\mathbb{D}}) \subset \mathbb{C}^2$$

extends holomorphically to a neighborhood of $\overline{\mathbb{D}} \times \overline{\mathbb{D}}$.

2.2. Biholomorphisms

Theorem 2.2.1. *Let $\Omega \subset \mathbb{C}^n$ be open and let $F = (F_1, \dots, F_n) : \Omega \longrightarrow \mathbb{C}^n$ be holomorphic. Then the following conditions are equivalent:*
(i) *$F(\Omega)$ is open and $F : \Omega \longrightarrow F(\Omega)$ is biholomorphic;*
(ii) *F is injective and $J_{\mathbb{C}}F(z) \neq 0$, $z \in \Omega$;*
(iii) *F is injective.*

Proof. The implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) are obvious.

The implication (ii) $\Longrightarrow$ (i) follows from the inverse mapping theorem (cf. Corollary 1.4.15).

(iii) $\Longrightarrow$ (ii) (Cf. [28]). We apply induction on n. The case $n = 1$ is well known. Assume that the result is true for $n - 1$ and let

$$A := \{a \in \Omega : F'(a) = 0\}.$$

Take an $a \in \Omega \setminus A$. We may assume that $a = F(a) = 0$ and $\frac{\partial F_n}{\partial z_n}(0) = c \neq 0$. Let

$$G(z) = G(z', z_n) := (z', F_n(z)), \quad z = (z', z_n) \in \Omega \subset \mathbb{C}^{n-1} \times \mathbb{C}.$$

Observe that $G(0) = 0$ and $J_{\mathbb{C}}G(0) = c \neq 0$. Hence, by the inverse mapping theorem (Corollary 1.4.15), there exists an open neighborhood U of 0, $U \subset \Omega$, such that $G(U)$ is open and $G|_U : U \longrightarrow G(U)$ is biholomorphic. Let

$$H = (H_1, \dots, H_n) := F \circ (G|_U)^{-1} : G(U) \longrightarrow \mathbb{C}^n.$$

Then H is an injective holomorphic mapping with $H_n(w) = w_n$. Take a $\mathbb{P}(r) \subset G(U)$ and define

$$\widetilde{H}(w') := (H_1(w', 0), \dots, H_{n-1}(w', 0)), \quad w' \in \mathbb{P}_{n-1}(r).$$

Observe that $\widetilde{H} : \mathbb{P}_{n-1}(r) \longrightarrow \mathbb{C}^{n-1}$ is injective. Moreover, $J_{\mathbb{C}}H(0', 0) = J_{\mathbb{C}}\widetilde{H}(0')$. By the inductive assumption we get $J_{\mathbb{C}}\widetilde{H}(0') \neq 0$. Hence $J_{\mathbb{C}}H(0) \neq 0$ and, consequently, $J_{\mathbb{C}}F(0) \neq 0$.

It remains to show that $A = \varnothing$. Suppose that $A \neq \varnothing$ and let $a \in A$. We may assume that $a = F(a) = 0$. Put

$$u(z) := \|F(z)\|^2 = |F_1(z)|^2 + \dots + |F_n(z)|^2, \quad z \in \Omega.$$

Notice that $u^{-1}(0) = \{0\}$. Fix a ball $\mathbb{B}(r) \subset\subset \Omega$ and let

$$t_0 := \min\{u(z) : z \in \partial\mathbb{B}(r)\}.$$

Put $U_t := \{z \in \mathbb{B}(r) : u(z) < t\}$. Observe that $U_t \subset\subset \mathbb{B}(r)$ for $0 < t < t_0$. By the Sard theorem (cf. [6], Th. 3.4.3) there exists a $t \in (0, t_0)$ such that $\operatorname{grad} u(z) \neq 0$ for all $z \in \Omega$ with $u(z) = t$. In particular, $A \cap \partial U_t = \varnothing$. Let D denote the connected component of U_t that contains 0. Then $K := A \cap D$ is a compact thin set with $0 \in K$. In particular, the set $D \setminus K$ is connected (Theorem 2.1.6). Observe that, by the first part of the proof, we have $A = \{z \in \Omega : J_{\mathbb{C}}F(z) = 0\}$. Consequently, by the Hartogs theorem (Corollary 2.1.2), the function $1/J_{\mathbb{C}}F$ extends holomorphically to D; contradiction (cf. Corollary 2.1.3). $\square$

2.3. Cartan theorems

Given an arbitrary domain $D \subset \mathbb{C}^n$ set

$$\operatorname{Aut}(D) := \{F : D \longrightarrow D : F \text{ is biholomorphic}\}, \qquad \operatorname{Aut}_a(D) := \{F \in \operatorname{Aut}(D) : F(a) = a\}, \quad a \in D.$$

Obviously, $\operatorname{Aut}(D)$ is a group and $\operatorname{Aut}_a(D)$ is a subgroup of $\operatorname{Aut}(D)$. The group $\operatorname{Aut}(D)$ is called the *group of automorphisms* of D. We say that $\operatorname{Aut}(D)$ *acts transitively* on D if for arbitrary $z', z'' \in D$ there exists an $F \in \operatorname{Aut}(D)$ such that $F(z') = z''$.

Notice that if $\Phi : D \longrightarrow G$ is biholomorphic, then

$$\operatorname{Aut}(D) \ni F \longmapsto \Phi \circ F \circ \Phi^{-1} \in \operatorname{Aut}(G)$$

is a group isomorphism.

Remark 2.3.1. (a) (cf. [4])

$$\begin{aligned}\operatorname{Aut}(\widehat{\mathbb{C}}) &= \{\widehat{\mathbb{C}} \ni z \longrightarrow \frac{az+b}{cz+d} \in \widehat{\mathbb{C}} : \quad a,b,c,d \in \mathbb{C},\ \det \begin{bmatrix} a,b \\ c,d \end{bmatrix} \neq 0\},\\ \operatorname{Aut}(\mathbb{C}) &= \{\mathbb{C} \ni z \longmapsto az+b \in \mathbb{C} : \quad a,b \in \mathbb{C},\ a \neq 0\},\\ \operatorname{Aut}(\mathbb{D}) &= \{\mathbb{D} \ni z \longmapsto e^{i\theta}\frac{z-a}{1-\overline{a}z} \in \mathbb{D} : \quad \theta \in \mathbb{R},\ a \in \mathbb{D}\};\end{aligned}$$

$\operatorname{Aut}(\widehat{\mathbb{C}})$, $\operatorname{Aut}(\mathbb{C})$, $\operatorname{Aut}(\mathbb{D})$ act transitively.
(b) Let

$$A := \{z \in \mathbb{C} : \frac{1}{R} < |z| < R\}.$$

Then

$$\operatorname{Aut}(A) = \{A \ni z \longmapsto e^{i\theta}z \in A : \quad \theta \in \mathbb{R}\} \cup \{A \ni z \longmapsto e^{i\theta}\frac{1}{z} \in A : \quad \theta \in \mathbb{R}\};$$

$\operatorname{Aut}(A)$ does not act transitively; cf. [29].

Indeed, fix an $F \in \operatorname{Aut}(A)$. Let $C := C(1)$ and let $A_- := \{1/R < |z| < 1\}$, $A_+ := \{1 < |z| < R\}$. Since $F^{-1}(C)$ is a compact subset of A, there exists an $R' \in (1,R)$ such that $F^{-1}(C) \subset \{z \in \mathbb{C} : 1/R' < |z| < R'\}$. Consider the set $B_+ := F(\{z \in \mathbb{C} : R' < |z| < R\})$. Since $B_+ \cap C = \varnothing$, the set B_+ is contained either in A_+ or in A_-. Taking $1/F$ instead of F (if necessary), we may assume that $B_+ \subset A_+$. Now consider the set $B_- := F(\{z \in \mathbb{C} : 1/R < |z| < 1/R'\})$. It is clear that $B_- \subset A_-$. Thus

$$\lim_{|z|\to 1/R} |F(z)| = 1/R, \qquad \lim_{|z|\to R} |F(z)| = R.$$

Hence, by the classical Hadamard three circles theorem (cf. [4], Th. 3.13; see also Proposition 3.2.37), we get

$$|F(z)| \leq (1/R)^{\frac{\log \frac{R}{|z|}}{\log \frac{R}{1/R}}} R^{\frac{\log \frac{|z|}{1/R}}{\log \frac{R}{1/R}}} = |z|, \quad z \in A.$$

Since F^{-1} has the same properties, we conclude that $|F^{-1}(w)| \leq |w|$ for any $w \in A$. Thus $|F(z)| = |z|$ for any $z \in A$, and, therefore, there exists a $\theta \in \mathbb{R}$ such that $F(z) = e^{i\theta}z$, $z \in A$.

Theorem 2.3.2 (Cartan)**.** *Let $D \subset \mathbb{C}^n$ be a bounded domain, let $a \in D$, and assume that $F : D \longrightarrow D$ is a holomorphic mapping such that $F(a) = a$ and $F'(a) = \mathrm{id}$. Then $F \equiv \mathrm{id}$.* (7)

Proof. Without loss of generality we may assume that $a = 0$. Suppose that $F \not\equiv \mathrm{id}$. Fix $r, R > 0$ such that $P := \mathbb{P}(r) \subset D \subset \mathbb{B}(R)$. We have

$$F(z) = \sum_{k=0}^{\infty} Q_k(z), \quad z \in P,$$

where $Q_k : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ is a homogeneous polynomial of degree k. By our assumptions, $Q_0 = 0$ and $Q_1 = \mathrm{id}$. Let $k_0 \geq 2$ be such that $Q_2 = \dots = Q_{k_0-1} = 0$, $Q_{k_0} \not\equiv 0$. Let $F^{\langle\nu\rangle}$ denote the ν-th iterate of the mapping F, i.e. $F^{\langle 0\rangle} := \mathrm{id}$, $F^{\langle\nu+1\rangle} := F^{\langle\nu\rangle} \circ F$. Then

$$F^{\langle\nu\rangle}(z) = z + \nu Q_{k_0} + \sum_{k=k_0+1}^{\infty} Q_{\nu,k}(z), \quad z \in P.$$

Hence, by the Cauchy integral formula, for any $z \in P$ we get

$$\|\nu Q_{k_0}(z)\| = \|\frac{1}{2\pi i}\int_{\mathbb{T}} \frac{F^{\langle\nu\rangle}(\zeta z)}{\zeta^{k_0+1}}\, d\zeta\| \leq \max\{\|F^{\langle\nu\rangle}(\zeta z)\| : \zeta \in \mathbb{T}\} \leq R, \quad \nu \geq 1.$$

Letting $\nu \longrightarrow +\infty$, we obtain $Q_{k_0} \equiv 0$; contradiction. □

(7) Notice that the assumption that D is bounded is essential (take for instance $D = \mathbb{C}$, $F(z) := z(1-z)$, $a := 0$).

Proposition 2.3.3 (Cartan)**.** *Let $D, G \subset \mathbb{C}^n$ be circular domains with $0 \in D$, $0 \in G$, such that D is bounded, and let $F : D \longrightarrow G$ be a biholomorphic mapping with $F(0) = 0$. Then F is a linear isomorphism.* (8) (9)

Proof. For $\theta \in \mathbb{R}$ put
$$H_\theta(z) := F^{-1}(e^{-i\theta}F(e^{i\theta}z)), \quad z \in D.$$
Then H_θ is well defined, $H_\theta \in \operatorname{Aut}(D)$, $H_\theta(0) = 0$, and $H'_\theta(0) = \mathrm{id}$. Therefore, by Theorem 2.3.2, $H_\theta = \mathrm{id}$, i.e.
$$F(e^{i\theta}z) = e^{i\theta}F(z), \quad z \in D,\ \theta \in \mathbb{R}.$$
Let
$$F(z) = \sum_{k=1}^{\infty} Q_k(z), \quad z \in \mathbb{P}(r),$$
be the expansion of F into the series of homogeneous polynomials in a polydisc $\mathbb{P}(r) \subset D$. Then
$$F(z) = \sum_{k=1}^{\infty} e^{i(k-1)\theta} Q_k(z), \quad z \in D,\ \theta \in \mathbb{R}.$$
This means (Exercise) that $Q_k = 0$ in $\mathbb{P}(r)$ for $k \geq 2$, and so, by the identity principle, $F \equiv Q_1$. Therefore F is a linear mapping. Since F is biholomorphic, it must be a linear isomorphism. $\square$

2.4. Automorphism group of $\mathbb{D}^n$

Given $a \in \mathbb{D}$ put
$$h_a(z) := \frac{z-a}{1-\bar{a}z}.$$

Theorem 2.4.1.
$$\operatorname{Aut}(\mathbb{D}^n) = \{\mathbb{D}^n \ni (z_1, \dots, z_n) \longmapsto (e^{i\theta_1}h_{a_1}(z_{\sigma(1)}), \dots, e^{i\theta_n}h_{a_n}(z_{\sigma(n)})) \in \mathbb{D}^n : \theta_j \in \mathbb{R},\ a_j \in \mathbb{D},\ j = 1 \dots, n,\ \sigma \in \mathfrak{S}_n\} =: \mathfrak{G}.$$
$$\operatorname{Aut}_0(\mathbb{D}^n) = \{\mathbb{D}^n \ni (z_1, \dots, z_n) \longmapsto (e^{i\theta_1}z_{\sigma(1)}, \dots, e^{i\theta_n}z_{\sigma(n)}) \in \mathbb{D}^n : \theta_j \in \mathbb{R},\ j = 1 \dots, n,\ \sigma \in \mathfrak{S}_n\} =: \mathfrak{G}_0,$$
where $\mathfrak{S}_n$ denotes the group of all permutations of n-elements.

The group $\operatorname{Aut}(\mathbb{D}^n)$ acts transitively.

Proof. It is easy to see that $\mathfrak{G}$ is a subgroup of $\operatorname{Aut}(\mathbb{D}^n)$, $\mathfrak{G}_0$ is a subgroup of $\operatorname{Aut}_0(\mathbb{D}^n)$, and $\mathfrak{G}$ acts transitively on $\mathbb{D}^n$. It remains to show that $\operatorname{Aut}_0(\mathbb{D}^n) \subset \mathfrak{G}_0$. By Lemma 1.4.26 and Proposition 2.3.3, any automorphism $F \in \operatorname{Aut}_0(\mathbb{D}^n)$ is a linear isomorphism such that $|F(z)| = |z|$, $z \in \mathbb{D}^n$. Let $[L_{j,k}]_{j,k=1,\dots,n}$ denote the matrix representation of F. We have
$$\max_{j=1,\dots,n}\{|\sum_{k=1}^{n} L_{j,k}z_k|\} = \max\{|z_1|, \dots, |z_n|\}, \quad z = (z_1, \dots, z_n) \in \mathbb{D}^n.$$
In particular,
$$\max\{|L_{1,k}|, \dots, |L_{n,k}|\} = 1, \quad k = 1, \dots, n, \qquad |L_{j,1}| + \dots + |L_{j,n}| \leq 1, \quad j = 1, \dots, n.$$
Thus the matrix $[L_{j,k}]$ has in each row, and each column, exactly one nonzero element (which must have absolute value 1). This means that $F \in \mathfrak{G}_0$. $\square$

---

(8) Notice that the assumption that D is bounded is essential (take for instance $D = G = \mathbb{C}^2$, $F(z_1, z_2) := (z_1 + f(z_2), z_2)$, where $f \in \mathcal{O}(\mathbb{C})$ is a nonlinear entire function such that $f(0) = 0$).

(9) Cf. Exercise 4.1.

2.5. Automorphism group of $\mathbb{B}_n$

For $a \in \mathbb{B}_n$ let

$$\chi_a(z) := \frac{1}{\|a\|^2} \frac{\sqrt{1-\|a\|^2}(\|a\|^2 z - \langle z, a\rangle a) - \|a\|^2 a + \langle z, a\rangle a}{1 - \langle z, a\rangle} \quad \text{if } a \neq 0,$$
$$\chi_0(z) := \mathrm{id},$$

where $\langle\cdot,\cdot\rangle$ denotes the standard complex scalar product in $\mathbb{C}^n$. Let $\mathfrak{U}(\mathbb{C}^n)$ denote the group of unitary automorphisms of $\mathbb{C}^n$.

Recall that a linear mapping $L : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ is unitary if $\langle L(z'), L(z'')\rangle = \langle z', z''\rangle$, $z', z'' \in \mathbb{C}^n$ (or, equivalently, $\|L(z)\| = \|z\|$, $z \in \mathbb{C}^n$).

Notice that the mapping χ_a is defined in the domain

$$D_a := \{z \in \mathbb{C}^n : \langle z, a\rangle \neq 1\} \supset \overline{\mathbb{B}}_n$$

and $\chi_a(a) = 0$.

Theorem 2.5.1.

$$\mathrm{Aut}(\mathbb{B}_n) = \{U \circ \chi_a : U \in \mathfrak{U}(\mathbb{C}^n),\ a \in \mathbb{B}_n\}, \qquad \mathrm{Aut}_0(\mathbb{B}_n) = \mathfrak{U}(\mathbb{C}^n).$$

The group $\mathrm{Aut}(\mathbb{B}_n)$ *acts transitively.*

Proof. The fact that $\mathrm{Aut}_0(\mathbb{B}_n) = \mathfrak{U}(\mathbb{C}_n)$ follows immediately from Lemma 1.4.26 and Proposition 2.3.3.

We move to the characterization of the full group $\mathrm{Aut}(\mathbb{B}_n)$. Since $\chi_a(a) = 0$, we only need to prove that $\chi_a \in \mathrm{Aut}(\mathbb{B}_n)$.

Fix an $a \in \mathbb{B}_n$. The case $a = 0$ is trivial, so assume that $a \neq 0$.

Direct calculations show (Exercise) that

$$1 - \langle \chi_a(z), \chi_a(w)\rangle = \frac{(1 - \langle a, a\rangle)(1 - \langle z, w\rangle)}{(1 - \langle z, a\rangle)(1 - \langle a, w\rangle)}, \quad z, w \in \overline{\mathbb{B}}_n$$

(cf. [30], Th. 2.2.2). Taking $w = z$, we conclude that $\chi_a(\mathbb{B}_n) \subset \mathbb{B}_n$ and $\chi_a(\partial\mathbb{B}_n) \subset \partial\mathbb{B}_n$.

In particular, $\chi_a \circ \chi_{-a}$ is well defined in a neighborhood of $\overline{\mathbb{B}}_n$. Using once again direct calculations, we prove (Exercise) that $\chi_a \circ \chi_{-a} = \mathrm{id}$. Hence $\chi_a \in \mathrm{Aut}(\mathbb{B}_n)$ and $(\chi_a)^{-1} = \chi_{-a}$. $\square$

2.6. Laurent series

Given $\boldsymbol{r}^- = (r_1^-, \dots, r_n^-)$, $\boldsymbol{r}^+ = (r_1^+, \dots, r_n^+) \in \mathbb{R}_+^n$ with $r_j^- < r_j^+$, $j = 1, \dots, n$, let

$$A = A(\boldsymbol{r}^-, \boldsymbol{r}^+) := A_1 \times \dots \times A_n \subset \mathbb{C}^n,$$

where

$$A_j := \{z \in \mathbb{C} : r_j^- \leq |z| \leq r_j^+\}.$$

Let f be holomorphic in a neighborhood of A. Given $\boldsymbol{r} = (r_1, \dots, r_n) \in \mathbb{R}_{>0}^n \cap A$, define

$$a_\alpha(\boldsymbol{r}) = a_\alpha^f(\boldsymbol{r}) := \frac{1}{(2\pi i)^n} \int_{\partial_0\mathbb{P}(\boldsymbol{r})} \frac{f(\zeta)}{\zeta^{\alpha+\boldsymbol{1}}}\, d\zeta, \quad \alpha \in \mathbb{Z}^n.$$

Proposition 2.6.1. (a) *The number* $a_\alpha = a_\alpha^f := a_\alpha^f(\boldsymbol{r})$ *is independent of* $\boldsymbol{r}$ *and*

$$|a_\alpha| \leq \frac{\|f\|_A}{\boldsymbol{r}^\alpha}, \quad \alpha \in \mathbb{Z}^n, \quad \boldsymbol{r} \in \mathbb{R}_{>0}^n \cap A. \tag{2.6.1}$$

(b) *If* $r_j^- = 0$ *for some* $j \in \{1, \dots, n\}$, *then* $a_\alpha = 0$ *for every* $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$ *with* $\alpha_j < 0$.

(c) *For every* $0 < \theta < 1$ *with* $\theta > \theta_0 := \max\{\sqrt{r_j^-/r_j^+} : j = 1, \dots, n\}$ *we have*

$$\|a_\alpha z^\alpha\|_{A_\theta} \leq \|f\|_A \theta^{|\alpha_1| + \dots + |\alpha_n|}, \quad \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n, \tag{2.6.2}$$

with

$$A_\theta := A(\frac{1}{\theta}\boldsymbol{r}^-, \theta\boldsymbol{r}^+).$$

(d) *The* Laurent series $\sum_{\alpha\in\mathbb{Z}^n} a_\alpha z^\alpha$ *converges locally normally in* $\operatorname{int} A$.
(e) $f(z) = \sum_{\alpha\in\mathbb{Z}^n} a_\alpha z^\alpha, \quad z \in \operatorname{int} A$.

Proof. (a) We apply induction on n. For $n = 1$ the result is well known (cf. [4], V.1.11). Assume that it is true for $n-1$. Let $\boldsymbol{r} = (r_1, \dots, r_n)$, $r' = (r'_1, \dots, r'_n) \in \mathbb{R}^n_{>0} \cap A$. Then

$$a^f_\alpha(\boldsymbol{r}') = \frac{1}{2\pi i}\int_{C(r'_n)} a^{f(\cdot,\zeta_n)}_{(\alpha_1,\dots,\alpha_{n-1})}((r'_1, \dots, r'_{n-1}))\frac{d\zeta_n}{\zeta_n^{\alpha_n+1}} = \frac{1}{2\pi i}\int_{C(r'_n)} a^{f(\cdot,\zeta_n)}_{(\alpha_1,\dots,\alpha_{n-1})}((r_1, \dots, r_{n-1}))\frac{d\zeta_n}{\zeta_n^{\alpha_n+1}}$$
$$= a^f_\alpha((r_1, \dots, r_{n-1}, r'_n)).$$

Similar argument with respect to the first variable shows that

$$a^f_\alpha((r_1, \dots, r_{n-1}, r'_n)) = a^f_\alpha(\boldsymbol{r}).$$

Directly from the definition of $a_\alpha(r)$ we get (2.6.1).

(b) Use (2.6.1) with $r_j \longrightarrow 0$.

(c) Fix $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$ and $\theta_0 < \theta < 1$. We may assume that $\alpha_1, \dots, \alpha_s \geq 0$, $\alpha_{s+1}, \dots, \alpha_n < 0$ with $0 \leq s \leq n$. Take a $z = (z_1, \dots, z_n) \in A_\theta$ with $z_1 \cdot \dots \cdot z_n \neq 0$. Let $\boldsymbol{r} := (|z_1|/\theta, \dots, |z_s|/\theta, \theta|z_{s+1}|, \dots, \theta|z_n|)$. Observe that $\boldsymbol{r} \in \mathbb{R}^n_{>0} \cap A$. Using (a), we get:

$$|a_\alpha z^\alpha| \leq \frac{\|f\|_A}{\boldsymbol{r}^\alpha}|z_1|^{\alpha_1} \cdot \dots \cdot |z_n|^{\alpha_n} = \|f\|_A \theta^{\alpha_1+\dots+\alpha_s-\alpha_{s+1}-\dots-\alpha_n} = \|f\|_A \theta^{|\alpha_1|+\dots+|\alpha_n|}.$$

(d) For every compact subset $K \subset \operatorname{int} A$ there exists $0 < \theta < 1$ such that $K \subset A_\theta$. Now we can apply (2.6.2) [10].

(e) We apply induction on n. For $n = 1$ the result is well known. Assume that it is true for $n-1$. Fix a $z = (z_1, z') \in \operatorname{int} A$ and observe that

$$f(z) = \sum_{\alpha_1\in\mathbb{Z}} a^{f(\cdot,z')}_{\alpha_1} z_1^{\alpha_1} = \sum_{\alpha_1\in\mathbb{Z}} \Big(\frac{1}{2\pi i}\int_{C(r_1)} \frac{f(\zeta_1, z')}{\zeta_1^{\alpha_1+1}}\, d\zeta_1\Big) z_1^{\alpha_1}$$
$$= \sum_{\alpha_1\in\mathbb{Z}} \Big(\frac{1}{2\pi i}\int_{C(r_1)} \frac{1}{\zeta_1^{\alpha_1+1}}\Big(\sum_{\alpha'\in\mathbb{Z}^{n-1}} a^{f(\zeta_1,\cdot)}_{\alpha'}(z')^{\alpha'}\Big)d\zeta_1\Big) z_1^{\alpha_1} = \sum_{\alpha\in\mathbb{Z}^n} \Big(\frac{1}{2\pi i}\int_{C(r_1)} \frac{a^{f(\zeta_1,\cdot)}_{\alpha'}}{\zeta_1^{\alpha_1+1}}\, d\zeta_1\Big) z^\alpha$$
$$= \sum_{\alpha\in\mathbb{Z}^n} \Big(\frac{1}{2\pi i}\int_{C(r_1)} \frac{1}{\zeta_1^{\alpha_1+1}}\Big(\frac{1}{(2\pi i)^{n-1}}\int_{C(r_2)\times\dots\times C(r_n)} \frac{f(\zeta_1, \zeta')}{(\zeta')^{\alpha'+\boldsymbol{1}}}d\zeta'\Big)d\zeta_1\Big) z^\alpha = \sum_{\alpha\in\mathbb{Z}^n} a_\alpha z^\alpha. \qquad \square$$

Put

$$\boldsymbol{V}_j := \mathbb{C}^{j-1} \times \{0\} \times \mathbb{C}^{n-j} \subset \mathbb{C}^n, \quad j = 1, \dots, n.$$

Proposition 2.6.2. *Let $D \subset \mathbb{C}^n$ be an n-circled domain and let $f \in \mathcal{O}(D)$. For $\boldsymbol{r} \in D \cap \mathbb{R}^n_{>0}$ put*

$$a_\alpha(\boldsymbol{r}) := \frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{P}(\boldsymbol{r})} \frac{f(\zeta)}{\zeta^{\alpha+\boldsymbol{1}}}\, d\zeta, \quad \alpha \in \mathbb{Z}^n.$$

Then:
(a) *The number $a_\alpha = a_\alpha(\boldsymbol{r})$ is independent of $\boldsymbol{r}$ ($\alpha \in \mathbb{Z}^n$).*
(b) *For any compact $K \subset D$ there exist $C > 0$ and $0 < \theta < 1$ such that*

$$\|a_\alpha z^\alpha\|_K \leq C\theta^{|\alpha_1|+\dots+|\alpha_n|}, \quad \alpha \in \mathbb{Z}^n.$$

In particular, the series $\sum_{\alpha\in\mathbb{Z}^n} a_\alpha z^\alpha$ converges locally normally in D.

---

[10] Cf. Example 1.3.2(b).

(c) $f(z) = \sum_{\alpha\in\mathbb{Z}^n} a_\alpha z^\alpha,\quad z \in D$. (d) *Assume that $D \cap \boldsymbol{V}_j \neq \varnothing$ for some $j \in \{1,\dots,n\}$. Then $a_\alpha = 0$ for every $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{Z}^n$ with $\alpha_j < 0$ and, consequently, f extends holomorphically to the domain*

$$\widetilde{D}^{(j)} := \{(z', \lambda z_j, z'') : (z', z_j, z'') \in D \subset \mathbb{C}^{j-1}\times\mathbb{C}\times\mathbb{C}^{n-j},\ \lambda \in \overline{\mathbb{D}}\}.$$

(e) *If $0 \in D$, then $f = T_0 f$ in D ([11]) and the function f extends holomorphically to the domain*

$$\widetilde{D} := \{(\lambda_1 z_1,\dots,\lambda_n z_n) : \lambda_1,\dots,\lambda_n \in \overline{\mathbb{D}},\ (z_1,\dots,z_n) \in D\}.\quad (^{12})$$

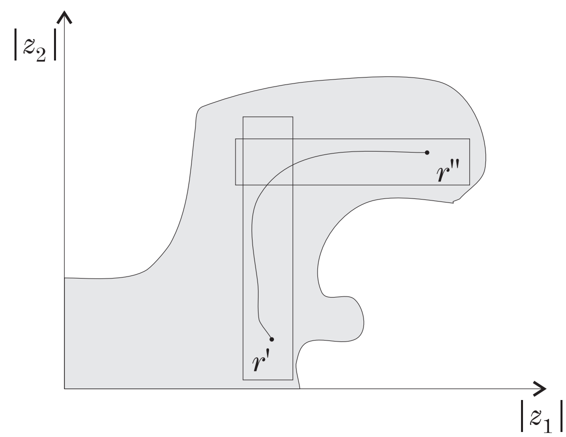

Figure 2.2.1

Proof. (a) Fix $\boldsymbol{r}' = (r_1',\dots,r_n')$, $\boldsymbol{r}'' = (r_1'',\dots,r_n'') \in D \cap \mathbb{R}^n_{>0}$. Since $D \cap \mathbb{R}^n_{>0}$ is connected, there exists a curve $\gamma : [0,1] \longrightarrow D \cap \mathbb{R}^n_{>0}$ such that $\gamma(0) = \boldsymbol{r}'$ and $\gamma(1) = \boldsymbol{r}''$. Since the set $\gamma([0,1])$ is compact, there exist $N \in \mathbb{N}$ and annuli $A^{(j)} = A(\boldsymbol{r}^-(j), \boldsymbol{r}^+(j))$, $j = 1,\dots,N$, such that

$\gamma([0,1]) \subset \operatorname{int} A^{(1)} \cup \dots \cup \operatorname{int} A^{(N)} \subset\subset D$,
$\boldsymbol{r}' \in \operatorname{int} A^{(1)}$, $\boldsymbol{r}'' \in \operatorname{int} A^{(N)}$,
$\operatorname{int} A^{(j)} \cap \operatorname{int} A^{(j+1)} \neq \varnothing$, $j = 1,\dots,N-1$.

By Proposition 2.6.1(a) we know that for fixed j the coefficient $a_\alpha(\boldsymbol{r})$ is independent of $\boldsymbol{r} \in A^{(j)} \cap \mathbb{R}^n_{>0}$. Put $a_\alpha^{(j)} := a_\alpha(\boldsymbol{r})$ for $\boldsymbol{r} \in A^{(j)} \cap \mathbb{R}^n_{>0}$. Consequently, since $\operatorname{int} A^{(j)} \cap \operatorname{int} A^{(j+1)} \neq \varnothing$, we get $a_\alpha(\boldsymbol{r}') = a_\alpha^{(1)} = \dots = a_\alpha^{(N)} = a_\alpha(\boldsymbol{r}'')$.

(b) Observe that for every compact subset $K \subset D$ there exist $N \in \mathbb{N}$, annuli $A^{(j)} = A(\boldsymbol{r}^-(j), \boldsymbol{r}^+(j))$, $j = 1,\dots,N$, and $0 < \theta < 1$ such that $K \subset A_\theta^{(1)} \cup \dots \cup A_\theta^{(N)} \subset\subset D$. Now we apply Proposition 2.6.1(c).

(c) follows immediately from Proposition 2.6.1(e).

(d) First observe that there exist $\boldsymbol{r}^-$ and $\boldsymbol{r}^+$ such that $r_j^- = 0$ and $A(\boldsymbol{r}^-, \boldsymbol{r}^+) \subset D$. Hence, by Proposition 2.6.1(b), $a_\alpha = 0$ whenever $\alpha_j < 0$. By (b), the series converges locally normally in $\widetilde{D}^{(j)}$ ([13]) and, therefore, its sum defines there a holomorphic extension of f.

(e) follows from (d). $\square$

Proposition 2.6.3. *Let $D \subset \mathbb{C}^n$ be a Hartogs domain over G with k-circled fibers. Then any function $f \in \mathcal{O}(D)$ can be represented by a* Hartogs–Laurent series

$$f(z,w) = \sum_{\beta\in\mathbb{Z}^k} f_\beta(z,w) w^\beta,\quad (z,w) \in D, \tag{2.6.3}$$

([11]) Cf. Corollary 1.4.7.

([12]) Observe that $\widetilde{D}$ is a complete n-circled domain.

([13]) If K, C, and θ are as in (b), then $\|a_\alpha z^\alpha\|_{\widetilde{K}^{(j)}} \le C\theta^{|\alpha_1|+\dots+|\alpha_n|}$, $\alpha \in \mathbb{Z}^n$, where $\widetilde{K}^{(j)} := \{(z', \lambda z_j, z'') : (z', z_j, z'') \in K,\ \lambda \in \overline{\mathbb{D}}\}$. Moreover, $\bigcup_{K\subset\subset D} \operatorname{int} \widetilde{K}^{(j)} = \widetilde{D}^{(j)}$.

where $f_\beta \in \mathcal{O}(D)$ and for any $z \in G$ the function $f_\beta(z,\cdot)$ is constant on any connected component of the fiber D_z, $\beta \in \mathbb{Z}^k$.

For any compact $K \subset D$ there exist $C > 0$ and $\theta \in (0,1)$ such that

$$|f_\beta(z,w)w^\beta| \le C\theta^{|\beta_1|+\dots+|\beta_k|}, \quad (z,w) \in K,\ \beta = (\beta_1,\dots,\beta_k) \in \mathbb{Z}^k.$$

In particular, the series converges locally normally in D.

Moreover, if $D \cap (G \times \{0\}^k) \neq \varnothing$, then $f_\beta \equiv 0$ for all $\beta \notin \mathbb{N}_0^k$ and (2.6.3) reduces to a Hartogs *series (cf. Proposition 1.6.5(a)).*

Proof. For $(z,w) \in D$ let $D_{z,w}$ denote the connected component of D_z such that $w \in D_{z,w}$. By Proposition 2.6.2 the function f can be represented in the form (2.6.3) with

$$f_\beta(z,w) := \frac{1}{(2\pi i)^k} \int_{\partial_0 \mathbb{P}(\boldsymbol{r})} \frac{f(z,\zeta)}{\zeta^{\beta+\mathbf{1}}} d\zeta, \quad \beta \in \mathbb{Z}^k, \tag{2.6.4}$$

where $\boldsymbol{r}$ is an arbitrary vector from $D_{z,w} \cap \mathbb{R}^k_{>0}$ (we know that the formula is independent of $\boldsymbol{r} \in D_{z,w} \cap \mathbb{R}^k_{>0}$). In particular, $f_\beta(z,\cdot)$ is constant on any connected component of D_z. Moreover, if $0 \in D_{z,w}$, then $f_\beta(z,w) = 0$ for all $\beta \notin \mathbb{N}_0^k$.

Observe that $\boldsymbol{r}$ in (2.6.4) can be chosen to be locally independent of (z,w). Hence $f_\beta \in \mathcal{O}(D)$, $\beta \in \mathbb{Z}^k$.

If $D \cap (G \times \{0\}^k) \neq \varnothing$, then by the identity principle, $f_\beta \equiv 0$ for all $\beta \notin \mathbb{N}_0^k$.

We pass to the proof of the estimate. It suffices to consider only the case where $K = K_0 \times A$, $K_0 \subset\subset G$,

$$A := \{(w_1,\dots,w_k) \in \mathbb{C}^k : r_j^- \le |w_j| \le r_j^+,\ j = 1,\dots,k\}$$

for some $0 \le r_j^- < r_j^+ < +\infty$, $j = 1,\dots,k$ (14). Let $\theta \in (0,1)$ be such that

$$L := K_0 \times \{(w_1,\dots,w_k) \in \mathbb{C}^k : \theta r_j^- \le |w_j| \le r_j^+/\theta,\ j = 1,\dots,k\} \subset D.$$

Then, by Proposition 2.6.1, we get

$$|f_\beta(z,w)w^\beta| \le \|f\|_L \theta^{|\beta_1|+\dots+|\beta_k|}, \quad (z,w) \in K,\ \beta \in \mathbb{Z}^k. \qquad \square$$

2.7. Domains of holomorphy

Recall that if $\Omega \subsetneq \mathbb{C}$, then each boundary point of Ω is a singular point for a function holomorphic in Ω. By virtue of the Hartogs theorem 2.1.2 a similar result is not true for $\Omega \subsetneq \mathbb{C}^n$ with $n \ge 2$. Thus, it is natural to look for characterizations of those open sets in $\mathbb{C}^n$, $n \ge 2$, which are regions of holomorphy.

On the other hand, even for $n = 1$, if we restrict the class of all holomorphic functions in Ω to, for instance, the class of all bounded holomorphic functions in Ω, then not all open sets $\Omega \subset \mathbb{C}$ are regions of existence of such functions; e.g. each function from $\mathcal{H}^\infty(\mathbb{D}_*)$ extends holomorphically to $\mathbb{D}$. The same situation appears for $n \ge 2$ if, for instance, $\Omega = D \setminus M$, where M is a relatively closed thin subset of D (cf. the Riemann theorem 2.1.6). Thus, it seems to be interesting to consider also the case where an open set $\Omega \subset \mathbb{C}^n$ is the region of holomorphy with respect to a family $\mathcal{F} \subset \mathcal{O}(\Omega)$.

Observe the following additional problem. Let $\Omega := \mathbb{C} \setminus \mathbb{R}_+$ and let $f := \operatorname{Log}$ be the principal branch of the logarithm. Then f cannot be holomorphically continued to a larger plane domain, but it is well known that f can be holomorphically continued to a Riemann surface which is no longer a plane domain.

One could expect that the above multivaluedness of holomorphic continuation disappears if we simultaneously extend all holomorphic functions. Unfortunately, this is not true — cf. Example 2.8.1. Consequently, to study maximal holomorphic extensions of open sets in $\mathbb{C}^n$ we have to consider 'multivalued' regions over $\mathbb{C}^n$ (called Riemann regions).

Definition 2.7.1. Let $\Omega \subset \mathbb{C}^n$ be open and let $\varnothing \neq \mathcal{F} \subset \mathcal{O}(\Omega)$. We say that Ω is an *$\mathcal{F}$-region of holomorphy* if there are no domains $\widetilde{\Omega}$, $\Omega_0 \subset \mathbb{C}^n$ such that

$\widetilde{\Omega} \not\subset \Omega$,

$\varnothing \neq \Omega_0 \subset \Omega \cap \widetilde{\Omega}$,

(14) Any compact subset of D can be covered by a finite number of compacts of the above type.

for every $f \in \mathcal{F}$ there exists an $\widetilde{f} \in \mathcal{O}(\widetilde{\Omega})$ such that $\widetilde{f} = f$ in Ω_0.

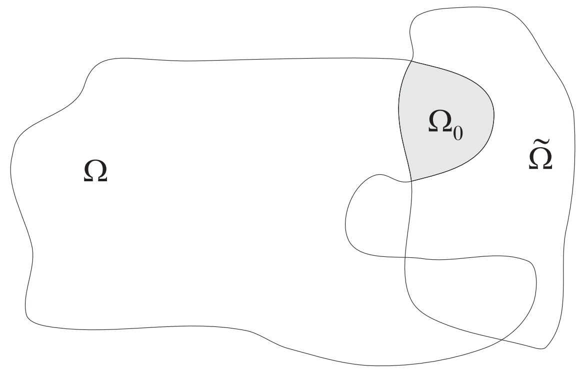

Figure 2.3.1

Observe that by the identity principle $\widetilde{f}$ is uniquely determined by f. Moreover, if Ω is not an $\mathcal{F}$-region of holomorphy, then we may always assume that Ω_0 is a connected component of $\Omega \cap \widetilde{\Omega}$.

If Ω is an $\{f\}$-region of holomorphy, then we say that Ω is the *region of existence of f* (or that the function f does not extend beyond Ω).

If Ω is an $\mathcal{O}(\Omega)$-region of holomorphy, then we shortly say that Ω is a *region of holomorphy.*

If Ω is connected, then we will say that Ω is an *$\mathcal{F}$-domain of holomorphy* (resp. the *domain of existence of f*, resp. a *domain of holomorphy*).

Remark 2.7.2. (a) If Ω is an $\mathcal{F}$-region of holomorphy, then Ω is a $\mathcal{G}$-region of holomorphy for any $\mathcal{G} \supset \mathcal{F}$. In particular, any $\mathcal{F}$-region of holomorphy is a region of holomorphy.
(b) Ω is an $\mathcal{F}$-region of holomorphy iff Ω is an $[\mathcal{F}]$-region of holomorphy, where $[\mathcal{F}]$ is the minimal subalgebra $\mathcal{A}$ of $\mathcal{O}(\Omega)$ such that
$\mathcal{F} \subset \mathcal{A}$,
$z_1, \dots, z_n \in \mathcal{A}$,
$\mathcal{A}$ is *∂-stable*, i.e. $\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n} \in \mathcal{A}$ for any $f \in \mathcal{A}$.
(c) Ω is an $\mathcal{F}$-region of holomorphy iff any connected component D of Ω is an $\mathcal{F}|_D$-domain of holomorphy.
(d) $\mathbb{C}^n$ is an $\mathcal{F}$-domain of holomorphy for every $\varnothing \neq \mathcal{F} \subset \mathcal{O}(\mathbb{C}^n)$.
(e) Every open subset $\Omega \subsetneq \mathbb{C}$ is an $\mathcal{F}$-domain of holomorphy with respect to the family

$$\mathcal{F} := \{\Omega \ni z \longmapsto \frac{1}{z-a},\ a \notin \Omega\}.$$

In particular, every open subset of $\mathbb{C}$ is a domain of holomorphy.
(f) Every open and fat subset $\Omega \subsetneq \mathbb{C}$ ([15]) is an $\mathcal{F}$-domain of holomorphy with respect to the family

$$\mathcal{F} := \{\Omega \ni z \longmapsto \frac{1}{z-a},\ a \notin \overline{\Omega}\}.$$

In particular, every open and fat subset of $\mathbb{C}$ is an $\mathcal{H}^\infty(\Omega)$-domain of holomorphy.
(g) Let K be a non-empty compact subset of a domain $D \subset \mathbb{C}^n$, $n \geq 2$, such that $D \setminus K$ is connected. Then, by Theorem 2.1.2, $D \setminus K$ is not a domain of holomorphy.
(h) Let M be a non-empty closed thin subset of a domain $D \subset \mathbb{C}^n$. Then, by Theorem 2.1.6, $D \setminus M$ is not an $\mathcal{H}^\infty(D \setminus M)$-domain of holomorphy.
(i) Let D be an n-circled domain with $0 \in D$ such that D is not complete. Then, by Proposition 2.6.2, D is not a domain of holomorphy.

([15]) Ω is *fat* if $\Omega = \operatorname{int} \overline{\Omega}$.

Proposition 2.7.3. (a) *Assume that Ω is not a region of holomorphy and let $\widetilde{\Omega}$, Ω_0 be as in Definition 2.7.1 with $\mathcal{F} := \mathcal{O}(\Omega)$. Then*

$$\widetilde{f}(\widetilde{\Omega}) \subset f(\Omega), \quad f \in \mathcal{O}(\Omega).$$

(b) *Assume that Ω is not an $\mathcal{H}^\infty(\Omega)$-region of holomorphy and let $\widetilde{\Omega}$, Ω_0 be as in Definition 2.7.1 with $\mathcal{F} := \mathcal{H}^\infty(\Omega)$. Then*

$$\|\widetilde{f}\|_{\widetilde{\Omega}} \leq \|f\|_\Omega, \quad f \in \mathcal{H}^\infty(\Omega).$$

Proof. (a) Suppose that for $f \in \mathcal{O}(\Omega)$ and $a \in \widetilde{\Omega}$ we have $\widetilde{f}(a) \notin f(\Omega)$. Let $g := 1/(f - \widetilde{f}(a))$. Obviously $g \in \mathcal{O}(\Omega)$. Since $g \cdot (f - \widetilde{f}(a)) \equiv 1$ in Ω, we get $\widetilde{g} \cdot (\widetilde{f} - \widetilde{f}(a)) \equiv 1$ in $\widetilde{\Omega}$; contradiction.

(b) Suppose that for some $f \in \mathcal{H}^\infty(\Omega)$ and $a \in \widetilde{\Omega}$ we have $|\widetilde{f}(a)| > \|f\|_\Omega$. Put $g := 1/(f - \widetilde{f}(a))$. Then $g \in \mathcal{H}^\infty(\Omega)$ and we can argue as in (a). □

Proposition 2.7.4. (a) *Assume that for every $a \in \partial\Omega$ there exists an $f_a \in \mathcal{O}(\Omega, \mathbb{D})$ such that $\lim_{z\to a} |f_a(z)| = 1$ (each such a function f_a is called a* barrier function*). Then Ω is an $\mathcal{H}^\infty(\Omega)$-region of holomorphy.*
(b) *Every convex domain $D \subset \mathbb{C}^n$ is an $\mathcal{H}^\infty(D)$-domain of holomorphy.*

Proof. (a) Let $\widetilde{\Omega}$ and Ω_0 be as in Definition 2.7.1 with $\mathcal{F} = \mathcal{H}^\infty(\Omega)$. We may assume that Ω_0 is a connected component of $\Omega \cap \widetilde{\Omega}$. Let $a \in \partial\Omega \cap \widetilde{\Omega} \cap \partial\Omega_0$. Then, by Proposition 2.7.3(b), $|\widetilde{f}_a| \leq 1$ in $\widetilde{\Omega}$. Moreover, $|\widetilde{f}_a(a)| = \lim_{\Omega_0 \ni z \to a} |f_a(z)| = 1$. Hence, by the maximum principle, $|f_a| = 1$ in Ω_0; contradiction.

(b) We may assume that $D \subsetneq \mathbb{C}^n$. We apply (a). Fix an $a \in \partial D$. Since D is convex, there exists a real affine function $\ell : \mathbb{C}^n \longrightarrow \mathbb{R}$ such that $\ell < 0$ in D and $\ell(a) = 0$. Let

$$\ell(z) = b_0 + \sum_{j=1}^n (b_j x_j + c_j y_j), \quad z = (x_1 + iy_1, \dots, x_n + iy_n),$$

with $b_0, \dots, b_n, c_1, \dots, c_n \in \mathbb{R}$. Put

$$L(z) := b_0 + \sum_{j=1}^n (b_j - ic_j) z_j.$$

Then $\ell = \operatorname{Re} L$. Let $f_a := e^L$. We have $|f_a| = e^{\operatorname{Re} L} = e^\ell < 1$ in D and $f_a(a) = 1$.

□

Proposition 2.7.5. *Let $A \subset \Omega$ be an arbitrary dense subset and let $\mathcal{F} \subset \mathcal{O}(\Omega)$. Then the following conditions are equivalent:*
(i) *Ω is an $\mathcal{F}$-region of holomorphy;*
(ii) *for any $a \in A$ and $r > d_\Omega(a)$ there exists an $f \in \mathcal{F}$ such that $d(T_a f) < r$, i.e.*

$$d_\Omega(a) = \inf\{d(T_a f) : f \in \mathcal{F}\}, \quad a \in A.$$

Proof. (i) $\Longrightarrow$ (ii). Suppose that for some $a \in A$ and $r > d_\Omega(a)$ condition (ii) does not hold. Then the polydiscs $\widetilde{\Omega} := \mathbb{P}(a, r)$, $\Omega_0 := \mathbb{P}(a, d_\Omega(a))$ satisfy the conditions of Definition 2.7.1; contradiction.

(ii) $\Longrightarrow$ (i). Suppose that Ω is not an $\mathcal{F}$-region of holomorphy and let $\widetilde{\Omega}$, Ω_0 be as in Definition 2.7.1. Take a point $a \in A \cap \Omega_0$ such that $d_\Omega(a) < d_{\widetilde{\Omega}}(a)$. Then $d(T_a f) = d(T_a \widetilde{f}) \geq d_{\widetilde{\Omega}}(a) > d_\Omega(a)$ for any $f \in \mathcal{F}$; contradiction. □

Definition 2.7.6. Let $\mathcal{F} \subset \mathcal{O}(\Omega)$ be a vector subspace. Assume that $\mathcal{F}$ is endowed with a Fréchet space topology. We say that $\mathcal{F}$ is a *natural Fréchet space* in $\mathcal{O}(\Omega)$ if the identity $\mathrm{id} : \mathcal{F} \longrightarrow \mathcal{O}(\Omega)$ is continuous, i.e. if a sequence $(f_\nu)_{\nu=1}^\infty \subset \mathcal{F}$ is convergent to $f_0 \in \mathcal{F}$ in the sense of the topology of $\mathcal{F}$, then $f_\nu \longrightarrow f_0$ locally uniformly in Ω.

Remark 2.7.7. $\mathcal{O}(\Omega)$, $\mathcal{H}^\infty(\Omega)$, $\mathcal{A}^k(\Omega)$, $L^p H(\Omega)$ are natural Fréchet spaces.

Let

$$\mathfrak{N}(\mathcal{F}) := \{f \in \mathcal{F} : f \text{ does not extend beyond } \Omega\}.$$

Proposition 2.7.8. *Let $\mathcal{F}$ be a natural Fréchet space in $\mathcal{O}(\Omega)$. Then the following conditions are equivalent:*
(i) *Ω is an $\mathcal{F}$-region of holomorphy;*
(ii) *$\mathfrak{N}(\mathcal{F}) \neq \varnothing$;*
(iii) *$\mathfrak{N}(\mathcal{F})$ is of second Baire category in $\mathcal{F}$* (16) .

Proof. The implications (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i) are obvious.

(i) $\Longrightarrow$ (iii). Let $A \subset \Omega$ be an arbitrary countable dense subset of Ω, $A = \{a_1, a_2, \dots\}$, and let

$$\mathcal{F}_{j,k} := \{f \in \mathcal{F} : d(T_{a_j} f) \geq d_\Omega(a_j) + \frac{1}{k}\}, \quad j, k \in \mathbb{N}.$$

By Proposition 2.7.5 we have

$$\mathcal{F} \setminus \mathfrak{N}(\mathcal{F}) = \bigcup_{j,k \in \mathbb{N}} \mathcal{F}_{j,k}.$$

$\mathcal{F}_{j,k}$ is a vector subspace of $\mathcal{F}$ and $\mathcal{F}_{j,k} \subsetneq \mathcal{F}$ for every $j, k \in \mathbb{N}$ (in virtue of (i)).

We define a topology on $\mathcal{F}_{j,k}$: a sequence $(f_\nu)_{\nu=1}^\infty \subset \mathcal{F}_{j,k}$ is convergent to $f_0 \in \mathcal{F}_{j,k}$ if $f_\nu \longrightarrow f_0$ in $\mathcal{F}$ and $T_{a_j} f_\nu \longrightarrow T_{a_j} f_0$ locally uniformly in the polydisc $\mathbb{P}(a_j, d_\Omega(a_j) + 1/k)$. The above topology is a Fréchet topology and the mapping $\mathrm{id} : \mathcal{F}_{j,k} \longrightarrow \mathcal{F}$ is clearly continuous. In particular, by the Banach theorem (17) , $\mathcal{F}_{j,k}$ is of the first category in $\mathcal{F}$. Consequently, $\mathfrak{N}(\mathcal{F})$ is of the second category. $\square$

Immediately from Definition 2.7.1 we obtain:

Remark 2.7.9. (a) If Ω_ι is an $\mathcal{F}_\iota$-region of holomorphy, then $\Omega := \operatorname{int} \bigcap_{\iota \in I} \Omega_\iota$ is an $\mathcal{F}$-region of holomorphy, where $\mathcal{F} := \bigcup_{\iota \in I} \mathcal{F}|_\Omega$.

Indeed,

$$\inf\{d(T_a f) : f \in \mathcal{F}\} = \inf_{\iota \in I} \inf\{d(T_a f) : f \in \mathcal{F}_i\} = \inf_{\iota \in I} d_{\Omega_\iota}(a) = d_\Omega(a), \quad a \in \Omega.$$

In particular, if Ω_ι, $\iota \in I$, are regions of holomorphy, then Ω is a region of holomorphy.

(b) If $\Omega_j \subset \mathbb{C}^{n_j}$ is an $\mathcal{F}_j$-region of holomorphy, $j = 1, \dots, N$, then $\Omega := \Omega_1 \times \dots \times \Omega_N$ is an $\mathcal{F}$-region of holomorphy, where

$$\mathcal{F} := \{f \circ \pi_j : f \in \mathcal{F}_j,\ j = 1, \dots, N\}$$

and $\pi_j : \mathbb{C}^{n_1 + \dots + n_N} \longrightarrow \mathbb{C}^{n_j}$ is the natural projection.

In particular, if $\Omega_1, \dots, \Omega_N$ are regions of holomorphy, then Ω is a region of holomorphy.

For example, if $\Omega_1, \dots, \Omega_n \subset \mathbb{C}$ are arbitrary open subsets, then $\Omega_1 \times \dots \times \Omega_n$ is a region of holomorphy in $\mathbb{C}^n$.

Indeed,

$$\begin{aligned}\inf\{d(T_a f) : f \in \mathcal{F}\} &= \min_{j=1,\dots,N} \inf\{d(T_a(f \circ \pi_j)) : f \in \mathcal{F}_j\} \\ &= \min_{j=1,\dots,N} \inf\{d(T_{a_j} f) : f \in \mathcal{F}_j\} = \min_{j=1,\dots,N} d_{\Omega_j}(a_j) = d_\Omega(a), \\ &\qquad a = (a_1, \dots, a_N) \in \Omega.\end{aligned}$$

(c) If Ω is a region of holomorphy, then for arbitrary functions $f_1, \dots, f_N \in \mathcal{O}(\Omega)$, the set $G := \{z \in \Omega : |f_j(z)| < 1,\ j = 1, \dots, N\}$ is a region of holomorphy.

Indeed, if $a \in G$ is such that $d_G(a) = d_\Omega(a)$, then

$$d_G(a) = d_\Omega(a) = \inf\{d(T_a f) : f \in \mathcal{O}(\Omega)\} \geq \inf\{d(T_a f) : f \in \mathcal{O}(G)\} \geq d_G(a).$$

If $d_G(a) < d_\Omega(a)$, then there exist $j \in \{1, \dots, N\}$ and $\zeta \in \mathbb{T}$ such that

$$\partial\mathbb{P}(a, d_G(a)) \cap \{z \in \Omega : f_j(z) = \zeta\} \neq \varnothing.$$

---

(16) Recall that a set $A \subset X$ is of the *first Baire category* in a topological space X if $A = \bigcup_{j=1}^\infty A_j$ with $\operatorname{int} \overline{A}_j = \varnothing$ for any j. We say that $A \subset X$ is of the *second Baire category* if $X \setminus A$ is of the first category.

(17) If X, Y are Fréchet spaces, $L : X \longrightarrow Y$ is linear and continuous, then either $L(X) = Y$ or $L(X)$ is of the first Baire category in Y; cf. [9], § 5.8.

Consequently, $g := 1/(f_j - \zeta) \in \mathcal{O}(G)$ and $d(T_a g) = d_G(a)$.
(d) If G is a domain of holomorphy, then for every function $f \in \mathcal{O}(G)$, the set $\Omega := G \setminus f^{-1}(0)$ is a domain of holomorphy.

Indeed, if $a \in \Omega$ is such that $d_\Omega(a) = d_G(a)$, then

$$d_\Omega(a) = \inf\{d(T_a f) : f \in \mathcal{O}(\Omega)\}$$

(exactly as above). If $d_\Omega(a) < d_G(a)$, then $\partial\mathbb{P}(a, d_\Omega(a)) \cap f^{-1}(0) \neq \varnothing$. Consequently, $d(T_a(1/f)) = d_\Omega(a)$.

Definition 2.7.10. Let $\mathcal{F} \subset \mathcal{O}(\Omega)$. For every compact $K \subset \Omega$ put

$$\widehat{K}_{\mathcal{F}} := \{z \in \Omega : \forall\, f \in \mathcal{F} : |f(z)| \leq \|f\|_K\}.$$

The set $\widehat{K}_{\mathcal{F}}$ is the *$\mathcal{F}$-hull* of K. If $\mathcal{F} = \mathcal{O}(\Omega)$, then $\widehat{K}_{\mathcal{O}(\Omega)}$ is the *holomorphic hull* of K. If $\Omega = \mathbb{C}^n$ and $\mathcal{F} = \mathcal{P}(\mathbb{C}^n)$, then $\widehat{K}_{\mathcal{P}(\mathbb{C}^n)}$ is the *polynomial hull* of K. If $K = \widehat{K}_{\mathcal{F}}$, then K is *$\mathcal{F}$-convex*. If K is $\mathcal{O}(\Omega)$-convex, then K is *holomorphically convex*. If $K = \widehat{K}_{\mathcal{P}(\mathbb{C}^n)}$, then K is *polynomially convex*; put $\widehat{K} := \widehat{K}_{\mathcal{P}(\mathbb{C}^n)}$.

We say that Ω is *$\mathcal{F}$-convex* if $\widehat{K}_{\mathcal{F}}$ is compact for every compact set $K \subset \Omega$. If Ω is $\mathcal{O}(\Omega)$-convex, then Ω is *holomorphically convex*.

Remark 2.7.11. (a) If $K_1 \subset K_2$ and $\mathcal{F}_1 \subset \mathcal{F}_2$, then $(\widehat{K_1})_{\mathcal{F}_2} \subset (\widehat{K_2})_{\mathcal{F}_1}$.
(b) The set $\widehat{K}_{\mathcal{F}}$ is closed in Ω.
(c) If $z_1, \dots, z_n \in \mathcal{F}$, then $\widehat{K}_{\mathcal{F}}$ is bounded.
(d) If $\widehat{K}_{\mathcal{F}}$ is compact, then $(\widehat{\widehat{K}_{\mathcal{F}}})_{\mathcal{F}} = \widehat{K}_{\mathcal{F}}$.
(e) $\widehat{K}_{\mathcal{F}} = \widehat{K}_{\mathcal{G}}$, where $\mathcal{G}$ denotes the closure in $\mathcal{O}(\Omega)$ of the family

$$\{af^\nu : a \in \mathbb{C},\ f \in \mathcal{F},\ \nu \in \mathbb{N}\}.$$

In particular, if $\Omega = \mathbb{C}^n$, then $\widehat{K}_{\mathcal{P}(\mathbb{C}^n)} = \widehat{K}_{\mathcal{O}(\mathbb{C}^n)}$.
(f) Ω is $\mathcal{F}$-convex iff there exists a sequence $(K_\nu)_{\nu=1}^\infty$ of $\mathcal{F}$-convex compact sets such that $K_\nu \subset \operatorname{int} K_{\nu+1}$ for any ν and $\Omega = \bigcup_{\nu=1}^\infty K_\nu$.

Indeed, the implication $\Longleftarrow$ is obvious. To prove $\Longrightarrow$ let $(L_j)_{j=1}^\infty$ be an arbitrary sequence of compact sets such that $L_j \subset \operatorname{int} L_{j+1}$ and $\Omega = \bigcup_{j=1}^\infty L_j$. Put $K_1 := (\widehat{L_1})_{\mathcal{F}}$. Since $\Omega = \bigcup_{j=1}^\infty \operatorname{int} L_j$, there exists a $j_2 > 1$ such that $K_1 \subset \operatorname{int} L_{j_2}$. Put $K_2 := (\widehat{L_{j_2}})_{\mathcal{F}}$. Now we take a $j_3 > j_2$ such that $K_2 \subset \operatorname{int} L_{j_3}$ etc.
(g) Let $\Omega_j \subset \mathbb{C}^{n_j}$ be open and let $K_j \subset \Omega_j$ be compact, $j = 1, 2$. Then

$$(\widehat{K_1 \times K_2})_{\mathcal{O}(\Omega_1 \times \Omega_2)} = (\widehat{K_1})_{\mathcal{O}(\Omega_1)} \times (\widehat{K_2})_{\mathcal{O}(\Omega_2)}.$$

In particular,

$$\widehat{K_1 \times K_2} = \widehat{K}_1 \times \widehat{K}_2.$$

Indeed, let $(z_1^0, z_2^0) \in (\widehat{K_1 \times K_2})_{\mathcal{O}(\Omega_1 \times \Omega_2)}$ and let $f \in \mathcal{O}(\Omega_j)$. The function f can be regarded as a holomorphic function on $\Omega_1 \times \Omega_2$. Thus

$$|f(z_j^0)| \leq \max_{K_j} |f|.$$

Conversely, let $(z_1^0, z_2^0) \in (\widehat{K_1})_{\mathcal{O}(\Omega_1)} \times (\widehat{K_2})_{\mathcal{O}(\Omega_2)}$ and let $f \in \mathcal{O}(\Omega_1 \times \Omega_2)$. Then

$$\begin{aligned}|f(z_1^0, z_2^0)| &\leq \max\{|f(z_1, z_2^0)| : z_1 \in K_1\} \\ &\leq \max\{\max\{|f(z_1, z_2)| : z_2 \in K_2\} : z_1 \in K_1\} = \max_{K_1 \times K_2} |f|.\end{aligned}$$

(h) Let $F : \Omega \longrightarrow \Omega'$ be biholomorphic. Then $(\widehat{F(K)})_{\mathcal{O}(\Omega')} = F(\widehat{K}_{\mathcal{O}(\Omega)})$ for any compact $K \subset \Omega$. In particular, Ω is holomorphically convex iff Ω' is holomorphically convex.

Consider the following conditions:
(HC1) Ω is an $\mathcal{F}$-region of holomorphy.
(HC2) Ω is $\mathcal{F}$-convex.
(HC3) For every compact set $K \subset \Omega$: $d_\Omega(\widehat{K}_{\mathcal{F}}) = d_\Omega(K)$.
(HC4) For every compact set $K \subset \Omega$: $d_\Omega(\widehat{K}_{\mathcal{F}}) > 0$.
(HC5) For every infinite subset $A \subset \Omega$ without accumulation points in Ω, there exists a function $f \in \mathcal{F}$ such that $\sup_A |f| = +\infty$.

Theorem 2.7.12. (a) (HC2) $\Longrightarrow$ (HC4), (HC3) $\Longrightarrow$ (HC4), (HC5) $\Longrightarrow$ (HC1).
(b) *If* $z_1, \dots, z_n \in \mathcal{F}$, *then* (HC4) $\Longrightarrow$ (HC2).
(c) *If* $\mathcal{F}$ *is a closed subalgebra of* $\mathcal{O}(\Omega)$, *then* (HC2) $\Longrightarrow$ (HC5).
(d) *If* $\mathcal{F}$ *is* ∂*-stable* (18) , *then* (HC1) $\Longrightarrow$ (HC3).
(e) *For* $\mathcal{F} = \mathcal{O}(\Omega)$ *all the conditions* (HC1), (HC2), (HC3), (HC4), (HC5) *are equivalent.*

$$(1) \xrightarrow{\partial\text{-stability}} (3) \longrightarrow (4) \xrightarrow{z_1,\dots,z_n\in\mathcal{F}} (2) \xrightarrow{\text{closed algebra}} (5) \longrightarrow (1)$$

Proof. (a) is obvious.

(b) follows from Remark 2.7.11(c).

(c) By Remark 2.7.11(f) there exists a sequence of $\mathcal{F}$-convex compact sets such that $K_\nu \subset \operatorname{int} K_{\nu+1}$ and $\bigcup_{\nu=1}^\infty K_\nu = \Omega$. We may assume that for a sequence $(a_\nu)_{\nu=1}^\infty \subset A$ we have $a_\nu \in K_{\nu+1} \setminus K_\nu$, $\nu \geq 1$. Since $a_1 \notin K_1$ and K_1 is $\mathcal{F}$-convex, there exists a function $f_1 \in \mathcal{F}$ such that $|f(a_1)| > \|f\|_{K_1}$. Replacing f_1 by $(af_1)^N$ with suitable $a > 0$ and $N \in \mathbb{N}$, we may assume that $|f_1(a_1)| \geq 1$, and $\|f_1\|_{K_1} \leq 1/2$ (here we use the fact that $\mathcal{F}$ is an algebra). Repeating the above argument for other points, we easily find a sequence $(f_\nu)_{\nu=1}^\infty \subset \mathcal{F}$ such that $|f_\nu(a_\nu)| \geq \nu + \sum_{\mu=1}^{\nu-1} |f_\mu(a_\nu)|$ and $\|f_\nu\|_{K_\nu} \leq 1/2^\nu$. Now put $f := \sum_{\nu=1}^\infty f_\nu$ (the series is locally normally convergent in Ω). Since $\mathcal{F}$ is a closed subalgebra we conclude that $f \in \mathcal{F}$. Moreover, $|f(a_\nu)| \geq \nu - 1$ for every ν.

(d) Suppose that for some $a \in \widehat{K}_{\mathcal{F}}$ we have $d_\Omega(a) < d_\Omega(K) =: r$. Let $0 < \sigma < r$. By the Cauchy inequalities we obtain

$$\|D^\alpha f\|_K \leq \frac{\alpha!}{\sigma^{|\alpha|}} \|f\|_{K^{(\sigma)}}, \quad f \in \mathcal{F}.$$

Hence (using the ∂-stability of $\mathcal{F}$) we get

$$|D^\alpha f(a)| \leq \frac{\alpha!}{\sigma^{|\alpha|}} \|f\|_{K^{(\sigma)}}, \quad f \in \mathcal{F}.$$

In particular, $d(T_a f) \geq \sigma$ and hence $d(T_a f) \geq r$, $f \in \mathcal{F}$. Finally, since Ω is an $\mathcal{F}$-region of holomorphy, we conclude that $\mathbb{P}(a, r) \subset \Omega$; contradiction.

(e) is an immediate consequence of (a) — (d). $\square$

Proposition 2.7.13. *If* $\Omega_1 \subset \mathbb{C}^n$, $\Omega_2 \subset \mathbb{C}^m$ *are holomorphically convex and* $f \in \mathcal{O}(\Omega_1, \mathbb{C}^m)$, *then* $\Omega := f^{-1}(\Omega_2)$ *is holomorphically convex.* (19)

Proof. Let $K \subset \Omega$ be compact. Then

$$\widehat{K}_{\mathcal{O}(\Omega)} \subset \Omega \cap \widehat{K}_{\mathcal{O}(\Omega_1)} \subset\subset \Omega_1.$$

Suppose that there exists a sequence $(z_\nu)_{\nu=1}^\infty \subset \widehat{K}_{\mathcal{O}(\Omega)}$ such that $z_\nu \longrightarrow z_0 \in \Omega_1 \cap \partial\Omega$. Observe that for any $z \in \widehat{K}_{\mathcal{O}(\Omega)}$ and $g \in \mathcal{O}(\Omega_2)$ we get

$$|g(f(z))| \leq \sup_K |g \circ f| = \sup_{f(K)} |g|.$$

Hence

$$f(\widehat{K}_{\mathcal{O}(\Omega)}) \subset \widehat{f(K)}_{\mathcal{O}(\Omega_2)} \subset\subset \Omega_2.$$

---

(18) Cf. Remark 2.7.2(b).
(19) Recall Remark 2.7.9(c), where $\Omega_2 = \mathbb{D}^m$.

In particular,

$$f(z_\nu) \in \widehat{f(K)}_{\mathcal{O}(\Omega_2)} \subset\subset \Omega_2, \quad \nu \geq 1,$$

and therefore $f(z_0) \in \Omega_2$; contradiction. $\square$

Proposition 2.7.14. *Let $\Omega \subset \mathbb{C}^n$ be a region of holomorphy and let V be an affine subspace of $\mathbb{C}^n$. Then $\Omega \cap V$ is a region of holomorphy.*

Proof. Let $V = a + \mathbb{C}v_1 + \dots + \mathbb{C}v_k$ $(k := \dim V)$,

$$\mathbb{C}^k \ni (z_1, \dots, z_k) \overset{f}{\longmapsto} a + z_1 v_1 + \dots + z_k v_k \in \mathbb{C}^n.$$

Then $\Omega \cap V \simeq f^{-1}(\Omega)$ and we can apply Proposition 2.7.13. $\square$

Proposition 2.7.15. *Let $D \subset \mathbb{C}^n$ be an n-circled domain. Then the following conditions are equivalent:*
(i) *D is a domain of holomorphy;*
(ii) *$\log D$ is a convex domain and*
($*$) *for every $j \in \{1, \dots, n\}$, if $D \cap \boldsymbol{V}_j \neq \varnothing$, then $\widetilde{D}^{(j)} \subset D$* [20] .

Note that the condition ($*$) is always satisfied if D is complete.

Proof. (i) $\Longrightarrow$ (ii). Take an $f \in \mathcal{O}(D)$. Then f can be represented by a Laurent series

$$f(z) = \sum_{\alpha \in \mathbb{Z}^n} a_\alpha z^\alpha, \quad z \in D \tag{2.7.1}$$

(Proposition 2.6.2(c)).

Fix $x_0', x_0'' \in \log D$ and let $U \subset\subset \log D$ be a domain with $x_0', x_0'' \in U$. We will prove that $\widetilde{U} := \operatorname{conv} U \subset \log D$.

Let $G, \widetilde{G} \subset (\mathbb{C}_*)^n$ be such that $\log G = U$ and $\log \widetilde{G} = \widetilde{U}$, respectively. Observe that $G \subset\subset D$. By Proposition 2.6.2(c) there exist $C > 0$ and $0 < \theta < 1$ such that

$$\|a_\alpha z^\alpha\| \leq C\theta^{|\alpha_1| + \dots + |\alpha_n|}, \quad z \in G,\ \alpha \in \mathbb{Z}^n.$$

For any $x', x'' \in U$ we get [21]

$$|a_\alpha| e^{\langle \alpha, tx' + (1-t)x'' \rangle} = (|a_\alpha| e^{\langle \alpha, x' \rangle})^t (|a_\alpha| e^{\langle \alpha, x'' \rangle})^{1-t} \leq C\theta^{|\alpha_1| + \dots + |\alpha_n|}, \quad 0 \leq t \leq 1,\ \alpha \in \mathbb{Z}^n.$$

Hence the series (2.7.1) is normally convergent in $\widetilde{G}$ and, therefore, its sum gives there a holomorphic extension of f. Since D is a domain of holomorphy, we conclude that $\widetilde{G} \subset D$ and, consequently, $\widetilde{U} \subset \log D$.

Now, suppose that $D \cap \boldsymbol{V}_j \neq \varnothing$. Then, by Proposition 2.6.2(d), the function f has a holomorphic extension to $\widetilde{D}^{(j)}$. Since D is a domain of holomorphy, we get $\widetilde{D}^{(j)} \subset D$, which gives ($*$).

(ii) $\Longrightarrow$ (i). Assume additionally that D is bounded.

Suppose that D is not a domain of holomorphy and let D_0, $\widetilde{D}$ be domains such that $\widetilde{D} \not\subset D$, $\varnothing \neq D_0 \subset D \cap \widetilde{D}$, and for every function $f \in \mathcal{O}(D)$ there exists a function $\widetilde{f} \in \mathcal{O}(\widetilde{D})$ such that $\widetilde{f} = f$ in D_0.

Put

$$\boldsymbol{V}_0 := \{(z_1, \dots, z_n) \in \mathbb{C}^n : z_1 \dots z_n = 0\}.$$

First, consider the case where $\widetilde{D} \setminus \boldsymbol{V}_0 \not\subset D$. Take an $a \in (\widetilde{D} \setminus \boldsymbol{V}_0) \setminus D$. Let $U \subset \widetilde{D} \setminus \boldsymbol{V}_0$ be a neighborhood of a. Then $\log U$ is a neighborhood of the point $(\log|a_1|, \dots, \log|a_n|) \notin \log D$. In particular, there exists a $b = (b_1, \dots, b_n) \in U$ such that $x_0 := (\log|b_1|, \dots, \log|b_n|) \notin \overline{\log D}$. Recall that $\log D$ is convex and $\log D \subset (-\infty, M)^n$ for some $M \in \mathbb{R}$ (because D is bounded). Hence there exists an $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbb{Z}^n)_*$ such that

$$\log D \subset \{x \in \mathbb{R}^n : \langle x - x_0, \alpha \rangle < 0\}. \tag{2.7.2}$$

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[20] $\boldsymbol{V}_j = \mathbb{C}^{j-1} \times \{0\} \times \mathbb{C}^{n-j}$, $\widetilde{D}^{(j)} = \{(z', \lambda z_j, z'') : (z', z_j, z'') \in D,\ \lambda \in \overline{\mathbb{D}}\}$.
[21] $\langle\ ,\ \rangle$ denotes the standard scalar product in $\mathbb{R}^n$.

Consequently,

$$|z^\alpha| \le |b^\alpha|, \quad z \in D \setminus \boldsymbol{V}_0.$$

Hence, by $(*)$, $D \cap \boldsymbol{V}_j = \varnothing$ for all j such that $\alpha_j < 0$ ([22]). Thus $D \subset \Omega(\alpha)$ (where $\Omega(\alpha)$ is as in Example 1.3.5) and

$$|z^\alpha| \le |b^\alpha|, \quad z \in D.$$

Let $\widetilde{f} \in \mathcal{O}(\widetilde{D})$ denote the extension of the function $D \ni z \longmapsto z^\alpha$. Then, by the identity principle, $\widetilde{f}(z) = z^\alpha$ for $z \in \widetilde{D} \setminus \boldsymbol{V}_0$. Now, by Proposition 2.7.3(b),

$$|z^\alpha| \le |b^\alpha|, \quad z \in \widetilde{D} \setminus \boldsymbol{V}_0.$$

Since $b \in \widetilde{D} \setminus \boldsymbol{V}_0$, we get a contradiction.

Now consider the case where $\widetilde{D} \setminus \boldsymbol{V}_0 \subset D$. Fix an $a = (a_1, \dots, a_n) \in \widetilde{D} \setminus D \subset \boldsymbol{V}_0$. Suppose that there exists a $j \in \{1, \dots, n\}$ such that $a_j = 0$ and $D \cap \boldsymbol{V}_j = \varnothing$. Then obviously the function $f(z) := 1/z_j$, $z \in D$, is holomorphic in D and cannot be extended to $\widetilde{D}$; contradiction.

Thus we may assume that $D \cap \boldsymbol{V}_j \neq \varnothing$ for all j such that $a_j = 0$. We may assume that $a_1, \dots, a_s \neq 0$ and $a_{s+1}, \dots, a_n = 0$ for some $0 \le s \le n-1$. Using $(*)$ we easily conclude that $(a_1, \dots, a_s, z_{s+1}, \dots, z_n) \notin D$ for any $z_{s+1}, \dots, z_n \in \mathbb{C}$ ([23]). In particular, $s \ge 1$.

Let $\pi : \mathbb{C}^n \longrightarrow \mathbb{C}^s$ denote the natural projection $(z_1, \dots, z_n) \longmapsto (z_1, \dots, z_s)$. Observe that $(\log|a_1|, \dots, \log|a_s|) \notin \pi(\log D)$. Hence there exists a point $b = (b_1, \dots, b_n) \in \widetilde{D}$ (in a neighborhood of a) such that $x_0' := (\log|b_1|, \dots, \log|b_s|) \notin \overline{\pi(\log D)}$. The set $\overline{\pi}(\log D)$ is closed and contained in $(-\infty, M)^s$ for some $M \in \mathbb{R}$. Consequently, there exists an $\alpha \in (\mathbb{Z}^s)_* \times \{0\}^{n-s}$ such that (2.7.2) holds. Now we can argue as above.

We pass to the case where D is unbounded. Let $D_N := D \cap \mathbb{P}(N)$, $N \in \mathbb{N}$. Then each D_N is n-circled, bounded, satisfies $(*)$, and $\log D_N = (\log D) \cap (-\infty, \log N)^n$ is convex. In particular, $D_N \setminus \boldsymbol{V}_0$ is a domain. Since $D_N \subset \overline{D_N \setminus \boldsymbol{V}_0}$, we conclude that D_N is a domain.

Thus, by the first part of the proof, D_N is a domain of holomorphy. Obviously, $D_N \subset D_{N+1}$.

Now, it would be sufficient to know that the union of an increasing sequence of regions of holomorphy is a region of holomorphy. This will be done in the sequel (cf. Proposition 4.1.2(c) and Theorem 5.3.2). $\square$

Corollary 2.7.16. *If D is an n-circled domain of holomorphy with $0 \in D$, then D is complete n-circled.*

Remark 2.7.17. In the case where D is complete the implication (ii) $\Longrightarrow$ (i) in Proposition 2.7.15 may be proved in a simpler way. Namely, we prove that D is holomorphically convex.

Fix a compact $K \subset D$ and an arbitrary point $a = (a_1, \dots, a_n)$ in the closure (in $\mathbb{C}^n$) of $\widehat{K}_{\mathcal{O}(D)}$. We want to show that $a \in D$. Without loss of generality we may assume that $a_1, \dots, a_s \neq 0$, $a_{s+1} = \dots = a_n = 0$, where $1 \le s \le n$. It is easily seen that there exists a finite number of points $\xi^{(1)}, \dots, \xi^{(N)} \in D \cap \mathbb{R}^n_{>0}$ such that

$$K \subset \bigcup_{j=1}^N \mathbb{P}(\xi^{(j)}).$$

By the definition of $\widehat{K}_{\mathcal{O}(D)}$ we have

$$|a_1^{\alpha_1} \cdot \dots \cdot a_s^{\alpha_s}| \le \max\{(\xi_1^{(j)})^{\alpha_1} \cdot \dots \cdot (\xi_s^{(j)})^{\alpha_s} : j = 1, \dots, N\}, \quad \alpha_1, \dots, \alpha_s \in \mathbb{N}_0.$$

Hence (we take the log and divide by $|\alpha_1| + \dots + |\alpha_s|$) we get

$$\sum_{\nu=1}^s t_\nu \log|a_\nu| \le \max\{\sum_{\nu=1}^s t_\nu \log \xi_\nu^{(j)} : j = 1, \dots, N\}, \quad t_1, \dots, t_s \in \mathbb{Q}_+,\ t_1 + \dots + t_s = 1,$$

([22]) Indeed, suppose that $\alpha_j < 0$ and $D \cap \boldsymbol{V}_j \neq \varnothing$. Fix a $z = (z_1, \dots, z_n) \in D \setminus \boldsymbol{V}_0$. Then, by $(*)$, $z(\lambda) := (z_1, \dots, z_{j-1}, \lambda z_j, z_{j+1}, \dots, z_n) \in D \setminus \boldsymbol{V}_0$ for any $\lambda \in \overline{\mathbb{D}} \setminus \{0\}$. Consequently, $|b^\alpha| \ge |z(\lambda)^\alpha| = |\lambda|^{\alpha_j} |z^\alpha|$, $\lambda \in \overline{\mathbb{D}} \setminus \{0\}$. Letting $\lambda \longrightarrow 0$ we get a contradiction.

([23]) Otherwise, $a = (a_1, \dots, a_s, 0, \dots, 0) \in D$.

and, consequently, by continuity, for all $t_1, \dots, t_s \in \mathbb{R}_+$ with $t_1 + \dots + t_s = 1$. Thus the point $(\log|a_1|, \dots, \log|a_s|)$ is in the convex hull C_s of the set

$$\{(\eta_1, \dots, \eta_s) \in \mathbb{R}^s : \exists\, j \in \{1, \dots, N\} : \eta_\nu \le \log \xi_\nu^{(j)},\ \nu = 1, \dots, s\}.$$

Clearly the set C_s is contained in the projection onto $\mathbb{R}^s$ of the convex hull C_n of the set

$$\{(\eta_1, \dots, \eta_n) \in \mathbb{R}^n : \exists\, j \in \{1, \dots, N\} : \eta_\nu \le \log \xi_\nu^{(j)},\ \nu = 1, \dots, n\}.$$

Since $C_n \subset \log D$, there exists a point $x \in \log D$, for which $|a_\nu| \le e^{x_\nu}$, $\nu = 1, \dots, n$. Hence $a \in D$.

Example 2.7.18. For $0 < a, b < 1$ let

$$D := \{(z, w) \in \mathbb{D}^2 : |z| < a \text{ or } |w| < b\}.$$

Then the smallest 2-circled domain of holomorphy containing D is of the form

$$\widetilde{D} := \{(z_1, z_2) \in \mathbb{D}^2 : |z_1|^{-\log b} |z_2|^{-\log a} < e^{-\log a \log b}\}.$$

2.8. Riemann regions over $\mathbb{C}^{\mathbf{n}}$

Example 2.8.1. (Cf. [33]) Let

$$D := \big(\mathbb{D} \times (2\mathbb{D})\big) \setminus \big(Q_1 \cup Q_2 \cup S\big),$$

where

$$Q_1 := \{(x + i0, w) \in \mathbb{D} \times (2\mathbb{D}) : x \ge 0,\ |w| \le 1\}, \quad Q_2 := \{(0 + iy, w) \in \mathbb{D} \times (2\mathbb{D}) : y \ge 0,\ |w| \ge 1\},$$
$$S := \{(x + iy, w) \in \mathbb{D} \times (2\mathbb{D}) : x \ge 0,\ y \ge 0,\ |w| = 1\}.$$

Notice that D is a Hartogs domain over $\mathbb{D}_*$ and

$$D_{x+iy} = \begin{cases} \{1 < |w| < 2\} & \text{if } x > 0 \text{ and } y = 0 \\ \mathbb{D} \cup \{1 < |w| < 2\} & \text{if } x > 0 \text{ and } y > 0 \\ \mathbb{D} & \text{if } x = 0 \text{ and } y > 0 \\ 2\mathbb{D} & \text{if } x < 0 \text{ or } y < 0 \end{cases}, \quad x + iy \in \mathbb{D}_*.$$

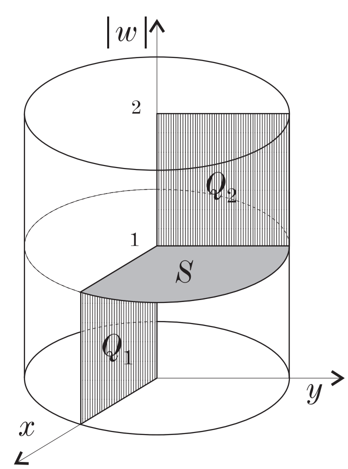

Figure 2.4.1

Put

$$\widetilde{D} := \{(x + iy, w) \in \mathbb{D} \times (2\mathbb{D}) : x \ne 0 \text{ or } y \notin [0, 1)\}$$

For $f \in \mathcal{O}(D)$ let

$$\widetilde{f}(z,w) := \frac{1}{2\pi i}\int_{C(r)} \frac{f(z,\zeta)}{\zeta - w}d\zeta, \quad (z,w) \in \widetilde{D},\ 1 < r < 2,\ |w| < r;$$

notice that $\widetilde{f}(z,w)$ is independent of r with $1 < r < 2$, $|w| < r$. Then $\widetilde{f} \in \mathcal{O}(\widetilde{D})$ and $\widetilde{f} = f$ in the domain

$$\{(x+iy,w) \in \mathbb{D} \times (2\mathbb{D}) : x < 0 \text{ or } y < 0\} \cup \{(x+iy,w) \in \mathbb{D} \times (2\mathbb{D}) : (x \neq 0 \text{ or } y \notin [0,1)),\ 1 < |w| < 2\}.$$

Consequently, every function holomorphic in D extends (in the sense of Definition 2.7.1) to $\widetilde{D}$.

Let $\mathrm{Log} : \mathbb{C} \setminus \mathbb{R}_- \longrightarrow \mathbb{C}$ denote the principal branch of the logarithm. Define

$$\ell_- : \mathbb{C} \setminus \mathbb{R}_+ \longrightarrow \mathbb{C}, \quad \ell_-(z) := \mathrm{Log}(-z), \qquad \ell_+ : \mathbb{C} \setminus i\mathbb{R}_+ \longrightarrow \mathbb{C}, \quad \ell_+(z) := \mathrm{Log}(iz) + i\pi/2.$$

Observe that $\ell_+ = \ell_-$ in the domain $\{x+iy : x < 0 \text{ or } y < 0\}$ and $\ell_+ = \ell_- + 2\pi i$ in the domain $\varDelta := \{x+iy : x > 0 \text{ and } y > 0\}$.

Define $f_0 : D \longrightarrow \mathbb{C}$,

$$f_0(z,w) := \begin{cases} \exp((1/2)\ell_-(z)) & \text{if } |w| \le 1 \\ \exp((1/2)\ell_+(z)) & \text{if } |w| \ge 1 \end{cases}.$$

One can easily prove that f_0 is well defined and holomorphic in D. It is also clear that $\widetilde{f_0}(z,w) = \exp((1/2)\ell_+(z))$, $(z,w) \in \widetilde{D}$. On the other hand, for $(z,w) \in \varDelta \times \mathbb{D}$ we have $\widetilde{f_0}(z,w) = \exp((1/2)\ell_+(z)) = -\exp((1/2)\ell_-(z)) = -f_0(z,w)$.

Consequently, the extension of the function f_0 cannot be univalent on $\widetilde{D}$.

A pair (X,p) is called a *Riemann region over* $\mathbb{C}^n$ if:

X is a topological Hausdorff space,

$p : X \longrightarrow \mathbb{C}^n$ is a local homeomorphism, i.e. an arbitrary point $a \in X$ has an open neighborhood U such that the set $p(U)$ is open in $\mathbb{C}^n$ and the mapping $p|_U : U \longrightarrow p(U)$ is a homeomorphism.

Any open set U with the above property will be called *univalent* or *schlicht*.

If X is connected, then we say that (X,p) is a *Riemann domain* over $\mathbb{C}^n$.

Remark 2.8.2. (a) If $\varOmega$ is an open subset of $\mathbb{C}^n$, then $(\varOmega, \mathrm{id})$ is a Riemann region.
(b) If (X,p) is a Riemann region and $Y \subset X$ is open, then $(Y, p|_Y)$ is a Riemann region.
(c) If (X_j,p_j) is a Riemann region over $\mathbb{C}^{n_j}$, $j = 1\dots,N$, then $(X_1 \times \dots \times X_N, p_1 \times \dots \times p_N)$ is a Riemann region over $\mathbb{C}^{n_1+\dots+n_N}$.

Definition 2.8.3. Let $(X,p),(Y,q)$ be Riemann regions over $\mathbb{C}^n$. A mapping $\varphi : X \longrightarrow Y$ is called a *morphism* if φ is continuous and $q \circ \varphi = p$.

$$\begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & \mathbb{C}^n & \end{tikzcd}$$

If, moreover, φ is bijective and φ^{-1} is also a morphism, then we say that φ is an *isomorphism*.

Remark 2.8.4. (a) The composition of morphisms is a morphism.

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[dr, "q"'] & Y \arrow[r, "\psi"] \arrow[d, "p"] & Z \arrow[dl, "r"] \\ & \mathbb{C}^n & \end{tikzcd}$$

(b) If (X,p) is a Riemann region over $\mathbb{C}^n$, then p is a morphism of (X,p) into $(\mathbb{C}^n, \mathrm{id})$.
(c) Every morphism is a local homeomorphism and, therefore, an open mapping. In particular, $\varphi(X)$ is an open subset of Y. Moreover, if $U \subset X$ is univalent, then $\varphi(U)$ is univalent in Y.
(d) If (X,p) is a Riemann region, then p is an open mapping.
(e) If a morphism φ is bijective, then it is an isomorphism.

Definition 2.8.5. Let (X,p), (Y,q) be Riemann regions over $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively. A mapping $f: X \longrightarrow Y$ is called *holomorphic* ($f \in \mathcal{O}(X,Y)$) if f is continuous and for any univalent open set $U \subset X$ the mapping

$$q \circ f \circ (p|_U)^{-1} : p(U) \longrightarrow \mathbb{C}^m$$

is holomorphic in the standard sense.

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[d, "q"] \\ U \arrow[u, "(p|_U)^{-1}"] & \mathbb{C}^m \end{tikzcd}$$

If $(Y,q) = (\mathbb{C}, \mathrm{id})$, then we write $f \in \mathcal{O}(X)$.

Remark 2.8.6. (a) In the case where $(X,p) = (\Omega, \mathrm{id})$ (with $\Omega \in \operatorname{top}\mathbb{C}^n$) and $(Y,q) = (G, \mathrm{id})$ (with $G \in \operatorname{top}\mathbb{C}^m$) the above definition of a holomorphic mapping is equivalent to the standard one.
(b) The composition of holomorphic mappings is holomorphic.
(c) Every morphism is a holomorphic mapping. In particular, if (X,p) is a Riemann region, then p is holomorphic.

Lemma 2.8.7 (Identity principle for liftings)**.** *Let (X,p) be a Riemann region, let T be a connected topological space, and let $\gamma_j : T \longrightarrow X$, $j = 1,2$, be continuous mappings such that $p \circ \gamma_1 \equiv p \circ \gamma_2$ and $\gamma_1(t_0) = \gamma_2(t_0)$ for some $t_0 \in T$. Then $\gamma_1 \equiv \gamma_2$.*

$$\begin{tikzcd} T \arrow[r, "\gamma_j"] \arrow[dr, "p\circ\gamma_j"'] & X \arrow[d, "p"] \\ & \mathbb{C}^n \end{tikzcd}$$

Proof. Let $T_0 := \{t \in T : \gamma_1(t) = \gamma_2(t)\}$. The set T_0 is closed and non-empty. It is sufficient to show that it is open. Fix a $t \in T_0$ and put $a := \gamma_1(t) = \gamma_2(t)$. Let U be a univalent neighborhood of a and let V be a neighborhood of t such that $\gamma_j(V) \subset U$, $j = 1,2$. Then $(p|_U) \circ \gamma_1 = (p|_U) \circ \gamma_2$ in V, and hence $V \subset T_0$. $\square$

Proposition 2.8.8 (Identity principle)**.** *Let (X,p), (Y,q) be Riemann regions over $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively, $f, g \in \mathcal{O}(X,Y)$. Assume that X is connected and $f = g$ on some non-empty open subset. Then $f \equiv g$.*

Proof. Because of Lemma 2.8.7 it is sufficient to show that $\widetilde{f} \equiv \widetilde{g}$, where $\widetilde{f} := q \circ f$, $\widetilde{g} := q \circ g$. Put

$$X_0 := \{x \in X : \widetilde{f} = \widetilde{g} \text{ in some neighborhood of } x\}.$$

The set X_0 is open and non-empty. It is sufficient to show that it is closed. Let a be an accumulation point of X_0 and let U be a univalent connected neighborhood of a. Note that $\widetilde{f} \circ (p|_U)^{-1} = \widetilde{g} \circ (p|_U)^{-1}$ on a non-empty and open set $p(X_0 \cap U)$. Hence, by the standard identity principle, $\widetilde{f} \circ (p|_U)^{-1} \equiv \widetilde{g} \circ (p|_U)^{-1}$, which shows that $U \subset X_0$. $\square$

For Riemann regions $(X,p), (Y,q)$ and a morphism $\varphi : X \longrightarrow Y$ let

$$\varphi^* : \mathcal{O}(Y) \longrightarrow \mathcal{O}(X), \quad \varphi^*(f) := f \circ \varphi.$$

Remark 2.8.9. The mapping φ^* is injective iff every connected component of Y intersects $\varphi(X)$.

Definition 2.8.10. Let $(X,p), (Y,q)$ be Riemann regions over $\mathbb{C}^n$, $\varnothing \neq \mathcal{F} \subset \mathcal{O}(X)$. We say that a morphism $\varphi : X \longrightarrow Y$ gives an *$\mathcal{F}$-extension* if φ^* is injective and $\mathcal{F} \subset \varphi^*(\mathcal{O}(Y))$.

We shortly write: $\varphi : X \longrightarrow Y$ is an $\mathcal{F}$-extension.

$\mathcal{O}(X)$-extensions are called *holomorphic extensions.*

An $\mathcal{F}$-extension $\varphi : X \longrightarrow Y$ is called a *maximal $\mathcal{F}$-extension* if for any $\mathcal{F}$-extension $\psi : X \longrightarrow Z$ there exists a morphism $\sigma : Z \longrightarrow Y$ such that $\sigma \circ \psi = \varphi$.

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[dr, "\psi"'] & Y \\ & Z \arrow[u, "\sigma"'] \end{tikzcd}$$

If $\varphi : X \longrightarrow Y$ is a maximal $\mathcal{F}$-extension, then (Y, q) is called an *$\mathcal{F}$-envelope of holomorphy* of (X, p).

If $\mathcal{F} = \mathcal{O}(X)$, then we say that $\varphi : X \longrightarrow Y$ is a *maximal holomorphic extension* and that (Y, q) is an *envelope of holomorphy* of (X, p).

We say that (X, p) is an *$\mathcal{F}$-region of holomorphy* if for any $\mathcal{F}$-extension $\varphi : X \longrightarrow Y$ the morphism φ is bijective.

Remark 2.8.11. (a) If $\varphi : X \longrightarrow Y$ is an $\mathcal{F}$-extension, then it is a $\mathcal{G}$-extension for any $\varnothing \neq \mathcal{G} \subset \mathcal{F}$.
(b) If $\varphi : X \longrightarrow Y$ is an $\mathcal{F}$-extension and $\psi : Y \longrightarrow Z$ is a $(\varphi^*)^{-1}(\mathcal{F})$-extension, then $\psi \circ \varphi : X \longrightarrow Z$ is an $\mathcal{F}$-extension.

Proposition 2.8.12. (a) *The maximal $\mathcal{F}$-extension is determined uniquely up to an isomorphism.*
(b) *If $\varphi : X \longrightarrow Y$ is the maximal $\mathcal{F}$-extension, then Y is a $\mathcal{G}$-region of holomorphy with $\mathcal{G} := (\varphi^*)^{-1}(\mathcal{F})$.*
(c) *If $\varphi : X \longrightarrow Y$ is the maximal $\mathcal{F}$-extension, then X is an $\mathcal{F}$-region of holomorphy iff φ is an isomorphism.*

Proof. (a) Suppose that $\varphi : X \longrightarrow Y$ and $\psi : X \longrightarrow Z$ are two maximal $\mathcal{F}$-extensions. Then there exist morphisms $\sigma : Z \longrightarrow Y$ and $\tau : Y \longrightarrow Z$ such that $\sigma \circ \psi = \varphi$, $\tau \circ \varphi = \psi$.

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[dr, "\psi"'] & Y \arrow[d, shift right, "\tau"'] \\ & Z \arrow[u, shift right, "\sigma"'] \end{tikzcd}$$

Consequently, $\sigma \circ \tau \circ \varphi = \varphi$, i.e. $\sigma \circ \tau = \mathrm{id}_Y$ on $\varphi(X)$. Since every connected component of Y intersects $\varphi(X)$, the identity principle implies that $\sigma \circ \tau = \mathrm{id}_Y$. Similarly, $\tau \circ \sigma = \mathrm{id}_Z$.

(b) Suppose that $\psi : Y \longrightarrow Z$ is a $\mathcal{G}$-extension. Then $\psi \circ \varphi : X \longrightarrow Z$ is an $\mathcal{F}$-extension. Consequently, since $\varphi : X \longrightarrow Y$ is maximal, there exists a morphism $\sigma : Z \longrightarrow Y$ such that $\sigma \circ (\psi \circ \varphi) = \varphi$. Hence, similarly as in (a), $\sigma \circ \psi = \mathrm{id}_Y$, $\psi \circ \sigma = \mathrm{id}_Z$. In particular, ψ is an isomorphism.

(c) The implication $\Longrightarrow$ is trivial. To prove $\Longleftarrow$ suppose that $\psi : X \longrightarrow Z$ is an arbitrary $\mathcal{F}$-extension. Since $\varphi : X \longrightarrow Y$ is maximal, there exists a morphism $\sigma : Z \longrightarrow Y$ such that $\sigma \circ \psi = \varphi$. We know that φ is an isomorphism. Consequently, ψ is an isomorphism with $\psi^{-1} = \varphi^{-1} \circ \sigma$. $\square$

Example 2.8.13 (Sheaf of germs)**.** (Cf. § 1.8.) Let $I \neq \varnothing$ be arbitrary. For $a \in \mathbb{C}^n$ let ${}^{(I)}\widetilde{\mathcal{O}}_a$ denote the set of all pairs $(U, \mathbf{F})$ such that U is a neighborhood of a and $\mathbf{F} : I \longrightarrow \mathcal{O}(U)$. We define an equivalence relation in ${}^{(I)}\widetilde{\mathcal{O}}_a$:

$$(U, \mathbf{F}) \overset{a}{\sim} (V, \mathbf{G}) \overset{\text{df}}{\Longleftrightarrow} \text{there exists a neighborhood } W \text{ of } a \ (W \subset U \cap V)$$
$$\text{such that } \mathbf{F}(\iota) = \mathbf{G}(\iota) \text{ in } W \text{ for any } \iota \in I.$$

Put

$${}^{(I)}\mathcal{O}_a := {}^{(I)}\widetilde{\mathcal{O}}_a / \overset{a}{\sim}, \quad {}^{(I)}\mathcal{O} := \bigcup_{a \in \mathbb{C}^n} \{a\} \times {}^{(I)}\mathcal{O}_a, \quad \pi_I : {}^{(I)}\mathcal{O} \longrightarrow \mathbb{C}^n, \quad \pi_I(a, \mathfrak{f}) := a.$$

We endow ${}^{(I)}\mathcal{O}$ with the topology in which the basis of neighborhoods of a point $(a, \mathfrak{f}) \in {}^{(I)}\mathcal{O}$ consists of all sets of the form

$$\mathfrak{U}(a, \mathfrak{f}, U) := \{(z, [(U, \mathbf{F})]_{\overset{z}{\sim}}) : z \in U\},$$

where $(U, \mathbf{F})$ is a representant of $\mathfrak{f}$. One can easily check that the topology is well defined and Hausdorff.

Indeed, if $(a, \mathfrak{f}) \neq (b, \mathfrak{g})$, $\mathfrak{f} = [(U, \mathbf{F})]_{\overset{a}{\sim}}$, $\mathfrak{g} = [(V, \mathbf{G})]_{\overset{b}{\sim}}$ then:

if $a \neq b$, then $\mathfrak{U}(a, \mathfrak{f}, U') \cap \mathfrak{U}(b, \mathfrak{g}, V') = \varnothing$ provided that $U' \subset U$, $V' \subset V$, and $U' \cap V' = \varnothing$,

if $a = b$, then, by the identity principle for holomorphic functions, $\mathfrak{U}(a, \mathfrak{f}, W) \cap \mathfrak{U}(a, \mathfrak{g}, W) = \varnothing$ for any connected neighborhood W of a such that $W \subset U \cap V$.

Moreover, the projection π_I maps homeomorphically $\mathfrak{U}(a, \mathfrak{f}, U)$ onto U and

$$(\pi_I|_{\mathfrak{U}(a,\mathfrak{f},U)})^{-1}(z) = (z, [(U, \mathbf{F})]_{\underset{\sim}{z}}), \quad z \in U.$$

Consequently, π_I is a local homeomorphism and $({}^{(I)}\mathcal{O}, \pi_I)$ is a Riemann region over $\mathbb{C}^n$.

Fix a $\iota \in I$ and let $\mathfrak{F}_\iota : {}^{(I)}\mathcal{O} \longrightarrow \mathbb{C}$ be given by the formula

$$\mathfrak{F}_\iota(a, \mathfrak{f}) := \mathbf{F}(\iota)(a),$$

where $(U, \mathbf{F})$ is a representant of $\mathfrak{f}$. Note that $\mathfrak{F}_\iota$ is well defined and

$$\mathfrak{F}_\iota \circ (\pi_I|_{\mathfrak{U}(a,\mathfrak{f},U)})^{-1} = \mathbf{F}(\iota).$$

Thus $\mathfrak{F}_\iota$ is holomorphic on ${}^{(I)}\mathcal{O}$.

Theorem 2.8.14 (Thullen theorem)**.** *For any Riemann region (X, p) and a family $\varnothing \neq \mathcal{F} \subset \mathcal{O}(X)$ there exists the maximal $\mathcal{F}$-extension.*

Proof. Fix (X, p) and $\mathcal{F}$. First we define a morphism $\varphi : X \longrightarrow {}^{(\mathcal{F})}\mathcal{O}$ (where $({}^{(\mathcal{F})}\mathcal{O}, \pi_{\mathcal{F}})$ is the Riemann region constructed in Example 2.8.13 with $I := \mathcal{F}$). For $x \in X$ we put

$$\varphi(x) := (p(x), [(p(U_x), \mathbf{F})]_{\underset{\sim}{p(x)}}),$$

where

U_x is a univalent neighborhood of x,

$\mathbf{F}(f) := f \circ (p|_{U_x})^{-1}$, $f \in \mathcal{F}$.

Observe that φ is well defined, $\pi_{\mathcal{F}} \circ \varphi = p$, and φ is continuous. Thus φ is a morphism.

Moreover, for an arbitrary $f \in \mathcal{F}$ the mapping $\mathfrak{F}_f$ (constructed in Example 2.8.13) is holomorphic on ${}^{(\mathcal{F})}\mathcal{O}$ and $\mathfrak{F}_f \circ \varphi = f$.

Let Y denote the union of those connected components of ${}^{(\mathcal{F})}\mathcal{O}$ that intersect $\varphi(X)$ and put $q := \pi_{\mathcal{F}}|_Y$. We have proved that $\varphi : X \longrightarrow Y$ is an $\mathcal{F}$-extension.

Now we show that the above extension is maximal. Let $\psi : X \longrightarrow Z$ be another $\mathcal{F}$-extension. Put $\mathcal{G} := (\psi^*)^{-1}(\mathcal{F})$ and let $\sigma : Z \longrightarrow {}^{(\mathcal{G})}\mathcal{O}$ be constructed in the same way as φ. Observe $({}^{(\mathcal{G})}\mathcal{O}, \pi_{\mathcal{G}}) = ({}^{(\mathcal{F})}\mathcal{O}, \pi_{\mathcal{F}})$ (because the mapping $\mathcal{G} \ni g \longmapsto g \circ \psi \in \mathcal{F}$ is bijective). For $x \in X$ we have

$$\sigma(\psi(x)) = (r(\psi(x)), [(r(\psi(U_x)), \mathbf{G})]_{\underset{\sim}{r(\psi(x))}}) = (p(x), [(p(U_x), \mathbf{F})]_{\underset{\sim}{p(x)}}) = \varphi(x),$$

where

U_x is a univalent neighborhood of x,

$\mathbf{G}(g) := g \circ (r|_{\psi(U_x)})^{-1} = g \circ \psi \circ (p|_{U_x})^{-1}$, $g \in \mathcal{G}$,

$\mathbf{F}(f) := f \circ (p|_{U_x})^{-1}$, $f \in \mathcal{F}$.

Moreover, since any connected component of Z intersects $\psi(X)$, we conclude that $\sigma(Z) \subset Y$. $\square$

Proposition 2.8.15. *Let $\Omega \subset \mathbb{C}^n$ be open and let $\varnothing \neq \mathcal{F} \subset \mathcal{O}(\Omega)$. Then Ω is an $\mathcal{F}$-region of holomorphy in the sense of Definition 2.7.1 iff (Ω, id) is an $\mathcal{F}$-region of holomorphy in the sense of Definition 2.8.10.*

Proof. $\Longrightarrow$. Let $\psi : \Omega \longrightarrow Z$ be an $\mathcal{F}$-extension. We have to show that ψ is bijective. Since $r \circ \psi = \mathrm{id}$, the mapping ψ must be injective. Suppose that ψ is not surjective and let $b \in Z$ be an arbitrary boundary point of $\psi(\Omega)$. Let $V \subset Z$ be a univalent and connected neighborhood of b. Put $\widetilde{\Omega} := r(V)$, $\Omega_0 := r(\boldsymbol{V}_0)$, where $\boldsymbol{V}_0$ is a connected component of $V \cap \psi(\Omega)$. Take an $f \in \mathcal{F}$. Let $g \in \mathcal{O}(Z)$ be such that $g \circ \psi = f$. Put $\widetilde{f} := g \circ (r|_V)^{-1}$. Then $\widetilde{f} \in \mathcal{O}(\widetilde{\Omega})$ and $\widetilde{f} = f$ on Ω_0. Consequently, $\widetilde{\Omega} \subset \Omega$; contradiction.

$\Longleftarrow$. Suppose that Ω is not an $\mathcal{F}$-region of holomorphy and let $\widetilde{\Omega}, \Omega_0$ be as in Definition 2.7.1 (for any $f \in \mathcal{F}$ there exists an $\widetilde{f} \in \mathcal{O}(\widetilde{\Omega})$ such that $\widetilde{f} = f$ on Ω_0). Let $\varphi : \Omega \longrightarrow Y$ be the $\mathcal{F}$-maximal extension constructed in the proof of the Thullen theorem. For $a \in \Omega_0$ we have

$$\varphi(a) = (a, [(\Omega_0, \mathbf{F})]_{\underset{\sim}{a}}),$$

where $\mathbf{F}(f) := f$, $f \in \mathcal{F}$. Hence

$$\{(a, [(\widetilde{\Omega}, \mathbf{G})]_{\underset{\sim}{a}}) : a \in \widetilde{\Omega}\} \subset Y,$$

where $\mathbf{G}(f) := \widetilde{f}$, $f \in \mathcal{F}$. In particular, $\widetilde{\Omega} \subset q(Y)$. Consequently, φ cannot be an isomorphism; contradiction. $\square$

Directly from the proof of the Thullen theorem we get the following corollaries.

Corollary 2.8.16. *Let (X, p), (Y, q) be Riemann domains over $\mathbb{C}^n$, let $\tau : X \longrightarrow Y$ be a morphism, and let $\mathcal{G} \subset \mathcal{O}(Y)$, $\tau^*(\mathcal{G}) \subset \mathcal{F} \subset \mathcal{O}(X)$. Assume that $\varphi : X \longrightarrow \widehat{X}$ and $\psi : Y \longrightarrow \widehat{Y}$ are the maximal $\mathcal{F}$- and $\mathcal{G}$-extension, respectively, where $(\widehat{X}, \widehat{p})$ and $(\widehat{Y}, \widehat{q})$ are Riemann domains over $\mathbb{C}^n$. Then there exists a morphism $\widehat{\tau} : \widehat{X} \longrightarrow \widehat{Y}$ such that $\widehat{\tau} \circ \varphi = \psi \circ \tau$. In particular, $\widehat{p}(\widehat{X}) \subset \widehat{q}(\widehat{Y})$.*

$$\begin{array}{ccc} X & \xrightarrow{\ \tau\ } & Y \\ {\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle\psi} \\ \widehat{X} & \xrightarrow{\ \widehat{\tau}\ } & \widehat{Y} \end{array}$$

Definition 2.8.17. If (X, p) is a Riemann region over $\mathbb{C}^n$ and $f \in \mathcal{O}(X)$, then for any $\alpha \in \mathbb{N}_0^n$ define

$$D^\alpha f(x) := D^\alpha(f \circ (p|_{U_x})^{-1})(p(x)), \quad x \in X,$$

where U_x is an arbitrary univalent neighborhood of x.

Observe that $D^\alpha f$ is well defined and holomorphic on X.

Corollary 2.8.18. *Let (X, p) be a Riemann region over $\mathbb{C}^n$, let $\mathcal{F} \subset \mathcal{O}(X)$, and let $\varphi : X \longrightarrow Y$ be the maximal $\mathcal{F}$-extension. Then φ is injective iff for any points $x_1, x_2 \in X$ with $x_1 \neq x_2$ and $p(x_1) = p(x_2)$ there exist $f \in \mathcal{F}$ and $\alpha \in \mathbb{N}_0^n$ such that $D^\alpha f(x_1) \neq D^\alpha f(x_2)$.*

In particular, if $\mathcal{F} = \mathcal{O}(X)$, then φ is injective iff $\mathcal{O}(X)$ separates points in X.

Consequently, if (X, p) is a region of holomorphy, then $\mathcal{O}(X)$ separates points in X.

Proposition 2.8.19. *Let (X, p), (Y, q) be Riemann domains over $\mathbb{C}^n$, let $\mathcal{F} \subset \mathcal{O}(X)$, and let $\tau : X \longrightarrow Y$ be an $\mathcal{F}$-extension. Assume that (Y, q) is a $\mathcal{G}$-domain of holomorphy, where $\mathcal{G} := (\tau^*)^{-1}(\mathcal{F})$. Then $\tau : X \longrightarrow Y$ is the maximal $\mathcal{F}$-extension.*

Proof. Let $\varphi : X \longrightarrow \widehat{X}$ be the maximal $\mathcal{F}$-extension. By definition there exists a morphism $\sigma : Y \longrightarrow \widehat{X}$ such that $\sigma \circ \tau = \varphi$. On the other hand, by Corollary 2.8.16, there exists a morphism $\widehat{\tau} : \widehat{X} \longrightarrow Y$ such that $\widehat{\tau} \circ \varphi = \tau$. Consequently, $(\widehat{\tau} \circ \sigma) \circ \tau = \tau$ and $(\sigma \circ \widehat{\tau}) \circ \varphi = \varphi$. Hence, by the identity principle, σ is an isomorphism and $\sigma^{-1} = \widehat{\tau}$. $\square$

It is clear that the notion of the natural Fréchet space (Definition 2.7.6) extends to Riemann regions over $\mathbb{C}^n$. Observe that if X is *countable at infinity* (i.e. $X = \bigcup_{\nu=1}^\infty K_\nu$, where $K_\nu \subset \operatorname{int} K_{\nu+1}$ and K_ν is compact, $\nu \in \mathbb{N}$), then $\mathcal{O}(X)$ with the topology of locally uniform convergence is a Fréchet space.

It is well known that any Riemann domain over $\mathbb{C}^n$ is countable at infinity (cf. [22]).

Remark 2.8.20. (a) Let (X, p), (Y, q) be countable at infinity Riemann regions over $\mathbb{C}^n$, let $\mathcal{F} \subset \mathcal{O}(X)$ and let $\varphi : X \longrightarrow Y$ be an $\mathcal{F}$-extension. Assume that $\mathcal{F}$ is a natural Fréchet space in $\mathcal{O}(X)$ and let $q_k : \mathcal{F} \longrightarrow \mathbb{R}_+$, $k \in \mathbb{N}$, be a family of seminorms defining the topology of $\mathcal{F}$. Let $(L_k)_{k=1}^\infty$ be a sequence of compact subsets of Y such that $L_k \subset \operatorname{int} L_{k+1}$ and $Y = \bigcup_{k=1}^\infty L_k$. Put $\mathcal{G} := (\varphi^*)^{-1}(\mathcal{F})$. We endow $\mathcal{G}$ with the topology generated by the following seminorms:

$$\mathcal{G} \ni g \longmapsto q_k(g \circ \varphi), \quad \mathcal{G} \ni g \longmapsto \|g\|_{L_k}, \quad k \in \mathbb{N}.$$

Then $\mathcal{G}$ is a Fréchet space. Moreover, by the Banach theorem, $\varphi^*|_{\mathcal{G}} : \mathcal{G} \longrightarrow \mathcal{F}$ is a topological isomorphism.

Indeed, let $(g_\nu)_{\nu=1}^\infty \subset \mathcal{G}$ be a Cauchy sequence. Then $(g_\nu \circ \varphi)_{\nu=1}^\infty$ is a Cauchy sequence in $\mathcal{F}$. Consequently, $g_\nu \circ \varphi \longrightarrow f_0$ in $\mathcal{F}$. On the other hand, $(g_\nu)_{\nu=1}^\infty \subset \mathcal{O}(Y)$ is a Cauchy sequence in the topology of locally uniform convergence on Y. Thus $g_\nu \longrightarrow g_0$ locally uniformly in Y. Clearly, $g_0 \circ \varphi = f_0$. Thus $g_0 \in \mathcal{G}$ and $g_\nu \longrightarrow g_0$ in $\mathcal{G}$.

(b) In the case where $\mathcal{F} = \mathcal{O}(X)$ (with the topology of locally uniform convergence on X) the continuity of $(\varphi^*)^{-1}$ means that for any compact $L \subset Y$ there exists a compact $K \subset X$ such that
$$\|g\|_L \le \|g \circ \varphi\|_K, \quad g \in \mathcal{O}(Y).$$

Indeed, since $(\varphi^*)^{-1}$ is continuous, for any compact $L \subset Y$ there exist $C > 0$ and a compact $K \subset X$ such that
$$\|g\|_L \le C\|g \circ \varphi\|_K, \quad g \in \mathcal{O}(Y).$$
In particular, taking g^k instead of g, we get
$$\|g\|_L \le C^{1/k}\|g \circ \varphi\|_K, \quad g \in \mathcal{O}(Y),\ k \in \mathbb{N}.$$
Letting $k \longrightarrow +\infty$ we get the required estimate.

For any domain $D \subset \mathbb{C}^n$ define
$$\widehat{D} := \operatorname{int} \bigcap_U U,$$
where the intersection is taken over all domains of holomorphy $U \subset \mathbb{C}^n$ with $D \subset U$. Observe that $\widehat{D}$ is the smallest domain of holomorphy containing D (Remark 2.7.9(a)).

Remark 2.8.21. (a) By Corollary 2.8.16, if $\varphi : D \longrightarrow \widehat{X}$ is the maximal holomorphic extension ($(\widehat{X}, \widehat{p})$ is a Riemann domain over $\mathbb{C}^n$), then $\widehat{p}(\widehat{X}) \subset \widehat{D}$.

Indeed, we take $(X, p) := (D, \mathrm{id})$, $\tau := \mathrm{id}$, $\mathcal{G} := \mathcal{O}(\widehat{D})$, $\mathcal{F} := \mathcal{O}(D)$, $(\widehat{Y}, \widehat{q}) := (\widehat{D}, \mathrm{id})$, $\psi := \mathrm{id}$ (Proposition 2.8.12). Then, by Corollary 2.8.16, there exists a morphism $\widehat{\tau} : \widehat{X} \longrightarrow \widehat{D}$ such that $\widehat{\tau} \circ \varphi = \mathrm{id}$ on D. Obviously, $\widehat{\tau} = \widehat{p}$. Hence $\widehat{p}(\widehat{X}) = \widehat{\tau}(\widehat{X}) \subset \widehat{D}$.

(b) If $\widehat{X}$ is univalent, then $\widehat{p}(\widehat{X}) = \widehat{D}$. Consequently, if the envelope of holomorphy of D is univalent, then $\widehat{D}$ is the envelope of holomorphy of D.

Indeed, if $\widehat{X}$ is univalent, then by Proposition 2.8.12(b) $\widehat{p}(\widehat{X})$ is a domain of holomorphy containing D ([24]) . Hence $\widehat{D} \subset \widehat{p}(\widehat{X})$.

(c) The envelope of D is univalent iff $\mathcal{O}(\widehat{D})|_D = \mathcal{O}(D)$.

Indeed, the implication $\Longleftarrow$ follows from Proposition 2.8.19.

Proposition 2.8.22. *Let $D \subset U$ be domains such that U has the univalent envelope of holomorphy and $\mathcal{O}(U)|_D$ is dense in $\mathcal{O}(D)$ in the topology of locally uniform convergence. Then the envelope of holomorphy of D is also univalent.*

Proof. Let $\varphi : D \longrightarrow \widehat{X}$ be the maximal holomorphic extension with $(\widehat{X}, \widehat{p})$ being a Riemann domain over $\mathbb{C}^n$ (the Thullen theorem). Recall (Corollary 2.8.18) that $\mathcal{O}(\widehat{X})$ separates points in $\widehat{X}$. Thus, to prove that $\widehat{X}$ univalent it suffices to prove that the space $\widehat{p}^*(\mathcal{O}(\widehat{p}(\widehat{X})))$ is dense in $\mathcal{O}(\widehat{X})$ in the topology of locally uniform convergence on $\widehat{X}$.

Take a $g \in \mathcal{O}(\widehat{X})$, let $f := g \circ \varphi \in \mathcal{O}(D)$, and let $(f_\nu)_{\nu=1}^\infty \subset \mathcal{O}(U)$ be such that $f_\nu \longrightarrow f$ locally uniformly in D. Let $\widehat{f}_\nu \in \mathcal{O}(\widehat{U})$ denote the continuation of f_ν. Recall that $\widehat{p}(\widehat{X}) \subset \widehat{D} \subset \widehat{U}$ (Remark 2.8.21). Put $g_\nu := \widehat{f}_\nu \circ \widehat{p} \in \widehat{p}^*(\mathcal{O}(\widehat{p}(\widehat{X})))$. Then $g_\nu \circ \varphi = f_\nu \longrightarrow f = g \circ \varphi$. Consequently, by Remark 2.8.20, $g_\nu \longrightarrow g$ locally uniformly in $\widehat{X}$. $\square$

Remark 2.8.23. Let $D \subset U \subset \mathbb{C}^n$ be domains such that U is a domain of holomorphy, and let $F : U \longrightarrow U$ be biholomorphic such that $F(D) = D$. Then $F(\widehat{D}) = \widehat{D}$.

Lemma 2.8.24. (a) *Let $D \subset \mathbb{C}^n$ be a domain. Then the following implications are true:*

D is n-circled $\Longrightarrow$ $\widehat{D}$ is n-circled.

D is complete n-circled $\Longrightarrow$ $\widehat{D}$ is complete n-circled.

D is balanced $\Longrightarrow$ $\widehat{D}$ is balanced.

([24]) $(\widehat{p}(\widehat{X}), \mathrm{id})$ is isomorphic with $(\widehat{X}, \widehat{p})$.

(b) *Let $D \subset \mathbb{C}^n$ be a Hartogs domain over a domain of holomorphy $G \subset \mathbb{C}^{n-k}$. Then the following implications are true:*

$\forall_{z\in G}$ D_z *is k-circled* $\Longrightarrow \forall_{z\in G}$ $\widehat{D}_z$ *is k-circled.*

$\forall_{z\in G}$ D_z *is complete k-circled* $\Longrightarrow \forall_{z\in G}$ $\widehat{D}_z$ *is complete k-circled.*

$\forall_{z\in G}$ D_z *is balanced* $\Longrightarrow \forall_{z\in G}$ $\widehat{D}_z$ *is balanced.*

Proof. (a) See the proof of (b).

(b) We apply Proposition 2.8.22 with $U = G \times \mathbb{C}^k$. In the first case we take $F(z,w) := (z, e^{i\theta_1}w_1, \dots, e^{i\theta_k}w_k)$ with $\theta_1, \dots, \theta_k \in \mathbb{R}$.

In the third case we put $F(z,w) := (z, \lambda w)$ with $0 < |\lambda| \le 1$.

In the second case we already know that the fibers of $\widehat{D}$ are k-circled and balanced. By Proposition 2.7.14, $\widehat{D}_z$ is a domain of holomorphy for any $z \in G$. Now we can use Corollary 2.7.16. □

Corollary 2.8.25. (a) *Any n-circled or balanced domain $D \subset \mathbb{C}^n$ has the univalent envelope of holomorphy.*
(b) *Let $D \subset \mathbb{C}^n$ be a Hartogs domain over a domain $G \subset \mathbb{C}^{n-k}$ such that the envelope of holomorphy of G is univalent. Each of the following conditions implies that the envelope of holomorphy of D is univalent:*

$\forall_{z\in G}$ D_z *is connected, k-circled, and* $D \cap (G \times \{0\}) \neq \varnothing$ (25) .

$\forall_{z\in G}$ D_z *is balanced.*

$\forall_{z\in G}$ D_z *is connected, k-circled, and* $D_z \subset (\mathbb{C}_*)^k$ (26) .

Proof. In all the cases we will apply Proposition 2.8.22.

(a) In the first case we put $U = U_1 \times \dots \times U_n$, where

$$U_j := \begin{cases} \mathbb{C}_* & \text{if } D \cap \boldsymbol{V}_j = \varnothing \\ \mathbb{C} & \text{if } D \cap \boldsymbol{V}_j \neq \varnothing \end{cases}, \quad j = 1, \dots, n$$

where $\boldsymbol{V}_j := \mathbb{C}^{j-1} \times \{0\} \times \mathbb{C}^{n-j}$, $j = 1, \dots, n$. Note that U is a domain of holomorphy. Now, Proposition 2.6.2 implies that $\mathcal{O}(U)|_D$ is dense in $\mathcal{O}(D)$.

In the second case we put $U := \mathbb{C}^n$ and we use Proposition 1.6.2 to check that $\mathcal{O}(U)|_D$ is dense in $\mathcal{O}(D)$.

(b) In the first case we take $U := G \times \mathbb{C}^k$. By Corollary 1.5.7 $\widehat{G} \times \mathbb{C}^k$ is the envelope of holomorphy of $G \times \mathbb{C}^k$. Now, by Proposition 2.6.3, $\mathcal{O}(U)|_D$ is dense in $\mathcal{O}(D)$.

In the second case take $U := G \times \mathbb{C}^k$ and use Proposition 1.6.5(b).

In the third case put $U := G \times (\mathbb{C}_*)^k$ and apply once again Proposition 2.6.3.

□

Exercises

2.1. Let $K \subset \mathbb{C}^n$, $n \ge 2$, be compact and let a function $f \in \mathcal{O}(\mathbb{C}^n \setminus K)$ be such that $\limsup_{|z|\to\infty} |f(z)| < +\infty$. Does it follow that $f = \text{const}$?

2.2. Let D be a domain in $\mathbb{C}^n$ and let $F : D \longrightarrow \mathbb{C}^n$ be a locally biholomorphic mapping (i.e. any point $a \in D$ has a neighborhood U such that $F(U)$ is open and $F|_U : U \longrightarrow F(U)$ is biholomorphic). Let $K \subset D$ be compact such that $D \setminus K$ is a domain. Assume that $F|_{D\setminus K} : D \setminus K \longrightarrow F(D \setminus K)$ is biholomorphic. Prove that $F : D \longrightarrow F(D)$ is biholomorphic.

2.3. Let

$$\mathcal{E}(1,m) := \{(z_1,z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^{2m} < 1\}, \quad m > 0.$$

Given $a \in \mathbb{D}$ put

$$\psi : \mathcal{E}(1,m) \ni (z_1,z_2) \longrightarrow \Big(\frac{z_1 - a}{1 - \bar{a}z_1}, \frac{(1-|a|^2)^{\frac{1}{2m}}}{(1-\bar{a}z_1)^{\frac{1}{m}}} z_2\Big) \in \mathbb{C}^2.$$

Prove that $\psi \in \operatorname{Aut}(\mathcal{E}(1,m))$.

(25) Notice that if the fibers D_z, $z \in G$, are not connected, then the envelope of holomorphy of D need not be univalent (Example 2.8.1).

(26) For instance, D is a Hartogs–Laurent domain over a domain with univalent envelope of holomorphy; cf. Remark 1.6.4(d).

2.4. Determine $\operatorname{Aut}(T)$, where T is Hartogs' triangle

$$T := \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < |z_2| < 1\}.$$

2.5. Find $\operatorname{Aut}(\mathbb{D}^2 \setminus ((1/2)\overline{\mathbb{D}})^2)$.

2.6. Does there exist a compact set $K \subset \mathbb{D}^2$, such that $\operatorname{int} K \neq \varnothing$, $0 \notin K$, and 0 is not a fixed point of $\operatorname{Aut}(\mathbb{D}^2 \setminus K)$?

2.7. (Analytic sets, cf. [3].) A subset M of an open set $\Omega \subset \mathbb{C}^n$ is an *analytic subset* of Ω if for any point $a \in \Omega$ there exist an open neighborhood $U \subset \Omega$ and $f_1, \dots, f_N \in \mathcal{O}(U)$ ($N = N(a)$) such that $M \cap U_a = \{z \in U : f_1(z) = \dots = f_N(z) = 0\}$.

Let M be an analytic subset of Ω. A point $a \in M$ is *regular* (we write $a \in \operatorname{Reg}(M)$) if there exists a neighborhood $U_a \subset \Omega$ such that $M \cap U_a$ is a complex manifold. Points from $\operatorname{Sing}(M) := M \setminus \operatorname{Reg}(M)$ are called *singular*. Notice that if $n = 1$, then $\operatorname{Sing}(M) = \varnothing$.

We say that M is *irreducible* if there are no analytic subsets M_1, M_2 of Ω such that $M = M_1 \cup M_2$ and $M_j \neq M$, $j = 1, 2$.

Verify the following statements:

(a) M is an analytic subset of Ω iff $M \cap C$ is an analytic subset of C for any connected component C of Ω.

(b) $\varnothing$ and Ω are analytic subsets of Ω. Any analytic subset of Ω is closed in Ω.

(c) If M is a complex submanifold of Ω, then M is an analytic subset of Ω with $\operatorname{Sing}(M) = \varnothing$.

(d) If M is an analytic subset of a domain $D \subset \mathbb{C}^n$, then either $M = D$ or M is thin in D (Definition 2.1.4). Consequently, $D \setminus M$ is a domain.

In particular, if M is an analytic subset of a domain $D \subset \mathbb{C}$, then either $M = D$ or M is a discrete subset of Ω.

(e) If $M_1, \dots, M_N$ are analytic subsets of Ω, then $M_1 \cap \dots \cap M_N$ is an analytic subset of Ω.

(f) If $M_1, \dots, M_N$ are analytic subsets of Ω, then $M_1 \cup \dots \cup M_N$ is an analytic subset of Ω.

(g) If $F : \Omega_1 \longrightarrow \Omega_2$ is holomorphic (Ω_j is an open set in $\mathbb{C}^{n_j}$, $j = 1, 2$) and M is an analytic subset of Ω_2, then $F^{-1}(M)$ is an analytic subset of Ω_1.

In particular, if $F : \Omega_1 \longrightarrow \Omega_2$ is biholomorphic ($n_1 = n_2$), then M is an analytic subset of Ω_2 iff $F^{-1}(M)$ is an analytic subset of Ω_1.

(h) If M_j is an analytic subset of $\Omega_j \subset \mathbb{C}^{n_j}$, $j = 1, \dots, N$, then $M_1 \times \dots \times M_N$ is an analytic subset of $\Omega_1 \times \dots \times \Omega_N$.

(i) Let $M := f^{-1}(0)$, $f \in \mathcal{O}(\Omega)$. Then the set $\operatorname{Reg}(M)$ is dense in M.

2.8. Verify whether the sets

$$\{(z_1, z_2) \in \mathbb{C}^2 : z_1^2 = z_2^3\}, \quad \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1 z_2 = z_3^2\}$$

are reducible. Determine their singular points.

2.9. Let $g : \mathbb{C} \to \mathbb{C}^2$, $g(z) := (z^2 - z, z^3 - z)$. Is $g(\mathbb{C}) \cap U$ an analytic subset of U for some neighborhood U of $(0, 0) \in \mathbb{C}^2$?

2.10. Let U be a neighborhood of a point $a \in \partial\mathbb{B}_n$ ($n \geq 2$) and let $D := U \setminus \overline{\mathbb{B}}_n$. Then there exists a neighborhood W of the point a such that every function holomorphic in D extends holomorphically to W.

2.11. Let D and G be convex domains in $\mathbb{C}$ containing 0 and let

$$K := [0, 1] \cdot (\partial D \times \partial G).$$

Prove that any function f holomorphic in a neighborhood of K, extends holomorphically to a neighborhood of $\overline{D} \times \overline{G}$.

2.12. Let M be a $\mathcal{C}^1$ submanifold of a domain $D \subset \mathbb{C}^n$, $\dim_{\mathbb{R}} M \leq 2n-1$. Show that $\mathcal{O}(D \setminus M) \cap \mathcal{C}(D) = \mathcal{O}(D)$, i.e. every function holomorphic in $D \setminus M$ and continuous in D is holomorphic in D.

2.13. Let $f \in \mathcal{O}(\mathbb{B}_n) \cap \mathcal{C}(\overline{\mathbb{B}}_n)$, $n \geq 2$. Suppose that $|f| = 1$ in $\partial\mathbb{B}_n$. Does it follow that $f = \text{const}$?

2.14. Check whether Proposition 2.7.4(a) remains true if the barrier function f_a exists only for all points a from a dense subset of $\partial\Omega$.

2.15. Let a domain D be bounded and convex. Construct a holomorphic function which does not extend beyond D (cf. Proposition 2.7.4(b)).

2.16. Let $\Omega \subset \mathbb{C}^n$ be open and let $\varphi : \Omega \longrightarrow \mathbb{R}$ be continuous. Put

$$L^2_h(\Omega, \varphi) := \{f \in \mathcal{O}(\Omega) : \int_\Omega |f|^2 e^{-\varphi} d\mathcal{L}^{2n} < +\infty\}.$$

(a) Prove that $L^2_h(\Omega, \varphi)$ with the scalar product

$$L^2_h(\Omega, \varphi) \times L^2_h(\Omega, \varphi) \ni (f, g) \longrightarrow \langle f, g\rangle_\varphi := \int_\Omega f\overline{g} e^{-\varphi} d\mathcal{L}^{2n} \in \mathbb{C}$$

is a complex Hilbert space; let $\| \ \|_\varphi$ denote the norm induced by the scalar product.

Moreover, $L^2_h(\Omega, \varphi)$ is a natural Hilbert space in $\mathcal{O}(\Omega)$.

(b) Let $\mathcal{F} \subset \mathcal{O}(\Omega)$ be locally uniformly bounded. Then there exists a continuous (even $\mathcal{C}^\infty$) function $\varphi : \Omega \longrightarrow \mathbb{R}$ such that

$$\mathcal{F} \subset \{f \in L^2_h(\Omega, \varphi) : \int_\Omega |f|^2 e^{-\varphi} d\mathcal{L}^{2n} \le 1\}.$$

(c) If $\Omega = \Omega_1 \times \Omega_2$, then the function φ in (b) can be found in the form $\varphi(z_1, z_2) = (\varphi_1 \oplus \varphi_2)(z_1, z_2) := \varphi_1(z_1) + \varphi_2(z_2)$, $(z_1, z_2) \in \Omega_1 \times \Omega_2$.

2.17. Let $b \in L^2_h(\Omega_1, \varphi_1)$, $f \in L^2_h(\Omega_1 \times \Omega_2, \varphi_1 \oplus \varphi_2)$,

$$g(z_2) := \int_{\Omega_1} f(\cdot, z_2)\overline{b} e^{-\varphi_1} d\mathcal{L}^{2n_1}, \quad z_2 \in \Omega_2.$$

Prove that $g \in L^2_h(\Omega_2, \varphi_2)$.

2.18. Let $(b_{j,k})_{k \in A_j} \subset L^2_h(\Omega, \varphi_j)$ be an othonormal basis ($\#A_j \le \aleph_0$), $j = 1, 2$. Prove that $(b_{1,k} \otimes b_{2,\ell})_{(k,\ell) \in A_1 \times A_2}$ [27] is an orthonormal basis in $L^2_h(\Omega_1 \times \Omega_2, \varphi_1 \oplus \varphi_2)$.

2.19. For $\mathcal{F}_j \subset \mathcal{O}(\Omega_j)$, $j = 1, 2$, let $\mathcal{F}_1 \otimes \mathcal{F}_2$ denote the vector subspace of $\mathcal{O}(\Omega_1 \times \Omega_2)$ spanned by the set

$$\{f_1 \otimes f_2 : (f_1, f_2) \in \mathcal{F}_1 \times \mathcal{F}_2\}.$$

Using Exercise 2.18 prove that the space $L^2_h(\Omega_1, \varphi_1) \otimes L^2_h(\Omega_2, \varphi_2)$ is dense in $L^2_h(\Omega_1 \times \Omega_2, \varphi_1 \oplus \varphi_2)$.

2.20. Using Exercises 2.16 and 2.18 prove that the space $\mathcal{O}(\Omega_1) \otimes \mathcal{O}(\Omega_2)$ is dense in $\mathcal{O}(\Omega_1 \times \Omega_2)$ (in the topology of locally uniform convergence).

2.21. Prove that the set $\{(z, w) \in \mathbb{C}^2 : |z|^2 + (\operatorname{Im} w)^2 > 1\}$ is not holomorphically convex.

2.22. Prove that every compact set $K \subset \mathbb{R}^n \subset \mathbb{C}^n$ is polynomially convex.

2.23. Let $G := \{z \in \mathbb{C}^n : 1 < |z| < 3\}$, $K := \{z \in \mathbb{C}^n : |z| = 2\}$. Determine the set $\widehat{K}_{\mathcal{O}(G)}$.

2.24. Let $\Omega = \mathbb{D}^2 \setminus \overline{\mathbb{P}}(1/2)$. Verify that for

$$K := \{(0, 3e^{it}/4) : 0 \le t < 2\pi\}$$

we have

$$\widehat{K}_{\mathcal{O}(\Omega)} = \{(0, re^{it}) : 0 \le t < 2\pi,\ 1/2 < r \le 3/4\}$$

(and, therefore, the set Ω is not holomorphically convex).

2.25. Let $G := \{(z, w) \in \mathbb{C}^2 : 0 < |z| < |w| < 1\}$. Is the set G holomorphically convex? Is it true that $\overline{G}$ has a neighborhood basis consisting of polynomially convex sets?

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[27] $(f \otimes g)(x, y) := f(x)g(y)$.

CHAPTER 3

Plurisubharmonic functions

Summary. In this chapter we consider properties of subharmonic and plurisubharmonic functions (cf. [18], [35]).

In Section 3.1 we collect the basic properties of harmonic functions. The results are standard, perhaps except Proposition 3.1.13 on the Dirichlet problem for the annulus.

Section 3.2 summarizes basic properties of subharmonic functions, e.g. the mean value property and the maximum principle. A more advanced result contained there is the removable singularities theorem for subharmonic functions. Various properties of subharmonic functions are obtained by their regularization. Another advanced result is the Oka theorem (Propositions 3.2.31, 3.2.32). The final part of Section 3.2 is devoted to results concerning logarithmic subharmonicity. The notions of harmonicity and subharmonicity (as well as main parts of Sections 3.1 and 3.2) can be generalized from open sets in $\mathbb{C} = \mathbb{R}^2$ to open sets in $\mathbb{R}^N$ (cf. [14], [18]).

In a short Section 3.3 we present basic properties of pluriharmonic functions.

In the next Section 3.4 plurisubharmonic functions are presented. Many of the basic properties of those functions, like the mean value property, the removable singularities theorem, the maximum principle, follow from their counterparts for subharmonic functions. § 3.4 ends with the introduction of strictly plurisubharmonic functions; the detailed discussion of this important class will not be pursued in this chapter.

3.1. Harmonic functions

Let $\Omega \subset \mathbb{R}^2 \simeq \mathbb{C}$ be open and let $h \in \mathcal{C}^2(\Omega, \mathbb{R})$. The function h is called *harmonic* in Ω ($h \in \mathcal{H}(\Omega)$) if

$$\Delta h = \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0 \text{ on } \Omega.$$

Remark 3.1.1. (a) $\mathcal{H}(\Omega)$ is a vector space.

(b) If $h : \Omega \longrightarrow \mathbb{R}$ is such that every point $a \in \Omega$ has a neighborhood $U_a \subset \Omega$ such that $h|_{U_a} \in \mathcal{H}(U_a)$, then $h \in \mathcal{H}(\Omega)$.

(c) $\Delta = 4\frac{\partial^2}{\partial z \partial \bar{z}}$.

(d) If $f = u + iv \in \mathcal{O}(\Omega)$, then $u, v \in \mathcal{H}(\Omega)$.

Indeed, by (c) we have $\Delta u + i\Delta v = \Delta f = 4\frac{\partial}{\partial z}(\frac{\partial f}{\partial \bar{z}}) \equiv 0$.

(e) If $f \in \mathcal{O}(\Omega)$ and $0 \notin f(\Omega)$, then $\log|f| \in \mathcal{H}(\Omega)$.

Indeed, $\log|f| = \operatorname{Re}\ell$, where ℓ is a local branch of the logarithm. Now we apply (d).

(f) Let Ω and Ω' be open sets in $\mathbb{C}$, $h \in \mathcal{H}(\Omega')$, $f \in \mathcal{O}(\Omega, \Omega')$. Then $h \circ f \in \mathcal{H}(\Omega)$.

Indeed,
$$\Delta(h \circ f) = ((\Delta h) \circ f)|f'|^2 = 0.$$

Proposition 3.1.2. *Let $D \subset \mathbb{C}$ be a simply connected domain and let $h : D \longrightarrow \mathbb{R}$. Then $h \in \mathcal{H}(D)$ iff there exists $f \in \mathcal{O}(D)$ such that $h = \operatorname{Re} f$.*

In particular, each harmonic function is locally the real part of a holomorphic function.

Proof. The implication $\Longleftarrow$ follows from Remark 3.1.1(d).

To prove $\Longrightarrow$ we assume first that D is star-shaped with respect to a point $a_0 \in D$, i.e. $[a_0, a] \subset D$ for every $a \in D$ (1) .

(1) $[a_0, a] := \{a_0 + t(a - a_0) : t \in [0, 1]\}$.

We may assume that $a_0 = 0$. Define

$$k(z) = \int_0^1 \Big(-\frac{\partial h}{\partial y}(tz)x + \frac{\partial h}{\partial x}(tz)y \Big)dt, \quad z = x + iy \in D.$$

Then $k \in \mathcal{C}^1(D, \mathbb{R})$. Moreover, differentiating under the integral sign, and using the fact that $\Delta u = 0$, we obtain

$$\begin{aligned}\frac{\partial k}{\partial x}(z) &= \int_0^1 \Big(-\frac{\partial^2 h}{\partial x \partial y}(tz)tx - \frac{\partial h}{\partial y}(tz) + \frac{\partial^2 h}{\partial x^2}(tz)ty \Big)dt \\ &= -\int_0^1 \Big(\frac{\partial^2 h}{\partial x \partial y}(tz)x + \frac{\partial^2 h}{\partial y^2}(tz)y \Big)t\,dt - \int_0^1 \frac{\partial h}{\partial y}(tz)\,dt = -\int_0^1 t\frac{d}{dt}\Big(\frac{\partial h}{\partial y}(tz)\Big)dt - \int_0^1 \frac{\partial h}{\partial y}(tz)\,dt \\ &= -t\frac{\partial h}{\partial y}(tz)\Big|_0^1 + \int_0^1 \frac{\partial h}{\partial y}(tz)\,dt - \int_0^1 \frac{\partial h}{\partial y}(tz)\,dt = -\frac{\partial h}{\partial y}(z).\end{aligned}$$

Similarly we check that $\frac{\partial k}{\partial y} = \frac{\partial h}{\partial x}$. This means that h and k satisfy the Cauchy–Riemann equations in D. Consequently, $f = h + ik \in \mathcal{O}(D)$.

Now let $D \subsetneq \mathbb{C}$ be an arbitrary simply connected domain. By the Riemann mapping theorem (cf. [4], Th. VII.4.2) there exists a biholomorphic mapping $\varphi : \mathbb{D} \longrightarrow D$. By Remark 3.1.1(f), $h \circ \varphi \in \mathcal{H}(\mathbb{D})$. Since $\mathbb{D}$ is star-shaped, there exists $g \in \mathcal{O}(\mathbb{D})$ such that $\operatorname{Re} g = h \circ \varphi$. Finally, $h = \operatorname{Re}(g \circ \varphi^{-1})$. $\square$

Corollary 3.1.3. $\mathcal{H}(\Omega) \subset \mathcal{C}^\infty(\Omega)$.

Proposition 3.1.4 (Identity principle)**.** *Let $D \subset \mathbb{C}$ be a domain and let $h \in \mathcal{H}(D)$. If $\operatorname{int}(h^{-1}(0)) \neq \varnothing$, then $h \equiv 0$ in D. In particular, if $h_1, h_2 \in \mathcal{H}(D)$ are equal on a non-empty open subset, then $h_1 \equiv h_2$ in D.* (2)

Proof. Let $D_0 = \{a \in D : h = 0 \text{ in a neighborhood of } a\}$. Obviously, D_0 is open and $D_0 \neq \varnothing$. To end the proof we need to show that D_0 is relatively closed in D. Suppose that $z_0 \in D$ is an accumulation point of D_0 in D. Take any $r > 0$ such that $K(z_0, r) \subset D$. By Proposition 3.1.2 there exists $f \in \mathcal{O}(K(z_0, r))$ such that $\operatorname{Re} f = h$. Then we have $\operatorname{Re} f = h = 0$ on the non-empty and open set $D_0 \cap K(z_0, r)$. By the usual identity principle for holomorphic functions ([4], Th. IV.3.7), $f \equiv \text{const}$ in $K(z_0, r)$. Since $h = 0$ in $D_0 \cap K(z_0, r)$, we have $h = 0$ in $K(z_0, r)$. Hence $z_0 \in D_0$. $\square$

Proposition 3.1.5 (Maximum principle)**.** *Let $D \subset \mathbb{C}$ be a domain and let $h \in \mathcal{H}(D)$, $h \not\equiv \text{const}$. Then h has no local maxima in D. If, moreover, D is bounded, then*

$$h(z) < \sup_{\zeta \in \partial D} \{\limsup_{D \ni w \to \zeta} h(w)\}, \quad z \in D.$$

If we replace h by $-h$, then we obtain the *minimum principle*.

Proof. Suppose that there exist $z_0 \in D$ and $r > 0$ such that $h(z) \leq h(z_0)$ for every $z \in K(z_0, r)$. By Proposition 3.1.2, there exists $f \in \mathcal{O}(K(z_0, r))$ such that $h = \operatorname{Re} f$. Then, for $z \in K(z_0, r)$, we get

$$|e^{f(z)}| = e^{h(z)} \leq e^{h(z_0)} = |e^{f(z_0)}|.$$

Hence, by the maximum principle for holomorphic functions (cf. [4], Th. IV.3.11), $e^f = \text{const}$ in $K(z_0, r)$. Consequently, $h = h(z_0)$ in $K(z_0, r)$ and, finally, by the identity principle (Proposition 3.1.4), we get $h \equiv h(z_0)$ in D; contradiction. $\square$

---

(2) Note that the function $h(x + iy) = x$ is harmonic in $\mathbb{C}$ and $h = 0$ on the line $i\mathbb{R}$. Therefore the assumption on the zero set of harmonic function cannot be so weak as in the identity principle for holomorphic functions.

Let $u : C(a,r) \longrightarrow [-\infty, +\infty)$ be measurable $\left({}^3\right)$ and bounded from above. Define

$$\mathbf{P}(u; a, r; z) := \frac{1}{2\pi} \int_0^{2\pi} \frac{r^2 - |z-a|^2}{|re^{i\theta} - (z-a)|^2} u(a + re^{i\theta})\, d\theta, \quad z \in K(a,r),$$

$$\mathbf{J}(u; a, r) := \mathbf{P}(u; a, r; a) = \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\theta})\, d\theta;$$

$\mathbf{J}(u; a, r)$ is the integral mean value of u over the circle $C(a,r)$.

Proposition 3.1.6. *Let $u : C(a,r) \longrightarrow \mathbb{R}$ be continuous. Then the function $K(a,r) \ni z \longmapsto \mathbf{P}(u; a, r; z)$ is harmonic.*

Proof. Consider the function

$$\mathbf{S}(u; a, r; z) := \frac{1}{2\pi} \int_0^{2\pi} \frac{re^{i\theta} + (z-a)}{re^{i\theta} - (z-a)} u(a + re^{i\theta}) d\theta, \quad z \in K(a,r).$$

Since u is real-valued, we have

$$\begin{aligned}\operatorname{Re} \mathbf{S}(u; a, r; z) &= \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re}\left(\frac{re^{i\theta} + (z-a)}{re^{i\theta} - (z-a)}\right) u(a + re^{i\theta})\, d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{r^2 - |z-a|^2}{|re^{i\theta} - (z-a)|^2} u(a + re^{i\theta})\, d\theta = \mathbf{P}(u; a, r; z).\end{aligned}$$

Moreover, the function $\mathbf{S}(u; a, r; \cdot)$ is holomorphic in $K(a,r)$. Now, the result follows from Remark 3.1.1(d). ☐

Lemma 3.1.7.

$$\mathbf{P}(1; a, r; z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{r^2 - |z-a|^2}{|re^{i\theta} - (z-a)|^2} d\theta = 1, \quad z \in K(a,r).$$

Proof. Since $\mathbf{P}(1; a, r; z) = \operatorname{Re} \mathbf{S}(1; a, r; z)$, it suffices to show that $\mathbf{S}(1; a, r; z) = 1$. We have

$$\frac{re^{i\theta} + (z-a)}{re^{i\theta} - (z-a)} = (re^{i\theta} + (z-a)) \sum_{n=0}^{\infty} \frac{(z-a)^n}{r^{n+1} e^{i(n+1)\theta}},$$

and for every $z \in K(a,r)$, the series converges uniformly for $\theta \in [0, 2\pi]$. Then

$$\begin{aligned}\mathbf{S}(1; a, r; z) &= \frac{1}{2\pi} \int_0^{2\pi} \frac{re^{i\theta} + (z-a)}{re^{i\theta} - (z-a)} d\theta \\ &= \frac{1}{2\pi} \sum_{n=0}^{\infty} \frac{(z-a)^n}{r^n} \int_0^{2\pi} e^{-in\theta} d\theta + \frac{1}{2\pi} \sum_{n=0}^{\infty} \frac{(z-a)^{n+1}}{r^{n+1}} \int_0^{2\pi} e^{-i(n+1)\theta} d\theta.\end{aligned}$$

The only non-vanishing term is the term for $n = 0$ in the first sum, which is equal to

$$\frac{1}{2\pi} \int_0^{2\pi} 1\, d\theta = 1.$$

☐

Definition 3.1.8. Let D be a bounded domain in $\mathbb{C}$ and let $b \in \mathcal{C}(\partial D, \mathbb{R})$. The *Dirichlet problem* for D and b is to find a function $h \in \mathcal{C}(\overline{D}) \cap \mathcal{H}(D)$ such that $h = b$ on ∂D.

By the maximum principle h is uniquely determined. It is called the *solution of the Dirichlet problem for D with boundary data b*. If the Dirichlet problem for D has a solution for any boundary data b, then we say that D is *regular with respect to the Dirichlet problem*.

In the sequel we need a solution of the Dirichlet problem in the case when D is a disc or an annulus in $\mathbb{C}$. In the first case the Dirichlet problem can be solved fairly explicitly.

$\left({}^3\right)$ That is, the function $[0, 2\pi) \ni \theta \longrightarrow u(a + re^{i\theta})$ is Lebesgue measurable; $C(a,r) := \partial K(a,r)$.

Proposition 3.1.9. *Let $b : C(a,r) \longrightarrow \mathbb{R}$ be continuous. Put*

$$h(z) := \begin{cases} b(z), & z \in C(a,r) \\ \mathbf{P}(b;a,r;z), & z \in K(a,r) \end{cases}.$$

Then h is the solution of the Dirichlet problem for $K(a,r)$ with boundary data b.

Thus all discs are regular with respect to the Dirichlet problem.

Proof. By Proposition 3.1.6 h is harmonic in $K(a,r)$. It remains to show that

$$\lim_{K(a,r)\ni z\to z_0} \mathbf{P}(b;a,r;z) = b(z_0), \quad z_0 \in C(a,r).$$

Fix $z_0 \in C(a,r)$ and observe that for $z \in K(a,r)$ we have by Lemma 3.1.7

$$|\mathbf{P}(b;a,r;z) - b(z_0)| = \left| \frac{1}{2\pi} \int_0^{2\pi} \frac{r^2 - |z-a|^2}{|re^{i\theta} - (z-a)|^2} (b(a+re^{i\theta}) - b(z_0))\, d\theta \right|. \tag{3.1.1}$$

We have $z_0 = a + re^{i\theta_0}$ for some $\theta_0 \in \mathbb{R}$. Fix $\varepsilon > 0$. There exists $\delta > 0$ such that for every θ with $|\theta - \theta_0| < \delta$, $|b(a+re^{i\theta}) - b(a+re^{i\theta_0})| < \varepsilon/2$. Let $\varGamma_1 := \{a + re^{i\theta} : |\theta - \theta_0| < \delta\}$, $\varGamma_2 := C(a,r) \setminus \varGamma_1$. Then the right-hand side of (3.1.1) is bounded by

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{r^2 - |z-a|^2}{|re^{i\theta} - (z-a)|^2} |b(a+re^{i\theta}) - b(a+re^{i\theta_0})| d\theta = \frac{1}{2\pi} \left(\int_{|\theta-\theta_0|\le\delta} \cdots + \int_{|\theta-\theta_0|\ge\delta} \cdots \right). \tag{3.1.2}$$

Moreover,

$$\frac{1}{2\pi} \int_{|\theta-\theta_0|\le\delta} \cdots \le \frac{1}{2\pi} \int_{|\theta-\theta_0|\le\delta} \frac{r^2 - |z-a|^2}{|re^{i\theta} - (z-a)|^2} \frac{\varepsilon}{2} d\theta \le \frac{\varepsilon}{2} \frac{1}{2\pi} \int_0^{2\pi} \frac{r^2 - |z-a|^2}{|re^{i\theta} - (z-a)|^2}\, d\theta = \frac{\varepsilon}{2}$$

by Lemma 3.1.7 (note that the estimate is independent of $z \in K(a,r)$).

To estimate the second integral in the right-hand side of (3.1.2) consider only $z = a + \varrho e^{i\tau} \in K(a,r)$ with $|\tau - \theta_0| < \delta/2$, and $\varrho_0 < \varrho < r$, where ϱ_0 is to be chosen. Put

$$m := \inf\{|re^{i\theta} - (z-a)| : z = a + \varrho e^{i\tau},\ |\tau - \theta_0| < \delta/2,\ 0 < \varrho < r,\ a + re^{i\theta} \in \varGamma_2\}.$$

Then $m > 0$ and

$$\frac{1}{2\pi} \int_{|\theta-\theta_0|\ge\delta} \cdots \le \frac{1}{2\pi} \int_0^{2\pi} \frac{r^2 - \varrho_0^2}{m^2} 2\|b\|_{C(a,r)} = \frac{2\|b\|_{C(a,r)}}{m^2} (r^2 - \varrho_0^2).$$

The last expression is smaller than $\varepsilon/2$ provided ϱ_0 is chosen sufficiently close to r. Thus, if $z \in K(a,r)$ is sufficiently close to z_0, then

$$|\mathbf{P}(b;a,r;z) - b(z_0)| < \varepsilon. \qquad \square$$

From the above proposition we obtain the following reproducing integral formula.

Proposition 3.1.10 (Poisson's formula)**.** *If $h \in \mathcal{C}(\overline{K}(a,r)) \cap \mathcal{H}(K(a,r))$, then*

$$h(z) = \mathbf{P}(h;a,r;z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{r^2 - |z-a|^2}{|re^{i\theta} - (z-a)|^2} h(a+re^{i\theta})\, d\theta, \quad z \in K(a,r).$$

In particular,

$$h(a) = \mathbf{J}(h;a,r) = \frac{1}{2\pi} \int_0^{2\pi} h(a+re^{i\theta})\, d\theta.$$

Proof. By Proposition 3.1.9, if

$$H(z) := \begin{cases} h(z), & z \in C(a,r) \\ \mathbf{P}(h;a,r;z), & z \in K(a,r) \end{cases},$$

then $H \in \mathcal{C}(\overline{K}(a,r)) \cap \mathcal{H}(K(a,r))$. Consequently, by the maximum principle for harmonic functions, $h \equiv H$ in $K(a,r)$. $\square$

Proposition 3.1.11 (1-st Harnack's theorem). *Let $\Omega \subset \mathbb{C}$ be open and let $(h_\nu)_{\nu=1}^\infty \subset \mathcal{H}(\Omega)$. If $h_\nu \longrightarrow h$ locally uniformly in Ω, then $h \in \mathcal{H}(\Omega)$.*

Proof. Fix $a \in \Omega$ and $r > 0$ such that $\overline{K}(a,r) \subset \Omega$. Then, by Proposition 3.1.10, we get

$$h_\nu(z) = \mathbf{P}(h_\nu; a, r; z), \quad z \in K(a,r),\ \nu \in \mathbb{N}.$$

Since $h_\nu \longrightarrow h$ uniformly on $C(a,r)$, we get $\mathbf{P}(h_\nu; a, r; z) \longrightarrow \mathbf{P}(h; a, r; z)$. On the other hand $h_\nu(z) \longrightarrow h(z)$. Thus

$$h(z) = \mathbf{P}(h; a, r; z), \quad z \in K(a,r).$$

Now, by Proposition 3.1.6, $h \in \mathcal{H}(K(a,r))$. □

Proposition 3.1.12 (2-nd Harnack's theorem). *Let D be a domain in $\mathbb{C}$, $(h_\nu)_{\nu=1}^\infty \subset \mathcal{H}(D)$, and $h_\nu \le h_{\nu+1}$, $\nu \ge 1$. If there exists $a \in D$ such that $\lim_{\nu\to+\infty} h_\nu(a)$ exists and is finite, then $(h_\nu)_{\nu=1}^\infty$ converges locally uniformly in D.*

Proof. Let

$$D_0 = \{z \in D : (h_\nu)_{\nu=1}^\infty \text{ is convergent uniformly in a neighborhood of } z\}.$$

If we show that D_0 is non-empty open and closed in D, then $D_0 = D$, which will end the proof.

The set D_0 is open by definition. To prove that $D_0 \neq \varnothing$ we show that $a \in D_0$. Choose $r > 0$ such that $\overline{K}(a,r) \subset D$. Note that

$$\frac{r^2 - |z-a|^2}{|re^{i\theta} - (z-a)|^2} \le \frac{r^2 - |z-a|^2}{(r - |z-a|)^2} = \frac{r + |z-a|}{r - |z-a|}, \quad z \in K(a,r). \tag{3.1.3}$$

Moreover, for $z \in K(a,r)$ and $\nu, \mu \in \mathbb{N}$, by Proposition 3.1.10 and (3.1.3), we have

$$\begin{aligned} 0 \le h_{\nu+\mu}(z) - h_\nu(z) &= \frac{1}{2\pi}\int_0^{2\pi} \frac{r^2 - |z-a|^2}{|re^{i\theta} - (z-a)|^2}(h_{\nu+\mu}(a + re^{i\theta}) - h_\nu(a + re^{i\theta}))\,d\theta \\ &\le \frac{1}{2\pi}\int_0^{2\pi} \frac{r + |z-a|}{r - |z-a|}(h_{\nu+\mu}(a + re^{i\theta}) - h_\nu(a + re^{i\theta}))\,d\theta = \frac{r + |z-a|}{r - |z-a|}(h_{\nu+\mu}(a) - h_\nu(a)). \end{aligned}$$

For $|z-a| < r/2$ this last expression is not greater than $3(h_{\nu+\mu}(a) - h_\nu(a))$. Therefore the sequence $(h_\nu)_{\nu=1}^\infty$ satisfies the uniform Cauchy condition in $K(a, r/2)$, and hence converges uniformly there. Thus $a \in D_0$.

Suppose now that $z_0 \in D$ is an accumulation point of the set D_0. Choose $r > 0$ such that $\overline{K}(z_0, r) \subset D$. There exists $b \in D_0 \cap K(z_0, r/3)$. Hence $\overline{K}(b, 2r/3) \subset D$. Since $b \in D_0$, the sequence $(h_\nu(b))_{\nu=1}^\infty$ is convergent. Similarly as above we prove that $(h_\nu)_{\nu=1}^\infty$ is convergent uniformly in $K(b, r/3)$. Hence $(h_\nu)_{\nu=1}^\infty$ is convergent uniformly in a neighborhood of z_0, and so $z_0 \in D_0$, which proves that D_0 is relatively closed. □

Proposition 3.1.13. *Any annulus*

$$A := \{z \in \mathbb{C} : r^- < |z| < r^+\}, \quad 0 < r^- < r^+ < +\infty,$$

is regular with respect to the Dirichlet problem.

Proof. First observe that the mapping

$$\mathbb{C} \ni z \longmapsto z/\sqrt{r^- r^+} \in \mathbb{C}$$

maps biholomorphically A onto the "symmetric" annulus

$$\{z \in \mathbb{C} : 1/R < |z| < R\}$$

with $R := \sqrt{r^+/r^-}$. Consequently, using Remark 3.1.1(f), we may assume that $r^- = 1/R$ and $r^+ = R$ for some $R > 1$.

By virtue of [14], it suffices to find the *Green function* for A, i.e. a function $g_A : A \times A \longrightarrow (0, +\infty]$ such that:

- $g_A(\cdot, a) \in \mathcal{H}(A \setminus \{a\})$,
- $\lim_{A \ni z \to \zeta} g_A(z, a) = 0$, $\zeta \in \partial A$,

• $\lim_{A\setminus\{a\}\ni z\to a}[g_A(z,a)+\log|z-a|]$ exists and is finite, $a\in A$.

By Remark 3.1.1(f) it suffices to construct $g_A(\cdot,a)$ only for $a\in A\cap\mathbb{R}_+$.

Fix $1/R<a<R$, put $q:=1/R^2$, and define

$$f(a,z):=(1-\frac{z}{a})\Pi(a,z),$$

where

$$\Pi(a,z):=\frac{\prod_{\nu=1}^{\infty}(1-\frac{z}{a}q^{2\nu})(1-\frac{a}{z}q^{2\nu})}{\prod_{\nu=1}^{\infty}(1-azq^{2\nu-1})(1-\frac{1}{az}q^{2\nu-1})}.$$

One can prove (cf. [5]) that:

• $f(a,\cdot)$ is meromorphic on $\mathbb{C}_*$,

• $f(a,\cdot)$ has simple poles at $z=R^{4k-2}/a$, $k\in\mathbb{Z}$,

• $f(a,\cdot)$ has simple zeros at $z=aR^{4k}$, $k\in\mathbb{Z}$.

In particular, $f(a,\cdot)$ is holomorphic on $\overline{A}$ and the only zero of $f(a,\cdot)$ in $\overline{A}$ is the simple zero at $z=a$. Moreover,

$$f(a,z)f(a,1/(R^2z))=1,\quad f(a,z)f(a,R^2/z)=R^2/a^2,\quad f(a,\overline{z})=\overline{f(a,z)}$$

and hence

$$|f(a,z)|=\begin{cases}1 & \text{if } |z|=1/R\\ R/a & \text{if } |z|=R\end{cases}. \tag{3.1.4}$$

Put

$$s(a):=\frac{1}{2}\Big(1-\frac{\log a}{\log R}\Big).$$

Then

$$g_A(z,a)=-\log|f(a,z)|+s(a)\log(R|z|),\quad z\in\overline{A}.$$

Indeed,

• $g_A(\cdot,a)\in\mathcal{H}(\overline{A}\setminus\{a\})$,

• $g_A(z,a)=0$ if $z\in\partial A$ (by (3.1.4)),

• $\lim_{A\setminus\{a\}\ni z\to a}[g_A(z,a)+\log|z-a|]=\log(a/\Pi(a,a))+s(a)\log(Ra)$. $\square$

Proposition 3.1.14 ([31]). *Let $u\in L^1(\Omega,\mathrm{loc})$ (4) be such that $\Delta u=0$ in the sense of distribution, i.e.*

$$\int_\Omega u\cdot(\Delta\varphi)\,d\mathcal{L}^2=0,\quad \varphi\in\mathcal{C}_0^\infty(\Omega).$$

Then there exists $h\in\mathcal{H}(\Omega)$ such that $u=h$ $\mathcal{L}^2$-a.e. on Ω.

3.2. Subharmonic functions

Definition 3.2.1. Let $\Omega\subset\mathbb{C}$ be open. A function $u:\Omega\longrightarrow[-\infty,+\infty)$ is called *subharmonic* in Ω (we write $u\in\mathcal{SH}(\Omega)$) if:

• u is upper semicontinuous in Ω ($u\in\mathcal{C}^\uparrow(\Omega)$),

• for every domain $D\subset\subset\Omega$ and for every function $h\in\mathcal{C}(\overline{D})\cap\mathcal{H}(D)$, if $u\le h$ on ∂D, then $u\le h$ in D.

In particular, the function $u\equiv-\infty$ is subharmonic.

The following properties are immediate consequences of the above definition and of the maximum principle for harmonic functions:

$\mathcal{H}(\Omega)\subset\mathcal{SH}(\Omega)$,

$\mathcal{SH}(\Omega)+\mathcal{H}(\Omega)=\mathcal{SH}(\Omega)$,

$\mathbb{R}_{>0}\cdot\mathcal{SH}(\Omega)=\mathcal{SH}(\Omega)$.

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(4) $L^1(\Omega,\mathrm{loc}):=\{u:\forall_{K\subset\subset\Omega}: u|_K\in L^1(K,\mathcal{L}^2)\}$.

Proposition 3.2.2 (Mean value property)**.** *If $u \in \mathcal{SH}(\Omega)$, then*

$$u(a) \le \mathbf{J}(u; a, r) = \frac{1}{2\pi}\int_0^{2\pi} u(a + re^{i\theta})\, d\theta, \quad a \in \Omega,\ 0 < r < d_\Omega(a).$$

Proof. Fix an $a \in \Omega$ and $0 < r < d_\Omega(a)$. Let $b_\nu : C(a, r) \longrightarrow \mathbb{R}$, $\nu \in \mathbb{N}$, be a sequence of continuous functions such that $b_\nu \searrow u$ pointwise on $C(a, r)$ (cf. [21]). Let h_ν be the solution of the Dirichlet problem for $K(a, r)$ with $h_\nu = b_\nu$ on $C(a, r)$ (cf. Proposition 3.1.9). Then $u \le h_\nu$ on $C(a, r)$ and hence on $K(a, r)$. Consequently, by Proposition 3.1.10, we get

$$u(a) \le h_\nu(a) = \mathbf{J}(h_\nu; a, r) = \mathbf{J}(b_\nu; a, r), \quad \nu \ge 1.$$

Since $b_\nu \searrow u$ on $C(a, r)$, the monotone convergence theorem implies that

$$\mathbf{J}(b_\nu; a, r) \longrightarrow \mathbf{J}(u; a, r). \qquad \square$$

Lemma 3.2.3. *Let $D \subset \mathbb{C}$ be a domain and let $v \in \mathcal{C}^{\uparrow}(D, [-\infty, +\infty))$, $v \not\equiv \mathrm{const}$. Assume that for every $a \in D$ there exists a number $0 < R(a) \le d_D(a)$ such that*

$$v(a) \le \mathbf{J}(v; a, r), \quad 0 < r < R(a).$$

Then v does not attain its global maximum in D.

Proof. Suppose that $v(z) \le v(z_0)$, $z \in D$ (for some $z_0 \in D$). Let $D_0 := v^{-1}(v(z_0))$. Then $D_0 \ne \varnothing$. Note that for every accumulation point $a \in D$ of D_0 we have

$$v(z_0) = \limsup_{D_0 \ni z \to a} v(z) \le \limsup_{D \ni z \to a} v(z) = v(a) \le v(z_0).$$

Hence $a \in D_0$, which means that D_0 is relatively closed in D. On the other hand, if $a \in D_0$, then

$$v(z_0) = v(a) \le \mathbf{J}(v; a, r) \le v(z_0), \quad 0 < r < R(a).$$

Now, since v is upper semicontinuous, we conclude that $v = v(z_0)$ on $C(a, r)$ with $0 < r < R(a)$. This implies that $K(a, R(a)) \subset D_0$, and therefore D_0 is open. Since D is connected, we have $D_0 = D$, which shows that $v \equiv v(z_0)$; contradiction. $\square$

From Proposition 3.2.2 and Lemma 3.2.3 we immediately obtain

Corollary 3.2.4 (Maximum principle)**.** *Let $D \subset \mathbb{C}$ be a domain and let $u \in \mathcal{SH}(D)$, $u \not\equiv \mathrm{const}$. Then u does not attain its global maximum in D. Moreover, if D is bounded, then*

$$u(z) < \sup_{\zeta \in \partial D} \{\limsup_{D \ni w \to \zeta} u(w)\}, \quad z \in D.$$

Notice that a subharmonic function can attain its global minimum.

Proposition 3.2.5. *Let $u : \Omega \longrightarrow [-\infty, +\infty)$. Then $u \in \mathcal{SH}(\Omega)$ iff $u \in \mathcal{C}^{\uparrow}(\Omega)$ and for every $a \in \Omega$ there exists an $R(a)$, $0 < R(a) \le d_\Omega(a)$, such that*

$$u(a) \le \mathbf{J}(u; a, r), \quad 0 < r < R(a). \tag{3.2.1}$$

Proof. The implication $\Longrightarrow$ follows from Proposition 3.2.2.

To prove the opposite, fix a domain $D \subset\subset \Omega$ and a function $h \in \mathcal{C}(\overline{D}) \cap \mathcal{H}(D)$ such that $u \le h$ on ∂D. Put $v(z) := u(z) - h(z)$, $z \in \overline{D}$. By Proposition 3.1.10 and (3.2.1) we have

$$v(a) \le \mathbf{J}(v; a, r), \quad 0 < r < \min\{R(a), d_D(a)\},\ a \in D.$$

Using Lemma 3.2.3, we conclude that $v \le 0$ in D, which shows that $u \le h$ in D. $\square$

Corollary 3.2.6. (a) *Let $u : \Omega \longrightarrow [-\infty, +\infty)$. Then $u \in \mathcal{SH}(\Omega)$ iff every point $a \in \Omega$ admits an open neighborhood $U_a \subset \Omega$ such that $u|_{U_a} \in \mathcal{SH}(U_a)$. In other words, subharmonicity is a local property.*
(b) $\mathcal{SH}(\Omega) + \mathcal{SH}(\Omega) = \mathcal{SH}(\Omega)$.

Proposition 3.2.7. *Let $u : \Omega \longrightarrow [-\infty, +\infty)$. Then $u \in \mathcal{SH}(\Omega)$ iff $u \in \mathcal{C}^{\uparrow}(\Omega)$ and for any $a \in \Omega$, $0 < r < d_\Omega(a)$, and $p \in \mathcal{P}(\mathbb{C})$, if $u \le \operatorname{Re} p$ on $C(a, r)$, then $u \le \operatorname{Re} p$ in $K(a, r)$.*

Proof. Since the function $\operatorname{Re} p$ is harmonic, the implication $\Longrightarrow$ is obvious.

We prove now the opposite. Fix $a \in \Omega$ and $0 < r < d_\Omega(a)$. In virtue of Proposition 3.2.5 and the proof of Proposition 3.2.2, it is sufficient to prove that for every continuous function $b : C(a,r) \longrightarrow \mathbb{R}$ such that $u \le b$ we have $u(a) \le \mathbf{J}(b; a, r)$. Fix a function b and let $\varphi_\nu : \mathbb{R} \longrightarrow \mathbb{R}$, $\nu \ge 1$, be a sequence of trigonometric polynomials (5) such that

$$|b(a + re^{i\theta}) + \frac{1}{\nu} - \varphi_\nu(\theta)| < \frac{1}{\nu}, \quad \theta \in \mathbb{R}$$

(cf. [29], the Fejèr theorem). Let $p_\nu \in \mathcal{P}(\mathbb{C})$ be such that $\varphi_\nu(\theta) = \operatorname{Re} p_\nu(a + re^{i\theta})$, $\theta \in \mathbb{R}$, $\nu \ge 1$. Then $u \le \operatorname{Re} p_\nu$ on $C(a,r)$ and hence

$$u(a) \le \operatorname{Re} p_\nu(a) = \mathbf{J}(\operatorname{Re} p_\nu; a, r) \le \mathbf{J}(b; a, r) + \frac{2}{\nu}, \quad \nu \ge 1.$$

(the first equality follows from the fact that the function $\operatorname{Re} p_\nu$ is harmonic). Letting $\nu \longrightarrow +\infty$, we end the proof. □

Proposition 3.2.8. *If $f \in \mathcal{O}(\Omega)$, then $\log|f| \in \mathcal{SH}(\Omega)$.*

Proof. Let $u := \log|f|$. Then $u \in \mathcal{C}^{\uparrow}(\Omega)$. By Proposition 3.2.5, it is enough to check that $u(a) \le \mathbf{J}(u; a, r)$, $a \in \Omega$, $0 < r < R(a)$. This is evident if $f(a) = 0$. If $f(a) \ne 0$, then $u \in \mathcal{H}(K(a, R(a)))$, where $R(a) := d_{\Omega \setminus f^{-1}(0)}(a)$ (cf. Remark 3.1.1(e)). □

Proposition 3.2.9. (a) *If $\mathcal{SH}(\Omega) \ni u_\nu \searrow u$, then $u \in \mathcal{SH}(\Omega)$.*
(b) *If $\mathcal{SH}(\Omega) \ni u_\nu \longrightarrow u$ locally uniformly in Ω, then $u \in \mathcal{SH}(\Omega)$.*

Proof. It is clear that in both cases $u \in \mathcal{C}^{\uparrow}(\Omega)$. For each ν we have

$$u_\nu(a) \le \mathbf{J}(u_\nu; a, r), \quad a \in \Omega,\ 0 < r < d_\Omega(a).$$

Letting $\nu \longrightarrow +\infty$ proves that u satisfies (3.2.1). □

Proposition 3.2.10. *If a family $(u_\iota)_{\iota \in I} \subset \mathcal{SH}(\Omega)$ is locally bounded from above (6), then the function*

$$u := (\sup_{\iota \in I} u_\iota)^*,$$

is subharmonic, where * denotes the upper regularization. (7)

In particular, $\max\{u_1, \dots, u_N\} \in \mathcal{SH}(\Omega)$ *for any* $u_1, \dots, u_N \in \mathcal{SH}(\Omega)$.

Proof. It is clear that u is upper semicontinuous. Let $D \subset\subset \Omega$, $h \in \mathcal{C}(\overline{D}) \cap \mathcal{H}(D)$, $u \le h$ on ∂D. Then $u_\iota \le h$ on ∂D for every $\iota \in I$, and hence $\sup_{\iota \in I} u_\iota \le h$ in D. Finally, since h is continuous, we get $u \le h$ in D. □

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(5) Recall that $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ is a *trigonometric polynomial* if

$$\varphi(\theta) = \alpha_0 + \sum_{j=1}^{k} (\alpha_j \cos j\theta + \beta_j \sin j\theta), \quad \theta \in \mathbb{R},$$

for some $\alpha_0, \dots, \alpha_k, \beta_1, \dots, \beta_k \in \mathbb{R}$. Observe that $\varphi(\theta) = \operatorname{Re} p(a + re^{i\theta})$, where

$$p(z) := q(\frac{z-a}{r}), \quad q(z) := \alpha_0 + \sum_{j=1}^{k} (\alpha_j - i\beta_j) z^j.$$

(6) Note that in general the function $\sup_{\iota \in I} u_\iota$ need not be upper semicontinuous.
(7) If $v : \Omega \longrightarrow [-\infty, +\infty)$ is locally bounded from above, then (cf. [21])

$$v^*(z) := \limsup_{z' \to z} v(z') = \inf\{\varphi(z) : \varphi \in \mathcal{C}(\Omega, \mathbb{R}),\ v \le \varphi\}, \quad z \in \Omega.$$

Proposition 3.2.11. *Let $G \subset \Omega \subset \mathbb{C}$ be open and let $v \in \mathcal{SH}(G)$, $u \in \mathcal{SH}(\Omega)$. Assume that*

$$\limsup_{G \ni z \to \zeta} v(z) \le u(\zeta), \quad \zeta \in (\partial G) \cap \Omega.$$

Let

$$\widetilde{u}(z) := \begin{cases} \max\{v(z), u(z)\}, & z \in G \\ u(z), & z \in \Omega \setminus G \end{cases}.$$

Then $\widetilde{u} \in \mathcal{SH}(\Omega)$.

Proof. It is evident that $\widetilde{u} \in \mathcal{C}^{\uparrow}(\Omega)$ and $\widetilde{u} \in \mathcal{SH}(\Omega \setminus \partial G)$. For $a \in \Omega \cap \partial G$ we have

$$\widetilde{u}(a) = u(a) \le \mathbf{J}(u; a, r) \le \mathbf{J}(\widetilde{u}; a, r), \quad 0 < r < d_\Omega(a). \qquad \square$$

Proposition 3.2.12. *Let $u : \Omega \longrightarrow [-\infty, +\infty)$. Then $u \in \mathcal{SH}(\Omega)$ iff $u \in \mathcal{C}^{\uparrow}(\Omega)$ and for every $a \in \Omega$ there exists an $R(a)$, $0 < R(a) \le d_\Omega(a)$, such that*

$$u(z) \le \mathbf{P}(u; a, r; z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{r^2 - |z-a|^2}{|re^{i\theta} - (z-a)|^2} u(a + re^{i\theta})\, d\theta, \quad 0 < r < R(a),\ z \in K(a, r). \tag{3.2.2}$$

Proof. Since $\mathbf{P}(u; a, r; a) = \mathbf{J}(u; a; r)$, the implication $\Longleftarrow$ follows from Proposition 3.2.5.

To prove the opposite, it is sufficient to argue as in the proof of Proposition 3.2.2 and use the Poisson formula

$$u(z) \le h_\nu(z) = \mathbf{P}(h_\nu; a, r; z) = \mathbf{P}(b_\nu; a, r; z) \searrow \mathbf{P}(u; a, r, z). \qquad \square$$

By Propositions 3.1.6 and 3.2.12 we get

Corollary 3.2.13. $\mathcal{SH}(\Omega) \cap (-\mathcal{SH}(\Omega)) = \mathcal{H}(\Omega)$.

Proposition 3.2.14. *If a sequence $(u_\nu)_{\nu=1}^\infty \subset \mathcal{SH}(\Omega)$ is locally bounded from above, then the function*

$$u := (\limsup_{\nu \to +\infty} u_\nu)^*.$$

is subharmonic. (8)

Proof. Of course, the function u is upper semicontinuous. Fix $a \in \Omega$ and $0 < r < d_\Omega(a)$. By Proposition 3.2.12 and Fatou's lemma we get

$$\limsup_{\nu \to +\infty} u_\nu(z) \le \limsup_{\nu \to +\infty} \mathbf{P}(u_\nu; a, r; z) \le \mathbf{P}(\limsup_{\nu \to +\infty} u_\nu; a, r; z) \le \mathbf{P}(u; a, r; z), \quad z \in K(a, r).$$

Since the right–hand side is a continuous function of z, we get $u(z) \le \mathbf{P}(u; a, r; z)$, $z \in K(a, r)$. $\square$

Let $u : K(a, r) \longrightarrow [-\infty, +\infty)$ be bounded from above and measurable. Define

$$\mathbf{A}(u; a, r) := \frac{1}{\pi r^2} \int_{K(a,r)} u\, d\mathcal{L}^2;$$

$\mathbf{A}(u; a, r)$ is the mean value of u on the disc $K(a, r)$.

Proposition 3.2.15 (Mean value property)**.** *Let $u : \Omega \longrightarrow [-\infty, +\infty)$. Then $u \in \mathcal{SH}(\Omega)$ iff $u \in \mathcal{C}^{\uparrow}(\Omega)$ and for every $a \in D$ there exists an $R(a)$, $0 < R(a) \le d_D(a)$, such that*

$$u(a) \le \mathbf{A}(u; a, r), \quad 0 < r < R(a).$$

(8) Note that in general the function $\limsup_{\nu \to +\infty} u_\nu$ need not be upper semicontinuous.

Proof. Let $u \in \mathcal{SH}(\Omega)$. Using polar coordinates, we have by Proposition 3.2.2

$$\mathbf{A}(u;a,r) = \frac{1}{\pi r^2}\int_0^r\int_0^{2\pi} u(a+\tau e^{i\theta})\tau\, d\theta\, d\tau$$
$$= \frac{2}{r^2}\int_0^r \mathbf{J}(u;a,\tau)\tau\, d\tau \geq \frac{2}{r^2}\int_0^r u(a)\tau\, d\tau = u(a), \quad a \in \Omega,\ 0 < r < d_\Omega(a).$$

To prove the opposite we check first that u does not attain its maximum (like in the proof of Lemma 3.2.3), and then we proceed as in the proof of Proposition 3.2.5. □

Proposition 3.2.16. *Let $D \subset \mathbb{C}$ be a domain and let $u \in \mathcal{SH}(D)$, $u \not\equiv -\infty$. Then $u \in L^1(D, \mathrm{loc})$. In particular, $\mathcal{L}^2(u^{-1}(-\infty)) = 0$.*

Proof. Suppose that for some $z_0 \in D$ we have $\int_U u\, d\mathcal{L}^2 = -\infty$ for any neighborhood U of z_0. Let $2r := d_D(z_0)$. By Proposition 3.2.15

$$u(z) \leq \mathbf{A}(u;z,r) = -\infty, \quad z \in K(z_0, r).$$

Let $D_0 := \{z \in D : u = -\infty \text{ in a neighborhood of } z\}$. The set D_0 is clearly open. We have already shown that it is non-empty ($z_0 \in D_0$). To obtain a contradiction, it is sufficient to note that proceeding exactly as above, we can prove that D_0 is relatively closed in D. □

Proposition 3.2.17 (Removable singularities)**.** *Let $D \subset \mathbb{C}$ be a domain and let $M \subset D$ be a relatively closed subset of D such that for every point $a \in M$ there exist a connected open neighborhood $U_a \subset D$ of a and a function $v_a \in \mathcal{SH}(U_a)$, $v_a \not\equiv -\infty$, such that $M \cap U_a = v_a^{-1}(-\infty)$. Let $u \in \mathcal{SH}(D \setminus M)$ be locally bounded from above in D (9) . Define*

$$\widetilde{u}(z) := \limsup_{D\setminus M \ni z' \to z} u(z'), \quad z \in D.$$

Then $\widetilde{u} \in \mathcal{SH}(D)$. In particular, the set $D \setminus M$ is connected.

Proof. By Proposition 3.2.16 the set M is nowhere dense and hence the function $\widetilde{u}$ is well defined for every $z \in D$. Note that $\widetilde{u} = (u_0)^*$, where $u_0 := u$ on $D \setminus M$ and $u_0 := -\infty$ on M. In particular, $\widetilde{u} \in \mathcal{C}^{\uparrow}(D)$. Moreover, $\widetilde{u} = u$ on $D \setminus M$.

It remains to prove that $\widetilde{u}$ is subharmonic. We may assume that $M = v^{-1}(-\infty)$, where $v \in \mathcal{SH}(D)$, $v \not\equiv -\infty$ and $v \leq 0$ in D. For $\varepsilon > 0$ let

$$u_\varepsilon(z) := \begin{cases} u(z) + \varepsilon v(z), & z \in D \setminus M \\ -\infty, & z \in M \end{cases}.$$

It is easy to see that $u_\varepsilon \in \mathcal{SH}(D)$ and that the family $(u_\varepsilon)_{\varepsilon>0}$ is locally bounded from above in D. Observe that $u_0 = \sup_{\varepsilon>0} u_\varepsilon$. Hence, by Proposition 3.2.10, $\widetilde{u} = (u_0)^* \in \mathcal{SH}(D)$.

To prove that $D \setminus M$ is connected, suppose that $D \setminus M = U_1 \cup U_2$, where U_1 and U_2 are disjoint and non–empty open sets. Then the function $u(z) := j$ for $z \in U_j$ would extend to a subharmonic function on D; contradiction. □

The above result can be generalized in the following way:

We say that a set $M \subset \mathbb{C}$ is *polar* if for every point $a \in M$ there exist a connected open neighborhood U_a and a function $v_a \in \mathcal{SH}(U_a)$, $v_a \not\equiv -\infty$, such that $M \cap U_a \subset v_a^{-1}(-\infty)$.

Note that the set M from Proposition 3.2.17 is polar. Every polar set has measure zero (by Proposition 3.2.16).

Lemma 3.2.18. *Let $M \subset \mathbb{C}$ be a polar set. Then for every $a \in \mathbb{C}$ there exists an $R(a) > 0$ such that*

$$\mathcal{L}^1(\{\theta \in [0, 2\pi) : a + re^{i\theta} \in M\}) = 0, \quad 0 < r < R(a).$$

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(9) That is, every point $a \in D$ admits an open neighborhood $V_a \subset D$ such that u is bounded from above in $V_a \setminus M$.

Proof. Suppose that for some $a \in \mathbb{C}$ it is not the case. Fix a disc $K(a,R)$ and a function $v \in \mathcal{SH}(K(a,R))$, $v \not\equiv -\infty$, such that $M \cap K(a,R) \subset v^{-1}(-\infty)$. Let $0 < r < R$ be such that

$$\mathcal{L}^1(\{\theta \in [0,2\pi) : a + re^{i\theta} \in M\}) > 0.$$

This means that $v(a+re^{i\theta}) = -\infty$ for θ in a set of positive measure. In particular, $v(z) \le \mathbf{P}(v;a,r;z) = -\infty$ for $z \in K(a,r)$, and so $v \equiv -\infty$ in $K(a,r)$; contradiction. □

Proposition 3.2.19 (Removable singularities). *Let $D \subset \mathbb{C}$ be a domain and let $M \subset D$ be a polar set. Assume that $u \in \mathcal{C}^{\uparrow}(D \setminus M)$ is locally bounded from above in D and for arbitrary $a \in D \setminus M$ there exists an $R(a)$, $0 < R(a) \le d_D(a)$, such that*

$$u(a) \le \mathbf{J}(u;a,r), \quad 0 < r < R(a). \quad (^{10})$$

Put

$$\widetilde{u}(z) := \limsup_{D \setminus M \ni z' \to z} u(z'), \quad z \in D.$$

Then $\widetilde{u} \in \mathcal{SH}(D)$. In particular, if M is closed in D, then $D \setminus M$ is a domain.

Proof. The function $\widetilde{u}$ is upper semicontinuous and $\widetilde{u} = u$ in $D \setminus M$. Let $G \subset\subset D$ be an arbitrary domain and let $h \in \mathcal{H}(G) \cap \mathcal{C}(\overline{G})$ be such that $\widetilde{u} \le h$ on ∂G. It is sufficient to check that $\widetilde{u} \le h$ in $G \setminus M$. Fix an $a \in G \setminus M$. One can prove (see for instance [14], Th. 5.11), that there exists a function v subharmonic in the neighborhood of $\overline{G}$ and such that $M \cap G \subset v^{-1}(-\infty)$, $v \le 0$, and $v(a) > -\infty$. Define $h_\varepsilon := \widetilde{u} + \varepsilon v - h$, $\varepsilon > 0$. Then $h_\varepsilon \in \mathcal{C}^{\uparrow}(\overline{G})$ and $h_\varepsilon \le 0$ on ∂G. One can easily check that $h_\varepsilon \in \mathcal{SH}(G)$ $(^{11})$. By the maximum principle (Corollary 3.2.4) it follows that $h_\varepsilon \le 0$ in G, $\varepsilon > 0$. In particular, $\widetilde{u}(a) - h(a) = \sup_{\varepsilon > 0}\{h_\varepsilon(a)\} \le 0$.

□

Proposition 3.2.20 (Hartogs lemma). *Let $(u_\nu)_{\nu=1}^{\infty} \subset \mathcal{SH}(\Omega)$ be locally bounded from above. Assume that for some $m \in \mathbb{R}$*

$$\limsup_{\nu \to +\infty} u_\nu \le m.$$

Then for any compact $K \subset \Omega$ and $\varepsilon > 0$ there exists a ν_0 such that

$$\max_K u_\nu \le m + \varepsilon, \quad \nu \ge \nu_0; \quad \textit{cf. Lemma 1.5.5.}$$

Proof. It is sufficient to show that for every $a \in \Omega$ the assertion holds for $K := \overline{K}(a, \delta(a))$, where $\delta(a) > 0$ is sufficiently small. Fix a and $0 < R < d_\Omega(a)/2$. We may assume that $u_\nu \le 0$ in $\overline{K}(a,2R)$, $\nu \ge 1$, and $m < 0$. By Fatou's lemma we have

$$\limsup_{\nu \to +\infty} \mathbf{A}(u_\nu;a,R) \le \mathbf{A}(\limsup_{\nu \to +\infty} u_\nu;a,R) \le \mathbf{A}(m;a,R) = m.$$

Let $0 < \delta < R/2$. By the above inequality, since $u_\nu \le 0$ on $\overline{K}(a,2R)$, we get

$$\limsup_{\nu \to +\infty} \max_{z \in \overline{K}(a,\delta)} u_\nu(z) \le \limsup_{\nu \to +\infty} \sup_{z \in \overline{K}(a,\delta)} \mathbf{A}(u_\nu; z, R+\delta) \le \limsup_{\nu \to +\infty} \frac{R^2}{(R+\delta)^2}\mathbf{A}(u_\nu;a,R) \le \frac{R^2}{(R+\delta)^2} m.$$

Now it is sufficient to take a $\delta = \delta(a)$ so small that the last term is smaller than $m + \varepsilon$. □

Proposition 3.2.21. *Let $I \subset \mathbb{R}$ be an open interval and let $\varphi : I \longrightarrow \mathbb{R}$ be non-decreasing and convex. Then $\varphi \circ u \in \mathcal{SH}(\Omega)$ for any subharmonic function $u : \Omega \longrightarrow I$. In particular,*

$e^u \in \mathcal{SH}(\Omega)$ for any function $u \in \mathcal{SH}(\Omega)$ $(^{12})$,

$u^p \in \mathcal{SH}(\Omega)$ for any subharmonic function $u : \Omega \longrightarrow \mathbb{R}_+$ and $p \ge 1$ $(^{13})$.

$(^{10})$ Note that if M is a closed subset of D, then every function $u \in \mathcal{SH}(D \setminus M)$ satisfies this condition (with $R(a) := d_{D \setminus M}(a)$). Moreover, by Lemma 3.2.18, the integral $\mathbf{J}(u;a,r)$ is well defined for small r.

$(^{11})$ We apply for instance Proposition 3.2.5: since $h_\varepsilon = -\infty$ on M, it is sufficient to observe that $h_\varepsilon(z_0) \le \mathbf{J}(h_\varepsilon; z_0, r)$ for $z_0 \in G \setminus M$.

$(^{12})$ First we consider $u : \Omega \longrightarrow \mathbb{R}$ and next we observe that in the general case we have $e^{\max\{u,-\nu\}} \searrow e^u$ when $\nu \nearrow +\infty$.

$(^{13})$ First we consider $u : \Omega \longrightarrow \mathbb{R}_{>0}$ and next we observe that in the general case we have $(u+\varepsilon)^p \searrow u^p$ when $\varepsilon \searrow 0$.

Proof. Since φ is convex, it is continuous (cf. [32]), and therefore $\varphi\circ u \in \mathcal{C}^{\uparrow}(\Omega)$. Fix $a \in \Omega$ and $0 < r < d_\Omega(a)$. By the monotonicity and convexity of φ and by Jensen's inequality (cf. [29]), we obtain

$$\varphi(u(a)) \le \varphi(\mathbf{J}(u; a, r)) \le \mathbf{J}(\varphi \circ u; a, r). \qquad \square$$

Proposition 3.2.22. *Let $u \in \mathcal{SH}(\Omega)$, $a \in \Omega$. Then the functions*

$$(-\infty, \log d_\Omega(a)) \ni t \longmapsto \mathbf{J}(u; a, e^t), \qquad (-\infty, \log d_\Omega(a)) \ni t \longmapsto \mathbf{A}(u; a, e^t)$$

are non–decreasing and convex. Moreover,

$$\mathbf{J}(u; a, r) \searrow u(a) \ \text{ when } r \searrow 0, \qquad \mathbf{A}(u; a, r) \searrow u(a) \ \text{ when } r \searrow 0.$$

Proof. We show first that it is sufficient to consider only the function $\mathbf{J}$. Note that if the function $\mathbf{J}(u; a, \cdot)$ is convex with respect to $\log r$, then it is continuous, and therefore we have

$$\mathbf{A}(u; a, r) = \frac{2}{r^2}\int_0^r \mathbf{J}(u; a, \tau)\tau\, d\tau = \lim_{N\to+\infty} \frac{2}{N^2}\sum_{j=1}^N j\mathbf{J}(u; a, \frac{jr}{N}) =: \lim_{N\to+\infty}\varphi_N(r).$$

If the function $\mathbf{J}(u; a, \cdot)$ is non–decreasing and convex with respect to $\log r$, then the same properties has every function φ_N, and so also the limit function $\mathbf{A}(u; a, .)$. Moreover,

$$u(a) \le \mathbf{A}(u; a, r) = \frac{2}{r^2}\int_0^r \mathbf{J}(u; a, \tau)\tau\, d\tau \le \sup_{0<\tau<r} \mathbf{J}(u; a, \tau) \le \mathbf{J}(u; a, r).$$

Therefore, if $\mathbf{J}(u; a, r) \longrightarrow u(a)$, then the same property has the function $\mathbf{A}$.

Now consider the function $\mathbf{J}$. Let $0 < r_1 < r_2 < d_\Omega(a)$, let $b_\nu \in \mathcal{C}(C(a, r_2), \mathbb{R})$, $b_\nu \searrow u$, and denote by h_ν the solution of the Dirichlet problem for $K(a, r_2)$ with boundary condition b_ν (cf. Proposition 3.1.9). Then

$$\mathbf{J}(u; a, r_1) \le \mathbf{J}(h_\nu; a, r_1) = h_\nu(a) = \mathbf{J}(h_\nu; a, r_2) = \mathbf{J}(b_\nu; a, r_2).$$

The last integral converges to $\mathbf{J}(u; a, r_2)$ when $\nu \longrightarrow +\infty$. Letting $\nu \longrightarrow +\infty$ we get the monotonicity of the function $\mathbf{J}(u; a, \cdot)$.

Note that by Fatou's lemma we have

$$u(a) \le \lim_{r\to 0} \mathbf{J}(u; a, r) \le \frac{1}{2\pi}\int_0^{2\pi} \limsup_{r\to 0} u(a + re^{i\theta})\, d\theta \le u(a).$$

This proves that $\mathbf{J}(u; a, r) \searrow u(a)$ when $r \searrow 0$.

It remains to check the convexity with respect to $\log r$, i.e. we want to prove the inequality

$$\mathbf{J}(u; a, r) \le \mathbf{J}(u; a, r_1) + \frac{\mathbf{J}(u; a, r_2) - \mathbf{J}(u; a, r_1)}{\log \frac{r_2}{r_1}} \log\frac{r}{r_1}, \quad 0 < r_1 < r < r_2 < d_\Omega(a).$$

Fix $0 < r_1 < r_2 < d_\Omega(a)$. Let $A := \{z \in \mathbb{C} : r_1 < |z| < r_2\}$, let $b_\nu \in \mathcal{C}(\partial A, \mathbb{R})$, $b_\nu \searrow u$, and let h_ν be the solution of the Dirichlet problem for the annulus A with boundary condition b_ν (cf. Proposition 3.1.13). Differentiating under the integral sign, we obtain

$$\begin{aligned}\frac{d}{dt}\mathbf{J}(h_\nu; a, e^t) = \frac{d}{dt}\frac{1}{2\pi}\int_0^{2\pi} h_\nu(a + e^t e^{i\theta})\, d\theta &= \frac{1}{2\pi}\int_0^{2\pi}\Big(\frac{\partial h_\nu}{\partial x}(a + e^t e^{i\theta})e^t\cos\theta + \frac{\partial h_\nu}{\partial y}(a + e^t e^{i\theta})e^t\sin\theta\Big)d\theta \\ &= \frac{1}{2\pi}\int_{C(a,e^t)} -\frac{\partial h_\nu}{\partial y}dx + \frac{\partial h_\nu}{\partial x}dy = \mathrm{const}(\nu).\end{aligned}$$

The last equality follows from the fact that the form

$$-\frac{\partial h_\nu}{\partial y}dx + \frac{\partial h_\nu}{\partial x}dy$$

is closed. Consequently, there exist $\alpha_\nu, \beta_\nu \in \mathbb{R}$ such that

$$\mathbf{J}(h_\nu; a, r) = \alpha_\nu \log r + \beta_\nu, \quad r_1 < r < r_2.$$

Finally,

$$\mathbf{J}(u;a,r) \le \mathbf{J}(h_\nu;a,r) = \mathbf{J}(h_\nu;a,r_1) + \frac{\mathbf{J}(h_\nu;a,r_2) - \mathbf{J}(h_\nu;a,r_1)}{\log\frac{r_2}{r_1}}\log\frac{r}{r_1}$$
$$= \mathbf{J}(b_\nu;a,r_1) + \frac{\mathbf{J}(b_\nu;a,r_2) - \mathbf{J}(b_\nu;a,r_1)}{\log\frac{r_2}{r_1}}\log\frac{r}{r_1}, \quad r_1 < r < r_2.$$

Letting $\nu \longrightarrow +\infty$ we end the proof. □

Corollary 3.2.23. *Let $u_1, u_2 \in \mathcal{SH}(\Omega)$. If $u_1 = u_2$ $\mathcal{L}^2$-almost everywhere in Ω, then $u_1 \equiv u_2$ in Ω.*

Corollary 3.2.24. *Let D and M be as in Proposition 3.2.17 or 3.2.19. Then for every function $u \in \mathcal{SH}(D)$ we have*

$$u(z) = \limsup_{D\setminus M \ni z' \to z} u(z'), \quad z \in D.$$

Fix a function $\Psi \in \mathcal{C}_0^\infty(\mathbb{C}, \mathbb{R}_+)$ such that
- $\operatorname{supp}\Psi = \overline{\mathbb{D}}$,
- $\Psi(z) = \Psi(|z|)$, $z \in \mathbb{C}$,
- $\int \Psi \, d\mathcal{L}^2 = 1$.

Let

$$\Psi_\varepsilon(z) := \frac{1}{\varepsilon^2}\Psi(\frac{z}{\varepsilon}), \quad z \in \mathbb{C}, \ \varepsilon > 0.$$

For every function $u \in L^1(\Omega, \mathrm{loc})$, we put

$$u_\varepsilon(z) := \int_\Omega u(w)\Psi_\varepsilon(z-w)\,d\mathcal{L}^2(w) = \int_{\mathbb{D}} u(z+\varepsilon w)\Psi(w)\,d\mathcal{L}^2(w), \quad z \in \Omega_\varepsilon := \{z \in \Omega : d_\Omega(z) > \varepsilon\}.$$

The function u_ε is called the ε-*regularization* of u.

Proposition 3.2.25. *If $u \in \mathcal{SH}(\Omega) \cap L^1(\Omega, \mathrm{loc})$, then $u_\varepsilon \in \mathcal{SH}(\Omega_\varepsilon) \cap \mathcal{C}^\infty(\Omega_\varepsilon)$ and $u_\varepsilon \searrow u$ when $\varepsilon \searrow 0$.*

Proof. Since we can differentiate under the integral sign in the first integral above, it is clear that $u_\varepsilon \in \mathcal{C}^\infty(\Omega_\varepsilon)$. For $a \in \Omega_\varepsilon$ and $0 < r < d_{\Omega_\varepsilon}(a)$ we have

$$\mathbf{J}(u_\varepsilon;a,r) = \frac{1}{2\pi}\int_0^{2\pi}\int_{\mathbb{D}} u(a + re^{i\theta} + \varepsilon w)\Psi(w)\,d\mathcal{L}^2(w)\,d\theta$$
$$= \int_{\mathbb{D}} \mathbf{J}(u;a+\varepsilon w, r)\Psi(w)\,d\mathcal{L}^2(w) \ge \int_{\mathbb{D}} u(a+\varepsilon w)\Psi(w)\,d\mathcal{L}^2(w) = u_\varepsilon(a),$$

which shows that $u_\varepsilon \in \mathcal{SH}(\Omega_\varepsilon)$. Note that

$$u_\varepsilon(a) = \int_{\mathbb{D}} u(a+\varepsilon w)\Psi(w)\,d\mathcal{L}^2(w) = \int_0^1\int_0^{2\pi} u(a+\varepsilon\tau e^{i\theta})\Psi(\tau)\tau\,d\theta\,d\tau = 2\pi\int_0^1 \mathbf{J}(u;a,\varepsilon\tau)\Psi(\tau)\tau\,d\tau.$$

Now, by Proposition 3.2.22 and monotone convergence theorem, we get $u_\varepsilon(a) \searrow u(a)$ when $\varepsilon \searrow 0$ for every $a \in \Omega$. □

Remark 3.2.26. It follows from the proof of Proposition 3.2.25 that for an arbitrary function $\Psi \in \mathcal{C}_0^\infty(\mathbb{C}, \mathbb{R}_+)$ such that $\operatorname{supp}\Psi = \overline{\mathbb{D}}$ and for every function $u \in \mathcal{SH}(\Omega)$, the functions

$$u_\varepsilon(z) := \int_{\mathbb{D}} u(z+\varepsilon w)\Psi(w)\,d\mathcal{L}^2(w), \quad z \in \Omega_\varepsilon, \quad \varepsilon > 0,$$

are subharmonic.

Proposition 3.2.27. *Let $u \in \mathcal{C}^2(\Omega, \mathbb{R})$. Then $u \in \mathcal{SH}(\Omega)$ iff $\Delta u \ge 0$ in Ω.*

Proof. $\Longleftarrow$. Assume first that $\Delta u > 0$ in Ω. Let $D \subset\subset \Omega$, $h \in \mathcal{C}(\overline{D}) \cap \mathcal{H}(D)$, $u \le h$ on ∂D. Put $v := u - h$ and let $z_0 \in \overline{D}$ be such that $v(z_0) = \max_{\overline{D}} v$. Suppose that $v(z_0) > 0$ (in particular, $z_0 \in D$). Then $(\Delta u)(z_0) \le 0$; contradiction.

For arbitrary u, take the sequence $v_\varepsilon(z) := u(z) + \varepsilon|z|^2$, $z \in \Omega$, $\varepsilon > 0$, and note that $\Delta v_\varepsilon = \Delta u + 4\varepsilon > 0$ and $v_\varepsilon \searrow u$.

$\Longrightarrow$. Suppose that $\Delta u < 0$ on some domain $D \subset \Omega$. Then, by the previous part of the proof, $-u \in \mathcal{SH}(D)$. Hence $u \in \mathcal{H}(D)$; contradiction. $\square$

Proposition 3.2.28. *If $u \in \mathcal{SH}(D)$ (D is a domain in $\mathbb{C}$), $u \not\equiv -\infty$, then $\Delta u \ge 0$ in D in the distribution sense, i.e. for every function $\varphi \in \mathcal{C}_0^\infty(D, \mathbb{R}_+)$ we have*

$$\int_D u \cdot (\Delta\varphi)\, d\mathcal{L}^2 \ge 0.$$

Conversely, if $u \in L^1(D, \mathrm{loc})$ is such that $\Delta u \ge 0$ in D in the distribution sense, then there exists a function $v \in \mathcal{SH}(D)$ such that $u = v$ $\mathcal{L}^2$-almost everywhere in D; cf. Proposition 3.1.14.

Proof. Note first that if $u \in \mathcal{C}^2(D)$, then, by the Stokes theorem, $\Delta u \ge 0$ in D in the distribution sense iff $\Delta u \ge 0$ in D in the usual sense.

$\Longrightarrow$. Let u_ε denote the regularization of the function u (as in Proposition 3.2.25). By Propositions 3.2.25 and 3.2.27, $\Delta u_\varepsilon \ge 0$ in D_ε in the distribution sense, i.e.

$$\int_{D_\varepsilon} u_\varepsilon \cdot (\Delta\varphi)\, d\mathcal{L}^2 \ge 0$$

for every test function $\varphi \in \mathcal{C}_0^\infty(D_\varepsilon, \mathbb{R}_+)$. Since $u_\varepsilon \searrow u$ (Proposition 3.2.25), we get

$$\int_D u \cdot (\Delta\varphi)\, d\mathcal{L}^2 \ge 0, \quad \varphi \in \mathcal{C}_0^\infty(D, \mathbb{R}_+).$$

$\Longleftarrow$. For every function $\varphi \in \mathcal{C}_0^\infty(D_\varepsilon, \mathbb{R}_+)$ we have

$$\begin{aligned}\int_{D_\varepsilon} u_\varepsilon \cdot (\Delta\varphi)\, d\mathcal{L}^2 = \int_{D_\varepsilon} (\Delta u_\varepsilon)\varphi\, d\mathcal{L}^2 &= \int_{D_\varepsilon} \Big(\int_D u(w)(\Delta\Psi_\varepsilon)(z-w)\, d\mathcal{L}^2(w)\Big)\varphi(z)\, d\mathcal{L}^2(z) \\ &= \int_{D_\varepsilon} \Big(\int_D u(w)(\Delta(\Psi_\varepsilon(z-\cdot)))(w)\, d\mathcal{L}^2(w)\Big)\varphi(z)\, d\mathcal{L}^2(z) \ge 0.\end{aligned}$$

This proves that $u_\varepsilon \in \mathcal{SH}(D_\varepsilon)$.

We show now that $u_\varepsilon \searrow$ when $\varepsilon \searrow 0$. Let $0 < \varepsilon_1 < \varepsilon_2$. By Proposition 3.2.25 applied for $z \in D_{\varepsilon_2}$ we have

$$\begin{aligned}u_{\varepsilon_2}(z) = \lim_{\varepsilon\to 0}(u_{\varepsilon_2})_\varepsilon(z) &= \lim_{\varepsilon\to 0}\int_{\mathbb{D}}\int_{\mathbb{D}} u(z + \varepsilon w + \varepsilon_2\xi)\Psi(\xi)\, d\mathcal{L}^2(\xi)\Psi(w)\, d\mathcal{L}^2(w) \\ &= \lim_{\varepsilon\to 0}\int_{\mathbb{D}}\int_{\mathbb{D}} u(z + \varepsilon w + \varepsilon_2\xi)\Psi(w)\, d\mathcal{L}^2(w)\Psi(\xi)\, d\mathcal{L}^2(\xi) \\ &= \lim_{\varepsilon\to 0}(u_\varepsilon)_{\varepsilon_2}(z) \ge \lim_{\varepsilon\to 0}(u_\varepsilon)_{\varepsilon_1}(z) = \lim_{\varepsilon\to 0}(u_{\varepsilon_1})_\varepsilon(z) = u_{\varepsilon_1}(z).\end{aligned}$$

Let $v := \lim_{\varepsilon\to 0} u_\varepsilon$. Then $v \in \mathcal{SH}(D)$. On the other hand, it is well known (cf. [29]) that $u_\varepsilon \longrightarrow u$ in $L^1(D, \mathrm{loc})$. In particular, $u_\varepsilon \longrightarrow u$ $\mathcal{L}^2$-almost everywhere in D. Hence $u = v$ $\mathcal{L}^2$-almost everywhere D. $\square$

Proposition 3.2.29. *For every $f \in \mathcal{O}(\Omega, G)$ (G is an open subset of $\mathbb{C}$) and $u \in \mathcal{SH}(G)$ we have $u \circ f \in \mathcal{SH}(\Omega)$.*

Proof. If $u \in \mathcal{C}^2(G)$ it is sufficient to note that

$$\Delta(u \circ f) = ((\Delta u) \circ f) \cdot |f'|^2,$$

and use Proposition 3.2.27. For the general case we use the regularizations $(u_\varepsilon)_{\varepsilon>0}$, cf. Proposition 3.2.25. Let $v_\varepsilon := u_\varepsilon \circ f$. Then $v_\varepsilon \in \mathcal{SH}(f^{-1}(G_\varepsilon))$, and $v_\varepsilon \searrow u \circ f$ in G, and so, by Proposition 3.2.9(a), $u \circ f \in \mathcal{SH}(\Omega)$. $\square$

Proposition 3.2.30 (Liouville type theorem). *If $u \in \mathcal{SH}(\mathbb{C})$ is bounded from above, then $u \equiv \mathrm{const}$.*

Proof. Let $v(z) := u(1/z)$, $z \in \mathbb{C}_*$. Then, by Proposition 3.2.29, $v \in \mathcal{SH}(\mathbb{C}_*)$ and v is bounded from above. Hence, by Proposition 3.2.17, v extends to a function $\widetilde{v} \in \mathcal{SH}(\mathbb{C})$. Now, by the maximum principle, for arbitrary $z \in \mathbb{C}$, we have

$$u(z) \le \max\{\max_{\mathbb{T}} u, \max_{\mathbb{T}} v\} = u(z_0)$$

for some $z_0 \in \mathbb{T}$. Using once again the maximum principle we conclude that $u \equiv \mathrm{const}$. □

Proposition 3.2.31 (Oka theorem). *For every function $u \in \mathcal{SH}(\Omega)$, and for every $\mathbb{R}$-analytic curve $\gamma : [0,1] \longrightarrow \Omega$ it holds*

$$u(\gamma(0)) = \limsup_{t\to 0+} u(\gamma(t)).$$

Proof. Since the curve γ is $\mathbb{R}$-analytic, there exists a function $f \in \mathcal{O}(G)$, where $G \subset \mathbb{C}$ is an open neighborhood of the interval $[0,1]$, such that $f = \gamma$ on $[0,1]$ and $f(G) \subset \Omega$. Put $u_1 := u \circ f$. To prove the assertion, it is sufficient to show that $\limsup_{x\to 0+} u_1(x) = u_1(0)$. Moreover, we may assume that $u_1 \le 0$.

Suppose that $\limsup_{x\to 0+} u_1(x) < C < u_1(0)$. Let

$$u_2 := -\frac{1}{C}\max\{u_1, C\} + 1.$$

Then $u_2 \in \mathcal{SH}(G)$, $0 \le u_2 \le 1$, $u_2(0) > 0$, and $u_2 = 0$ on $(0, \delta]$ for some $0 < \delta \ll 1$. We may assume that $\delta\overline{\mathbb{D}} \subset G$. Define $v(z) := u_2(\delta z)$, $z \in \overline{\mathbb{D}}$. Then $v \in \mathcal{SH}$, $0 \le v \le 1$, $v(0) > 0$, and $v = 0$ on $(0,1]$. Let

$$S_\nu := \{re^{i\theta} : 0 < r < 1,\ 0 < \theta < \frac{2\pi}{\nu}\},$$

$$v_\nu(z) := \begin{cases} v(z^\nu), & \text{for } z \in S_\nu \\ 0, & \text{for } z \in \mathbb{D}_* \setminus S_\nu \end{cases}, \quad \nu \in \mathbb{N}.$$

It is not difficult to check that $v_\nu \in \mathcal{SH}(\mathbb{D}_*)$ (cf. Proposition 3.2.11). Since $v_\nu \le 1$, the function v_ν extends to a subharmonic function on $\mathbb{D}$; denote the extension also by v_ν. Observe that

$$v_\nu(0) = \limsup_{\mathbb{D}_*\ni z\to 0} v_\nu(z) = \limsup_{S_\nu\ni z\to 0} v(z^\nu) = \limsup_{\mathbb{D}_*\ni z\to 0} v(z) = v(0).$$

Finally, for any $0 < r < 1$, we have

$$v(0) = v_\nu(0) \le \mathbf{J}(v_\nu; 0, r) = \frac{1}{2\pi}\int_0^{2\pi/\nu} v(r^\nu e^{i\nu\theta})\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} v(r^\nu e^{i\theta})\frac{1}{\nu}\, d\theta \le \frac{1}{\nu}.$$

Letting $\nu \longrightarrow +\infty$ we obtain $v(0) = 0$; contradiction. □

The above result can be generalized as follows:

Proposition 3.2.32 (Oka theorem). *For any $u \in \mathcal{SH}(\Omega)$ and a curve $\gamma : [0,1] \longrightarrow \Omega$ we have*

$$u(\gamma(0)) = \limsup_{t\to 0+} u(\gamma(t)).$$

Proof. Cf. [35]. We may assume that $\gamma(0) = 0 \in \Omega$. Suppose that

$$u(0) > A > \limsup_{t\to 0+} u(\gamma(t)).$$

Take $r > 0$ and $0 < t_0 \le 1$ such that:
- $K(r) \subset\subset \Omega$,
- $|\gamma(t)| < r$ for $0 \le t < t_0$,
- $|\gamma(t_0)| = r$,
- $u(\gamma(t)) < A$ for $0 < t \le t_0$.

We may assume that $t_0 = 1$. Let $\Omega_0 := \{z \in \Omega : u(z) < A\}$. Observe that Ω_0 is open and $\gamma((0,1]) \subset \Omega_0$. Let G denote the connected component of Ω_0 that contains $\gamma((0,1])$. For $0 < \varrho < r$ let $0 < t_\varrho < 1$ be such that $|\gamma(t_\varrho)| = \varrho$. Take a Jordan arc $\sigma_\varrho : [0,1] \longrightarrow G$ such that $\sigma_\varrho(0) = \gamma(t_\varrho)$, $\sigma_\varrho(1) = \gamma(1)$. There exist $0 \le \tau_0 < \tau_1 \le 1$ such that
- $|\sigma_\varrho(\tau_0)| = \varrho$,

- $\varrho < |\sigma_\varrho(t)| < r$ for $\tau_0 < t < \tau_1$,
- $|\sigma_\varrho(\tau_1)| = r$.

We may assume that $\tau_0 = 0$, $\tau_1 = 1$. Let $L_\varrho := \sigma_\varrho([0,1])$, $D_\varrho := K(r) \setminus L_\varrho$. One can prove that D_ϱ is simply connected (Exercise). Let $\varphi_\varrho : \mathbb{D} \longrightarrow D_\varrho$ be a biholomorphic mapping (from the Riemann theorem) with $\varphi_\varrho(0) = 0$ and $\varphi'_\varrho(0) \in \mathbb{R}_{>0}$. By the Carathéodory theorem (cf. [35]), the mapping φ_ϱ extends continuously to $\overline{\mathbb{D}}$ (we denote this extension also by φ_ϱ) and $\varphi_\varrho(\mathbb{T}) \subset \partial D_\varrho$. Let

$$T_\varrho := \{\theta \in [0, 2\pi) : \varphi_\varrho(e^{i\theta}) \in L_\varrho\}$$

(observe that T_ϱ is relatively closed in $[0, 2\pi)$) and let $m_\varrho := \mathcal{L}^1(T_\varrho)/(2\pi)$. Notice that $|\varphi_\varrho(e^{i\theta})| = r$ for $\theta \in T'_\varrho := [0, 2\pi) \setminus T_\varrho$. The function

$$\psi_\varrho(z) := \begin{cases} \varphi_\varrho(z)/z, & z \neq 0 \\ \varphi'_\varrho(0), & z = 0 \end{cases}$$

is holomorphic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. Moreover, $\psi_\varrho(z) \neq 0$, $z \in \overline{\mathbb{D}}$. In particular, $\log|\psi_\varrho|$ is harmonic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$ and, therefore,

$$\begin{aligned}\log \varphi'_\varrho(0) = \log|\psi_\varrho(0)| = \mathbf{J}(\log|\psi_\varrho|; 0, 1) = \mathbf{J}(\log|\varphi_\varrho|; 0, 1)\\ = \frac{1}{2\pi}\Big(\int_{T_\varrho} \log|\varphi_\varrho(e^{i\theta})|\, d\theta + \int_{T'_\varrho} \log|\varphi_\varrho(e^{i\theta})|\, d\theta\Big) \geq m_\varrho \log\varrho + (1 - m_\varrho)\log r.\end{aligned}$$

On the other hand, by the Koebe theorem (cf. [35]), since $\overline{K}(\varrho) \not\subset \varphi_\varrho(\mathbb{D})$, we get $\varphi'_\varrho(0) \leq 4\varrho$. Hence

$$4\varrho^{1-m_\varrho} \geq r^{1-m_\varrho},$$

and, consequently, $\lim_{\varrho\to 0} m_\varrho = 1$.

Since $u \circ \varphi_\varrho$ is subharmonic in $\mathbb{D}$ and upper semicontinuous in $\overline{\mathbb{D}}$, we get

$$u(0) \leq \mathbf{J}(u \circ \varphi_\varrho; 0, 1) = \frac{1}{2\pi}\Big(\int_{T_\varrho} u(\varphi_\varrho(e^{i\theta}))\, d\theta + \int_{T'_\varrho} u(\varphi_\varrho(e^{i\theta}))\, d\theta\Big) \leq m_\varrho A + (1 - m_\varrho)c,$$

where $c := \sup_{\overline{K}(r)} u$. Letting $\varrho \longrightarrow 0$ gives $u(0) \leq A$; contradiction □

Proposition 3.2.33. *Let $u \in \mathcal{C}^\uparrow(\Omega, \mathbb{R}_+)$. Then $\log u \in \mathcal{SH}(\Omega)$ (14) iff for every polynomial $p \in \mathcal{P}(\mathbb{C})$ the function $|e^p|u$ is subharmonic. In particular, if $\log u_1$, $\log u_2 \in \mathcal{SH}(\Omega)$, then $\log(u_1 + u_2) \in \mathcal{SH}(\Omega)$.*

Proof. $\Longrightarrow$. Let $v(z) := |e^{p(z)}|u(z)$, $z \in \Omega$. Then $\log v = \operatorname{Re} p + \log u$ and hence $\log v \in \mathcal{SH}(\Omega)$; therefore also $v \in \mathcal{SH}(\Omega)$.

$\Longleftarrow$. We use Proposition 3.2.7. Let $a \in \Omega$, $0 < r < d_\Omega(a)$ and let $p \in \mathcal{P}(\mathbb{C})$ be such that $\log u \leq \operatorname{Re} p$ on $C(a, r)$. Then $v := |e^{-p}|u \leq 1$ on $C(a, r)$. Since the function v is subharmonic, it follows from the maximum principle that $v \leq 1$ in $K(a, r)$, which means that $\log u \leq \operatorname{Re} p$ in $K(a, r)$. □

Proposition 3.2.33 can be generalized in the following way:

Proposition 3.2.34. *Let $u \in \mathcal{C}^\uparrow(\Omega, \mathbb{R}_+)$. Then $\log u \in \mathcal{SH}(\Omega)$ iff for every $a \in \mathbb{C}$ the function $|e^{az}|u(z)$ is subharmonic.*

Proof. It is clear that the problem is to prove $\Longleftarrow$. Suppose first that $u \in \mathcal{C}^2(\Omega, \mathbb{R}_{>0})$. It is sufficient to check that $\Delta \log u \geq 0$ in Ω. Note that

$$\Delta \log u = \frac{1}{u}\Big(\Delta u - \frac{(\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2}{u}\Big).$$

Let $a = \alpha + i\beta$ and put $v_a := |e^{az}|u$. Then

$$0 \leq \Delta v_a = |e^{az}|\Big(\Delta u + |a|^2 u + 2(\alpha\frac{\partial u}{\partial x} - \beta\frac{\partial u}{\partial y})\Big).$$

(14) That is u is *logarithmically subharmonic*.

Fix a $z_0 \in \Omega$ and put

$$\alpha := -\frac{\frac{\partial u}{\partial x}(z_0)}{u(z_0)}, \quad \beta := \frac{\frac{\partial u}{\partial y}(z_0)}{u(z_0)}.$$

Then

$$(\Delta \log u)(z_0) = \frac{|e^{-az_0}|}{u(z_0)} \Delta v_a(z_0) \geq 0.$$

Now consider the general case. Note that the function u is subharmonic (because $u = |e^{0z}|u$). Let $(u_\varepsilon)_{\varepsilon>0}$ denote the regularizations of the function u. Since $u_\varepsilon + \varepsilon \searrow u$, it suffices to show that $\log(u_\varepsilon + \varepsilon) \in \mathcal{SH}(\Omega_\varepsilon)$, $\varepsilon > 0$. Fix an $\varepsilon > 0$. In virtue of the first part of the proof it remains to show that $|e^{az}|u_\varepsilon \in \mathcal{SH}(\Omega_\varepsilon)$ for every $a \in \mathbb{C}$. Fix an $a \in \mathbb{C}$. Then

$$|e^{az}|u_\varepsilon(z) = \int_{\mathbb{D}} |e^{a(z+\varepsilon w)}|u(z+\varepsilon w)\Psi(w)|e^{-a\varepsilon w}|\, d\mathcal{L}^2(w), \quad z \in \Omega_\varepsilon.$$

Now we apply Corollary 3.2.26. $\square$

Proposition 3.2.35 (Schwarz type lemma)**.** *Let* $u : \mathbb{D} \longrightarrow [0,1]$ *be such that* $\log u \in \mathcal{SH}(\mathbb{D})$, $u(0) = 0$, *and*

$$\limsup_{\mathbb{D}_* \ni z \to 0} \frac{u(z)}{|z|} < +\infty.$$

Then

$$u(z) \leq |z|,\ z \in \mathbb{D}, \quad \textit{and} \quad \limsup_{\mathbb{D}_* \ni z \to 0} \frac{u(z)}{|z|} \leq 1.$$

Moreover, if

$$\exists_{z_0 \in \mathbb{D}_*} :\ u(z_0) = |z_0| \quad \textit{or} \quad \limsup_{\mathbb{D}_* \ni z \to 0} \frac{u(z)}{|z|} = 1,$$

then $u(z) = |z|$ *for all* $z \in \mathbb{D}$.

Proof. Let $v(z) := u(z)/|z|$, $z \in \mathbb{D}_*$. Since $\log v = \log u - \log|z|$, it follows that $\log v \in \mathcal{SH}(\mathbb{D}_*)$, and hence $v \in \mathcal{SH}(\mathbb{D}_*)$. By the assumption we conclude that the function v is locally bounded in $\mathbb{D}$. Hence, putting $v(0) := \limsup_{\mathbb{D}_* \ni z \to 0} v(z)$, and using Proposition 3.2.17, we obtain a function subharmonic in $\mathbb{D}$. By the maximum principle we get $v \leq 1$, which gives the required inequalities.

Moreover, if $v(z_0) = 1$ for some $z_0 \in \mathbb{D}$, then $v \equiv 1$. $\square$

Proposition 3.2.36. *Let* $D \subset \mathbb{C}$ *be a convex domain and let* $u : D \longrightarrow \mathbb{R}$ *be a convex function* (15) *. Then* $u \in \mathcal{SH}(D)$.

Proof. Since u is convex, it is also continuous (cf. [32]). Fix $a \in D$ and $0 < r < d_D(a)$. Then we have

$$\mathbf{J}(u; a, r) = \lim_{N\to+\infty} \sum_{j=1}^{N} \frac{1}{N} u(a + re^{i\frac{2\pi j}{N}}) \geq \lim_{N\to+\infty} u\Big(\sum_{j=1}^{N} \frac{1}{N}(a + re^{i\frac{2\pi j}{N}})\Big) = u(a).$$

It remains to apply Proposition 3.2.5. $\square$

Proposition 3.2.37 (Hadamard's three circles theorem)**.** *Let*

$$A := \{z \in \mathbb{C} : r_1 < |z| < r_2\}$$

($0 < r_1 < r_2 < +\infty$) *and let* $\log u \in \mathcal{SH}(A)$. *Assume that*

$$\limsup_{|z|\to r_j} u(z) \leq M_j, \quad j = 1, 2.$$

Then

$$u(z) \leq M_1^{\frac{\log \frac{r_2}{|z|}}{\log \frac{r_2}{r_1}}} M_2^{\frac{\log \frac{|z|}{r_1}}{\log \frac{r_2}{r_1}}}, \quad z \in A.$$

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(15) That is, $u(t_1z_1+\cdots+t_Nz_N) \leq t_1u(z_1)+\cdots+t_Nu(z_N)$ for any $z_1, \ldots, z_N \in D$ and $t_1, \ldots, t_N \geq 0$ with $t_1+\cdots+t_N = 1$; cf. Exercise 3.11.

Proof. For $\alpha \in \mathbb{R}$ put $u_\alpha(z) := |z|^\alpha u(z)$, $z \in A$. Observe that u_α is logarithmically subharmonic on A. Now, by the maximum principle (Corollary 3.2.4), we get

$$|z|^\alpha u(z) = u_\alpha(z) \le \max\{r_1^\alpha M_1, r_2^\alpha M_2\}, \quad z \in A.$$

Taking $\alpha \in \mathbb{R}$ so that $r_1^\alpha M_1 = r_2^\alpha M_2$, we get the required estimate. □

3.3. Pluriharmonic functions

Definition 3.3.1. Let Ω be an open subset of $\mathbb{C}^n$. A function $u \in \mathcal{C}^2(\Omega, \mathbb{R})$ is *pluriharmonic* on Ω ($u \in \mathcal{PH}(\Omega)$) if

$$\frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}(z) = 0, \quad z \in \Omega,\ j,k = 1, \dots, n. \tag{3.3.1}$$

Remark 3.3.2. (a) If $n = 1$, then $\mathcal{PH}(\Omega) = \mathcal{H}(\Omega)$ (cf. § 3.1).
(b) $\mathcal{PH}(\Omega)$ is a vector space.
(c) Condition (3.3.1) is equivalent to the following system of equations

$$\frac{\partial^2 u}{\partial x_j \partial y_k}(z) = \frac{\partial^2 u}{\partial x_k \partial y_j}(z), \quad \frac{\partial^2 u}{\partial x_j \partial x_k}(z) + \frac{\partial^2 u}{\partial y_j \partial y_k}(z) = 0, \quad z \in \Omega,\ j,k = 1, \dots, n.$$

In particular,

$$\frac{\partial^2 u}{\partial x_j^2}(z) + \frac{\partial^2 u}{\partial y_j^2}(z) = 0, \quad z \in \Omega,\ j = 1, \dots, n,$$

which shows that any function $u \in \mathcal{PH}(\Omega)$ is *separately harmonic* on Ω, i.e.

$$\mathcal{PH}(\Omega) \subset \mathcal{H}_s(\Omega) := \{u \in \mathcal{C}^2(\Omega, \mathbb{R}) : \forall_{a \in \Omega}\ \forall_{j \in \{1,\dots,n\}} :\ u_{a,e_j} \in \mathcal{H}(\Omega_{a,e_j})\}. \quad (^{16})$$

Obviously every separately harmonic function is harmonic as a function of $(2n)$-variables. Thus $\mathcal{PH}(\Omega) \subset \mathcal{H}_s(\Omega) \subset \mathcal{H}(\Omega)$. In particular, $\mathcal{PH}(\Omega) \subset \mathcal{C}^\infty(\Omega)$. Notice that for $n = 1$ we have $\mathcal{PH}(\Omega) = \mathcal{H}_s(\Omega) = \mathcal{H}(\Omega)$. If $n \ge 2$, then $\mathcal{PH}(\Omega) \subsetneq \mathcal{H}_s(\Omega) \subsetneq \mathcal{H}(\Omega)$ (17) .
(d) If $f = u + iv \in \mathcal{O}(\Omega)$, then $u \in \mathcal{PH}(\Omega)$.

Proposition 3.3.3. *If $D \subset \mathbb{C}^n$ is a star-shaped domain with respect to a point $a \in D$, then for any $u \in \mathcal{PH}(D)$ there exists an $f \in \mathcal{O}(D)$ such that $u = \operatorname{Re} f$.*

In particular, any pluriharmonic function is locally the real part of a holomorphic function (18) .

Proof. Observe that the function $z \longrightarrow u(z+a)$ is pluriharmonic on $D-a$ and the domain $D-a$ is star-shaped with respect to 0. Thus we may assume that $a = 0$. Define

$$v(z) := -i \int_0^1 \sum_{j=1}^n \Big(z_j \frac{\partial u}{\partial z_j}(tz) - \overline{z}_j \frac{\partial u}{\partial \overline{z}_j}(tz) \Big) dt, \quad z \in D.$$

Then $v \in \mathcal{C}^1(D)$ and using (3.3.1) we get

$$\frac{\partial(u+iv)}{\partial \overline{z}_k}(z) = \frac{\partial u}{\partial \overline{z}_k} + \int_0^1 \Big(\sum_{j=1}^n \Big(z_j \frac{\partial^2 u}{\partial \overline{z}_k \partial z_j}(tz)t - \overline{z}_j \frac{\partial^2 u}{\partial \overline{z}_k \partial \overline{z}_j}(tz)t \Big) - \frac{\partial u}{\partial \overline{z}_k}(tz) \Big) dt$$

$$= \frac{\partial u}{\partial \overline{z}_k} - \int_0^1 \Big(t \sum_{j=1}^n \Big(z_j \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}(tz) + \overline{z}_j \frac{\partial^2 u}{\partial \overline{z}_j \partial \overline{z}_k}(tz) \Big) + \frac{\partial u}{\partial \overline{z}_k}(tz) \Big) dt = \frac{\partial u}{\partial \overline{z}_k}(z) - \int_0^1 \frac{d}{dt}\Big(t \frac{\partial u}{\partial \overline{z}_k}(tz) \Big) dt = 0,$$

$$k = 1, \dots, n. \quad \square$$

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(16) One can prove (but it is much more difficult) that if $u : \Omega \longrightarrow \mathbb{R}$ is such that $u_{a,e_j} \in \mathcal{H}(\Omega_{a,e_j})$ for any $a \in \Omega$ and $j = 1, \dots, n$, then $u \in \mathcal{C}^2(\Omega, \mathbb{R})$.

(17) Let $u_1(z_1, z_2) = u_1(x_1, y_1, x_2, y_2) := x_1 x_2$, $u_2(z_1, z_2) = u_2(x_1, y_1, x_2, y_2) := x_1^2 - x_2^2$. Then $u_1 \in \mathcal{H}_s(\mathbb{C}^2) \setminus \mathcal{PH}(\mathbb{C}^2)$, $u_2 \in \mathcal{H}(\mathbb{C}^2) \setminus \mathcal{H}_s(\mathbb{C}^2)$.

(18) We have got another proof of the inclusion $\mathcal{PH}(\Omega) \subset \mathcal{C}^\infty(\Omega)$.

Corollary 3.3.4. *Let $\Omega_j \subset \mathbb{C}^{n_j}$ be open, $j = 1, 2$, and let $F \in \mathcal{O}(\Omega_1, \Omega_2)$. Then $u \circ F \in \mathcal{PH}(\Omega_1)$ for any $u \in \mathcal{PH}(\Omega_2)$.*

For an arbitrary function $u \in \mathcal{C}^2(\Omega)$ denote by $\mathcal{L}u : \Omega \times \mathbb{C}^n \longrightarrow \mathbb{C}$ the *Levi form* (called also the *complex Hessian*) of u, i.e.

$$(\mathcal{L}u)(a; X) := \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}(a) X_j \overline{X}_k.$$

Observe that

$$(\mathcal{L}u)(a; X) = \frac{\partial^2 u_{a,X}}{\partial \lambda \partial \overline{\lambda}}(0).$$

In particular, if $n = 1$, then

$$(\mathcal{L}u)(a; X) = \frac{1}{4}(\Delta u(a))|X|^2.$$

Remark 3.3.5. For a function $u \in \mathcal{C}^2(\Omega, \mathbb{R})$ the following conditions are equivalent:
(i) $u \in \mathcal{PH}(\Omega)$;
(ii) $u_{a,X} \in \mathcal{H}(\Omega_{a,X})$ for any $a \in \Omega$ and $X \in \mathbb{C}^n$;
(iii) $\mathcal{L}u(a; X) = 0$ for any $a \in \Omega$ and $X \in \mathbb{C}^n$.

Let $u : \partial_0 \mathbb{P}(a, \boldsymbol{r}) \longrightarrow [-\infty, +\infty)$ be bounded from above and measurable (19) . For $z = (z_1, \dots, z_n) \in \mathbb{P}(a, \boldsymbol{r})$, $a = (a_1, \dots, a_n)$, $\boldsymbol{r} = (r_1, \dots, r_n)$, define

$$\mathbf{P}(u; a, \boldsymbol{r}; z) = \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{r_1^2 - |z_1 - a_1|^2}{|r_1 e^{i\theta_1} - (z_1 - a_1)|^2} \cdots \frac{r_n^2 - |z_n - a_n|^2}{|r_n e^{i\theta_n} - (z_n - a_n)|^2} \times u(a_1 + r_1 e^{i\theta_1}, \dots, a_n + r_n e^{i\theta_n})\, d\theta_1 \dots d\theta_n.$$

Remark 3.3.6. $\mathbf{P}(u; a, \boldsymbol{r}; \cdot) \in \mathcal{H}_s(\mathbb{P}(a, \boldsymbol{r}))$.

For any affine $\mathbb{C}$-isomorphism $L : \mathbb{C}^n \longrightarrow \mathbb{C}^n$, put

$$\Omega_L := L^{-1}(\Omega), \qquad u_L(z) := u(L(z)),\ z \in \Omega_L.$$

By Corollary 3.3.4 we have $u \in \mathcal{PH}(\Omega)$ iff $u_L \in \mathcal{PH}(\Omega_L)$.

Proposition 3.3.7. *For $u \in \mathcal{C}(\Omega, \mathbb{R})$ the following conditions are equivalent:*
(i) *$u \in \mathcal{PH}(\Omega)$;*
(ii) *for every affine isomorphism L and for every $a \in \Omega_L$ there exists an $R(a)$, $0 < R(a) \le d_{\Omega_L}(a)$, such that for any $\boldsymbol{r} = (r_1, \dots, r_n)$ with $0 < r_j < R(a)$, $j = 1, \dots, n$, we have*

$$u_L(z) = \mathbf{P}(u_L; a, \boldsymbol{r}; z), \quad z \in \mathbb{P}(a, \boldsymbol{r}). \tag{3.3.2}$$

Moreover, if $u \in \mathcal{PH}(\Omega)$, then (3.3.2) holds for any $\mathbb{P}(a, \boldsymbol{r}) \subset\subset \Omega_L$.

Proof. Assume that $u \in \mathcal{PH}(\Omega)$. Fix an affine isomorphism L. Recall that $u_L \in \mathcal{PH}(\Omega_L) \subset \mathcal{H}_s(\Omega)$. Take a $\mathbb{P}(a, \boldsymbol{r}) \subset\subset \Omega_L$. Then, by the Poisson formula (Proposition 3.1.10), for $z = (z_1, \dots, z_n) \in \mathbb{P}(a, \boldsymbol{r})$ we have

$$\begin{aligned} u_L(z) &= \mathbf{P}(u_L(\cdot, z_2, \dots, z_n); a_1, r_1; z_1) = \frac{1}{2\pi} \int_0^{2\pi} \frac{r_1^2 - |z_1 - a_1|^2}{|r_1 e^{i\theta_1} - (z_1 - a_1)|^2}\, u_L(a_1 + r_1 e^{i\theta_1}, z_2, \dots, z_n)\, d\theta_1 \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{r_1^2 - |z_1 - a_1|^2}{|r_1 e^{i\theta_1} - (z_1 - a_1)|^2}\, \frac{1}{2\pi} \int_0^{2\pi} \frac{r_2^2 - |z_2 - a_2|^2}{|r_2 e^{i\theta_2} - (z_2 - a_2)|^2} u_L(a_1 + r_1 e^{i\theta_1}, a_2 + r_2 e^{i\theta_2}, z_3, \dots, z_n)\, d\theta_2 d\theta_1 \\ &= \dots = \mathbf{P}(u_L; a, \boldsymbol{r}; z). \end{aligned}$$

Conversely, assume that (ii) is satisfied. In particular, $u \in \mathcal{C}^2(\Omega, \mathbb{R})$. By Remark 3.3.5 it suffices to prove that $u_{a,X} \in \mathcal{H}(\Omega_{a,X})$ for any $a \in \Omega$ and $X \in \mathbb{C}^n$. Fix a and $X \neq 0$. Let L be an affine isomorphism such that $u_{a,X} = u_L(0', \cdot)$. Now, by virtue of (3.3.2) and Remark 3.3.6, $u_L(0', \cdot)$ is harmonic. $\square$

(19) That is, the function $[0, 2\pi)^n \ni (\theta_1, \dots, \theta_n) \longrightarrow u(a_1 + r_1 e^{i\theta_1}, \dots, a_n + r_n e^{i\theta_n})$ is Lebesgue measurable.

3.4. Plurisubharmonic functions

Definition 3.4.1. Let $\Omega \subset \mathbb{C}^n$ be open. A function $u : \Omega \longrightarrow [-\infty, +\infty)$ is called *plurisubharmonic* (shortly *psh*) in Ω ($u \in \mathcal{PSH}(\Omega)$) if:
- $u \in \mathcal{C}^{\uparrow}(\Omega)$,
- for every $a \in \Omega$ and $X \in \mathbb{C}^n$ the function

$$\Omega_{a,X} \ni \lambda \overset{u_{a,X}}{\longrightarrow} u(a + \lambda X)$$

is subharmonic.

Notice that the function $u \equiv -\infty$ is psh.

The properties of psh functions mentioned just below follow directly from the definition and corresponding properties of subharmonic functions.

Proposition 3.4.2. $\mathcal{PH}(\Omega) \subset \mathcal{PSH}(\Omega)$, $\mathcal{PSH}(\Omega) + \mathcal{PSH}(\Omega) = \mathcal{PSH}(\Omega)$, $\mathbb{R}_{>0} \cdot \mathcal{PSH}(\Omega) = \mathcal{PSH}(\Omega)$.

Proposition 3.4.3. *Plurisubharmonicity is a local property, i.e. a function $u : \Omega \longrightarrow [-\infty, +\infty)$ is psh in Ω iff every point $a \in \Omega$ admits an open neighborhood $U_a \subset \Omega$ such that $u|_{U_a} \in \mathcal{PSH}(U_a)$.*

Proposition 3.4.4. *Let $u : \Omega \longrightarrow [-\infty, +\infty)$ be upper semicontinuous. Then $u \in \mathcal{PSH}(\Omega)$ iff for any $a \in \Omega$, $X \in \mathbb{C}^n$, and $r > 0$ such that $a + r\overline{\mathbb{D}} \cdot X \subset \Omega$ we have*

$$u(a) \leq \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\theta} X)\, d\theta.$$

Proposition 3.4.5. *Let $f \in \mathcal{O}(\Omega)$. Then $\log |f| \in \mathcal{PSH}(\Omega)$.*

Proposition 3.4.6. *Let $I \subset \mathbb{R}$ be an open interval and let $\varphi : I \longrightarrow \mathbb{R}$ be convex and non–decreasing. Then for every psh function $u : \Omega \longrightarrow I$, the function $\varphi \circ u$ is psh. In particular,*
— if $u \in \mathcal{PSH}(\Omega)$, then $e^u \in \mathcal{PSH}(\Omega)$,
— if $u : \Omega \longrightarrow \mathbb{R}_+$ is psh, then for every $p \geq 1$, the function u^p is psh in Ω.

Proposition 3.4.7. *If $\log u_1$, $\log u_2 \in \mathcal{PSH}(\Omega)$, then $\log(u_1 + u_2) \in \mathcal{PSH}(\Omega)$.* ([20])

Proposition 3.4.8. *If $(u_\nu)_{\nu=1}^{\infty} \subset \mathcal{PSH}(\Omega)$ and $u_\nu \searrow u$, then $u \in \mathcal{PSH}(\Omega)$.*

Proposition 3.4.9. *If $(u_\nu)_{\nu=1}^{\infty} \subset \mathcal{PSH}(\Omega)$ and $u_\nu \longrightarrow u$ locally uniformly in Ω, then $u \in \mathcal{PSH}(\Omega)$.*

Proposition 3.4.10 (Liouville type theorem)**.** *If a function $u \in \mathcal{PSH}(\mathbb{C}^n)$ is globally bounded from above, then $u \equiv$ const.*

Proposition 3.4.11. *Let $u \in \mathcal{C}^2(\Omega, \mathbb{R})$. Then $u \in \mathcal{PSH}(\Omega)$ iff $(\mathcal{L}u)(a; X) \geq 0$ for any $a \in \Omega$ and $X \in \mathbb{C}^n$; cf. Proposition 3.3.5.*

Proposition 3.4.12. *Let $G \subset \Omega$ be open subsets of $\mathbb{C}^n$ and let $v \in \mathcal{PSH}(G)$, $u \in \mathcal{PSH}(\Omega)$. Assume that*

$$\limsup_{G \ni z \to \zeta} v(z) \leq u(\zeta), \quad \zeta \in (\partial G) \cap \Omega.$$

Let

$$\widetilde{u}(z) := \begin{cases} \max\{v(z), u(z)\}, & z \in G \\ u(z), & z \in \Omega \setminus G \end{cases}.$$

Then $\widetilde{u} \in \mathcal{PSH}(\Omega)$.

Proof. Exercise — cf. the proof of Proposition 3.2.11 (use Proposition 3.4.4). □

([20]) If $\log u \in \mathcal{PSH}(\Omega)$, then u is called *logarithmically plurisubharmonic*.

We move now to more advanced properties of psh functions. Recall that the most part of the properties of subharmonic functions follows from Propositions 3.2.5, 3.2.12, and 3.2.15, i.e. from characterization of subharmonic functions by different mean value theorems. We try to obtain a similar characterization for psh functions.

Let $u : \partial_0\mathbb{P}(a,\boldsymbol{r}) \longrightarrow [-\infty,+\infty)$ be bounded from above and measurable. Define

$$\mathbf{J}(u;a,\boldsymbol{r}) := \mathbf{P}(u;a,\boldsymbol{r};a).$$

If $u : \mathbb{P}(a,\boldsymbol{r}) \longrightarrow [-\infty,+\infty)$ is bounded from above and measurable, $a = (a_1,\dots,a_n) \in \mathbb{C}^n$, and $\boldsymbol{r} = (r_1,\dots,r_n) \in (\mathbb{R}_{>0})^n$, then we put

$$\mathbf{A}(u;a,\boldsymbol{r}) := \frac{1}{(\pi r_1^2)\dots(\pi r_n^2)} \int_{\mathbb{P}(a,\boldsymbol{r})} u\, d\mathcal{L}^{2n}.$$

Note that $u \in \mathcal{PSH}(\Omega)$ iff $u_L \in \mathcal{PSH}(\Omega_L)$ for an arbitrary affine $\mathbb{C}$-isomorphism $L : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ $\left({}^{21}\right)$.

Proposition 3.4.13. *Let $u : \Omega \longrightarrow [-\infty,+\infty)$ be upper semicontinuous. Then the following conditions are equivalent:*

(i) $u \in \mathcal{PSH}(\Omega)$;

(ii) *for every affine isomorphism L and every $a \in \Omega_L$, there exists an $R(a)$, $0 < R(a) \le d_{\Omega_L}(a)$, such that for every $\boldsymbol{r} = (r_1,\dots,r_n)$, $0 < r_j < R(a)$, $j = 1,\dots,n$, we have*

$$u_L(z) \le \mathbf{P}(u_L;a,\boldsymbol{r};z), \quad z \in \mathbb{P}(a,\boldsymbol{r});$$

(iii) *for every affine isomorphism L and for every $a \in \Omega_L$, there exists an $R(a)$, $0 < R(a) \le d_{\Omega_L}(a)$, such that for every $\boldsymbol{r} = (r_1,\dots,r_n)$, $0 < r_j < R(a)$, $j = 1,\dots,n$, we have*

$$u_L(a) \le \mathbf{J}(u_L;a,\boldsymbol{r});$$

(iv) *for every affine isomorphism L and for every $a \in \Omega_L$, there exists an $R(a)$, $0 < R(a) \le d_{\Omega_L}(a)$, such that for every $\boldsymbol{r} = (r_1,\dots,r_n)$, $0 < r_j < R(a)$, $j = 1,\dots,n$, we have*

$$u_L(a) \le \mathbf{A}(u_L;a,\boldsymbol{r}).$$

Moreover, if $u \in \mathcal{PSH}(\Omega)$, then the inequalities in (ii), (iii), *and* (iv) *hold for every $\boldsymbol{r} = (r_1,\dots,r_n)$ such that $\overline{\mathbb{P}}(a,\boldsymbol{r}) \subset \Omega_L$.*

Proof. (i) $\Longrightarrow$ (ii). Fix L, a, $\boldsymbol{r} = (r_1,\dots r_n)$ such that $\overline{\mathbb{P}}(a,\boldsymbol{r}) \subset \Omega_L$, and $z = (z_1,\dots,z_n) \in \mathbb{P}(a,\boldsymbol{r})$. Since $u_L \in \mathcal{PSH}(\Omega_L)$, we obtain (applying n-times Proposition 3.2.12)

$$\begin{aligned}
&u_L(z) \le \mathbf{P}(u_L(\cdot,z_2,\dots,z_n);a_1,r_1;z_1) \le \frac{1}{2\pi}\int_0^{2\pi} \frac{r_1^2-|z_1-a_1|^2}{|r_1e^{i\theta_1}-(z_1-a_1)|^2}\, u_L(a_1+r_1e^{i\theta_1},z_2,\dots,z_n)\,d\theta_1\\
&\le \frac{1}{2\pi}\int_0^{2\pi} \frac{r_1^2-|z_1-a_1|^2}{|r_1e^{i\theta_1}-(z_1-a_1)|^2}\, \frac{1}{2\pi}\int_0^{2\pi} \frac{r_2^2-|z_2-a_2|^2}{|r_2e^{i\theta_2}-(z_2-a_2)|^2} u_L(a_1+r_1e^{i\theta_1},a_2+r_2e^{i\theta_2},z_3,\dots,z_n)\,d\theta_2 d\theta_1\\
&\le \dots \le \mathbf{P}(u_L;a,\boldsymbol{r};z).
\end{aligned}$$

The implication (ii) $\Longrightarrow$ (iii) is evident.

(iii) $\Longrightarrow$ (iv).

$$\begin{aligned}
u_L(a) &= \frac{2}{r_1^2}\dots\frac{2}{r_n^2}\int_0^{r_1}\dots\int_0^{r_n} u_L(a)\tau_1\dots\tau_n\,d\tau_1\dots d\tau_n\\
&\le \frac{2}{r_1^2}\dots\frac{2}{r_n^2}\int_0^{r_1}\dots\int_0^{r_n} \mathbf{J}(u_L;a,(\tau_1,\dots,\tau_n))\tau_1\dots\tau_n\,d\tau_1\dots d\tau_n = \mathbf{A}(u_L;a,\boldsymbol{r}).
\end{aligned}$$

$\left({}^{21}\right)$ Recall that $\Omega_L := L^{-1}(\Omega)$, $u_L := u \circ L$.

(iv) $\Longrightarrow$ (i). Fix $a \in \Omega$, $X \in \mathbb{C}^n$, $\|X\| = 1$. It is sufficient to show that $u(a) \leq \mathbf{A}(u_{a,X}; 0, r)$ for $r > 0$ sufficiently small. Let L be an affine isometry such that $L(a + \lambda e_n) = a + \lambda X$, $\lambda \in \mathbb{C}$. By Fatou's lemma, for $r_n > 0$ sufficiently small, we have

$$\begin{aligned}
&u(a) = u_L(a) \leq \limsup_{r_1,\dots,r_{n-1}\to 0} \mathbf{A}(u_L; a, (r_1, \dots, r_{n-1}, r_n)) \\
&= \limsup_{r_1,\dots,r_{n-1}\to 0} \frac{1}{\pi^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \int_0^1 \cdots \int_0^1 u_L(a_1 + t_1 r_1 e^{i\theta_1}, \dots, a_n + t_n r_n e^{i\theta_n}) t_1 \dots t_n \, dt_1 \dots dt_n \, d\theta_1 \dots d\theta_n \\
&\leq \frac{1}{\pi^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \int_0^1 \cdots \int_0^1 \limsup_{r_1,\dots,r_{n-1}\to 0} u_L(a_1 + t_1 r_1 e^{i\theta_1}, \dots, a_n + t_n r_n e^{i\theta_n}) t_1 \dots t_n \, dt_1 \dots dt_n \, d\theta_1 \dots d\theta_n \\
&\leq \frac{1}{\pi^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \int_0^1 \cdots \int_0^1 u_L(a_1, \dots, a_{n-1}, a_n + t_n r_n e^{i\theta_n}) t_1 \dots t_n \, dt_1 \dots dt_n \, d\theta_1 \dots d\theta_n \\
&= \frac{1}{\pi} \int_0^{2\pi} \int_0^1 u_L(a_1, \dots, a_{n-1}, a_n + t_n r_n e^{i\theta_n}) t_n \, dt_n \, d\theta_n = \mathbf{A}(u_{a,X}; 0, r_n).
\end{aligned}$$

□

Propositions 3.4.13 and 3.3.7 imply the following corollary (cf. Corollary 3.2.13).

Corollary 3.4.14. $\mathcal{PSH}(\Omega) \cap (-\mathcal{PSH}(\Omega)) = \mathcal{PH}(\Omega)$.

Proposition 3.4.15. *Let $D \subset \mathbb{C}^n$ be a domain and let $u \in \mathcal{PSH}(D)$, $u \not\equiv -\infty$. Then $u \in L^1(D, \mathrm{loc})$. Consequently, $\mathcal{L}^{2n}(u^{-1}(-\infty)) = 0$.*

In particular, if M is a thin subset of an open set $\Omega \subset \mathbb{C}^n$ (cf. Definition 2.1.4), then $\mathcal{L}^{2n}(M) = 0$.

Proof. It is sufficient to apply the method of the proof of Proposition 3.2.16 and use Proposition 3.4.13(iv) (with $L = \mathrm{id}$). □

Proposition 3.4.16 (Maximum principle)**.** *Let $D \subset \mathbb{C}^n$ be a domain and let $u \in \mathcal{PSH}(D)$, $u \not\equiv \mathrm{const}$. Then u does not attain its global maximum in D. If, moreover, D is bounded, then*

$$u(z) < \sup_{\zeta \in \partial D} \{\limsup_{D \ni z \to \zeta} u(z)\}, \quad z \in D.$$

Proof. Exercise — cf. the proof of Lemma 3.2.3. □

Proposition 3.4.17. *If a family $(u_\iota)_{\iota \in I} \subset \mathcal{PSH}(\Omega)$ is locally bounded from above, then the function*

$$u := (\sup_{\iota \in I} u_\iota)^*$$

is psh in Ω.

In particular, $\max\{u_1, \dots, u_N\} \in \mathcal{PSH}(\Omega)$ *for any $u_1, \dots, u_N \in \mathcal{PSH}(\Omega)$.*

Proof. We use Proposition 3.4.13(ii). Fix L, $a \in \Omega_L$, and $r = (r_1, \dots, r_n)$, $0 < r_j < d_{\Omega_L}(a)$, $j = 1, \dots, n$. Note that $u_L = (\sup_{\iota \in I} (u_\iota)_L)^*$. For every $\iota \in I$ we have $(u_\iota)_L(z) \leq \mathbf{P}((u_\iota)_L; a, \boldsymbol{r}, z)$, $z \in \mathbb{P}(a, \boldsymbol{r})$, and consequently, $\sup_{\iota \in I} (u_\iota)_L(z) \leq \mathbf{P}(u_L; a, \boldsymbol{r}; z)$, $z \in \mathbb{P}(a, \boldsymbol{r})$. Now it is sufficient to observe that the right-hand side is a continuous function of the variable z, and hence $u_L(z) \leq \mathbf{P}(u_L; a, \boldsymbol{r}; z)$, $z \in \mathbb{P}(a, \boldsymbol{r})$. □

Proposition 3.4.18. *If a sequence $(u_\nu)_{\nu=1}^\infty \subset \mathcal{PSH}(\Omega)$ is locally bounded from above, then the function*

$$u := (\limsup_{\nu \to \infty} u_\nu)^*$$

is psh on Ω.

Proof. Exercise — cf. the proof of Proposition 3.2.14. □

Proposition 3.4.19 (Removable singularities)**.** *Let $D \subset \mathbb{C}^n$ be a domain, and let $M \subset D$ be a closed subset of D such that for every point $a \in M$ there exist a connected open neighborhood $U_a \subset D$ and a function $v_a \in \mathcal{PSH}(U_a)$, $v_a \not\equiv -\infty$, such that $M \cap U_a = v_a^{-1}(-\infty)$. Assume also that $u \in \mathcal{PSH}(D \setminus M)$ is an arbitrary function locally bounded from above in D. Let*

$$\widetilde{u}(z) := \limsup_{D \setminus M \ni z' \to z} u(z'), \quad z \in D.$$

Then $\widetilde{u} \in \mathcal{SH}(D)$. In particular, the set $D \setminus M$ is connected.

Proof. Exercise — cf. the proof of Proposition 3.2.17. □

Proposition 3.4.20 (Hartogs lemma)**.** *Let $(u_\nu)_{\nu=1}^\infty \subset \mathcal{PSH}(\Omega)$ be a sequence locally bounded from above. Assume that for some $m \in \mathbb{R}$*

$$\limsup_{\nu \to +\infty} u_\nu \le m.$$

Then for every compact subset $K \subset \Omega$ and for every $\varepsilon > 0$, there exists a ν_0 such that

$$\max_K u_\nu \le m + \varepsilon, \quad \nu \ge \nu_0.$$

Proof. Exercise; cf. the proof of Proposition 3.2.20 (use Proposition 3.4.13(iv)).

□

Proposition 3.4.21. *Let $u \in \mathcal{PSH}(\Omega)$, $a \in \Omega$. Then*

$$\mathbf{J}(u; a, \boldsymbol{r}') \le \mathbf{J}(u; a, \boldsymbol{r}'') \text{ for } \boldsymbol{r}' \le \boldsymbol{r}'' \text{ and } \mathbf{J}(u; a, \boldsymbol{r}) \longrightarrow u(a) \text{ when } \boldsymbol{r} \longrightarrow 0,$$
$$\mathbf{A}(u; a, \boldsymbol{r}') \le \mathbf{A}(u; a, \boldsymbol{r}'') \text{ for } \boldsymbol{r}' \le \boldsymbol{r}'' \text{ and } \mathbf{A}(u; a, \boldsymbol{r}) \longrightarrow u(a) \text{ when } \boldsymbol{r} \longrightarrow 0.$$

Proof. Exercise — cf. Proposition 3.2.22. □

Corollary 3.4.22. *Let $u_1, u_2 \in \mathcal{PSH}(\Omega)$. If $u_1 = u_2$ $\mathcal{L}^{2n}$-almost everywhere in Ω, then $u_1 \equiv u_2$ in Ω.*

Corollary 3.4.23. *Let D and M be as in Proposition 3.4.19. Then for every function $u \in \mathcal{PSH}(D)$ we have*

$$u(z) := \limsup_{D \setminus M \ni z' \to z} u(z'), \quad z \in D.$$

Let $\Phi(z_1, \dots, z_n) := \Psi(z_1) \cdot \dots \cdot \Psi(z_n)$, $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, where Ψ is a regularization function from § 3.2. Put

$$\Phi_\varepsilon(z) := \frac{1}{\varepsilon^{2n}} \Phi(\frac{z}{\varepsilon}), \quad z \in \mathbb{C}^n, \ \varepsilon > 0.$$

Notice that:
- $\Phi_\varepsilon \in \mathcal{C}_0^\infty(\mathbb{C}^n, \mathbb{R}_+)$,
- $\operatorname{supp} \Phi_\varepsilon = \overline{\mathbb{P}}(\varepsilon)$,
- $\Phi_\varepsilon(z_1, \dots, z_n) = \Phi_\varepsilon(|z_1|, \dots, |z_n|)$, $z = (z_1, \dots, z_n) \in \mathbb{C}^n$,
- $\int_{\mathbb{C}^n} \Phi_\varepsilon d\mathcal{L}^{2n} = 1$.

For every function $u \in L^1(\Omega, \mathrm{loc})$, define

$$u_\varepsilon(z) := \int_\Omega u(w) \Phi_\varepsilon(z - w)\, d\mathcal{L}^{2n}(w) = \int_{\mathbb{D}^n} u(z + \varepsilon w) \Phi(w)\, d\mathcal{L}^{2n}(w), \quad z \in \Omega_\varepsilon := \{z \in \Omega : d_\Omega(z) > \varepsilon\}.$$

The function u_ε is called the *ε-regularization* of u.

Proposition 3.4.24. *If $u \in \mathcal{PSH}(\Omega) \cap L^1(\Omega, \mathrm{loc})$, then $u_\varepsilon \in \mathcal{PSH}(\Omega_\varepsilon) \cap \mathcal{C}^\infty(\Omega_\varepsilon)$ and $u_\varepsilon \searrow u$ in Ω when $\varepsilon \searrow 0$.*

Proof. We have $u_\varepsilon \in \mathcal{C}^\infty(\Omega_\varepsilon)$. The monotonicity and convergence follow from the identity

$$u_\varepsilon(a) = (2\pi)^n \int_0^1 \cdots \int_0^1 \mathbf{J}(u; a, \varepsilon(\tau_1, \dots, \tau_n))\Phi(\tau_1, \dots, \tau_n)\tau_1 \dots \tau_n \, d\tau_1 \dots d\tau_n$$

and from Proposition 3.4.21. To show that the function u_ε is psh, we use Proposition 3.4.4. Fix $a \in \Omega_\varepsilon$, $X \in \mathbb{C}^n$, and $r > 0$ such that $a + r \cdot \overline{\mathbb{D}} \cdot X \subset \Omega_\varepsilon$. Then

$$\frac{1}{2\pi}\int_0^{2\pi} u_\varepsilon(a + re^{i\theta}X)\, d\theta = \int_{\mathbb{D}^n} \Big(\frac{1}{2\pi}\int_0^{2\pi} u(a + re^{i\theta}X + \varepsilon w)\, d\theta\Big)\Phi(w)\, d\mathcal{L}^{2n}(w)$$
$$\geq \int_{\mathbb{D}^n} u(a + \varepsilon w)\Phi(w)\, d\mathcal{L}^{2n}(w) = u_\varepsilon(a). \qquad \square$$

Proposition 3.4.25. *Let Ω_j be an open subset of $\mathbb{C}^{n_j}$, $j = 1, 2$, and suppose that $F : \Omega_1 \longrightarrow \Omega_2$ is a holomorphic mapping. Then for any $u \in \mathcal{PSH}(\Omega_2)$, the function $u \circ F$ is in $\mathcal{PSH}(\Omega_1)$.*

Proof. Exercise — cf. the proof of Proposition 3.2.29; we consider first the case $u \in \mathcal{C}^2(\Omega)$ and use the formula

$$(\mathcal{L}(u \circ F))(a; X) = (\mathcal{L}u)(F(a); F'(a)(X)).$$

In the general case we apply the regularization (Proposition 3.4.24). $\square$

Proposition 3.4.26. *If $q : \mathbb{C}^n \longrightarrow \mathbb{R}_+$ is a $\mathbb{C}$-seminorm, then $\log q \in \mathcal{PSH}(\mathbb{C}^n)$.*

Proof. Given arbitrary a, $X \in \mathbb{C}^n$, the function $\mathbb{C} \ni \lambda \longrightarrow q(a + \lambda X)$ is convex, and hence subharmonic (Proposition 3.2.36). This means that $q \in \mathcal{PSH}(\mathbb{C}^n)$. Moreover, for every polynomial $p \in \mathcal{P}(\mathbb{C})$ we have

$$v(\lambda) := |e^{p(\lambda)}|q(a + \lambda X) = q(e^{p(\lambda)}(a + \lambda X))$$

and so, by Proposition 3.4.25, $v \in \mathcal{SH}(\mathbb{C})$. Consequently, by Proposition 3.2.33, $\log q \in \mathcal{PSH}(\mathbb{C}^n)$. $\square$

Proposition 3.4.27. *Let $h : \mathbb{C}^n \longrightarrow \mathbb{R}_+$ be such that $h(\lambda z) = |\lambda| h(z)$, $\lambda \in \mathbb{C}, z \in \mathbb{C}^n$. Then h is psh in $\mathbb{C}^n$ iff $\log h$ is psh in $\mathbb{C}^n$.*

Proof. Exercise — cf. the proof of Proposition 3.4.26. $\square$

Proposition 3.4.28. *If $u \in \mathcal{PSH}(D)$ (D is a domain in $\mathbb{C}^n$), $u \not\equiv -\infty$, then $\mathcal{L}u \geq 0$ in D in the distribution sense, i.e. for every $\varphi \in \mathcal{C}_0^\infty(D, \mathbb{R}_+)$ we have*

$$\int_D u(z)(\mathcal{L}\varphi)(z; X)\, d\mathcal{L}^{2n}(z) \geq 0, \quad X \in \mathbb{C}^n.$$

Conversely, if $u \in L^1(D, \mathrm{loc})$ is such that $\mathcal{L}u \geq 0$ in D in the distribution sense, then there exists a function $v \in \mathcal{PSH}(D)$ such that $u = v$ $\mathcal{L}^{2n}$-almost everywhere in D.

Proof. Exercise — cf. the proof of Proposition 3.2.28. $\square$

Definition 3.4.29. A function $u \in \mathcal{C}(\Omega, \mathbb{R})$ is called *strictly plurisubharmonic* if for every domain $D \subset\subset \Omega$ there exists an $\varepsilon > 0$ such that the function $D \ni z \longrightarrow u(z) - \varepsilon\|z\|^2$ is psh.

Proposition 3.4.30. *A function $u \in \mathcal{C}^2(\Omega)$ is strictly psh iff*

$$(\mathcal{L}u)(a; X) > 0, \quad a \in \Omega,\ X \in \mathbb{C}^n,\ X \neq 0.$$

Proof. $\Longleftarrow$. Fix a domain $D \subset\subset \Omega$ and let

$$\varepsilon := \min\{(\mathcal{L}u)(a; X) : a \in D,\ \|X\| = 1\}, \quad v(z) := u(z) - \varepsilon\|z\|^2,\ z \in D.$$

Then

$$(\mathcal{L}v)(a; X) = (\mathcal{L}u)(a; X) - \varepsilon\|X\|^2 \geq 0, \quad a \in \Omega,\ X \in \mathbb{C}^n,$$

and so $v \in \mathcal{PSH}(D)$ (Proposition 3.4.11).

$\Longrightarrow$. Fix an $a \in \Omega$ and let $D \subset\subset \Omega$, $\varepsilon > 0$ be such that $a \in D$ and the function $v(z) = u(z) - \varepsilon\|z\|^2$, $z \in D$, is psh in D. Then

$$(\mathcal{L}u)(a; X) = (\mathcal{L}v)(a; X) + \varepsilon\|X\|^2 \geq \varepsilon\|X\|^2 > 0, \quad X \neq 0. \qquad \square$$

Remark 3.4.31. In the case where $u \in \mathcal{PSH}(\Omega)$ is continuous Proposition 3.4.24 may be generalized in the following way (cf. [27]):

If $u \in \mathcal{PSH}(\Omega)$ is continuous, then for any continuous function $\eta : \Omega \longrightarrow \mathbb{R}_{>0}$ there exists a strictly psh function $v \in \mathcal{C}^\infty(\Omega)$ such that $u \leq v \leq u + \eta$ on Ω.

Proposition 3.4.24 may be also 'globalized' in the case where Ω is a region of holomorphy (cf. [7]):

If Ω is a region of holomorphy, then for any $u \in \mathcal{PSH}(\Omega)$ there exists a sequence $(u_\nu)_{\nu=1}^\infty \subset \mathcal{PSH}(\Omega) \cap \mathcal{C}^\infty(\Omega)$ such that $u_\nu \searrow u$ pointwise on Ω.

Definition 3.4.32. We say that a bounded domain $D \subset \mathbb{C}^n$ is *hyperconvex* if there exists a psh continuous function $u : D \longrightarrow (-\infty, 0)$ such that

$\{z \in D : u(z) \leq t\} \subset\subset D$ for any $t < 0$. (*)

We say that D is *weakly hyperconvex* if there exists a $u : D \longrightarrow [-\infty, 0)$ with (*).

It is clear that any hyperconvex domain is weakly hyperconvex.

Proposition 3.4.33. *Let D be a bounded domain in $\mathbb{C}^n$. Then D is hyperconvex iff D is weakly hyperconvex.*

Proof. (Cf. [36].) Let $u : D \longrightarrow [-\infty, 0)$ be a psh function with (*). We will construct a continuous psh function $v_0 : D \longrightarrow (-\infty, 0)$ with (*).

Fix a ball $K := \overline{\mathbb{B}}(a, r) \subset D$ and let

$$h_{D,K}(z) := \sup\{h(z) : h \in \mathcal{PSH}(D),\ h \leq 1,\ h|_K \leq 0\}, \quad z \in D.$$

Obviously $0 \leq h_{D,K} \leq 1$ and $h_{D,K} = 0$ on K. Put $v := h^*_{D,K}$. It is clear that $v = 0$ in $\mathbb{B}(a, r)$. Observe that $v \in \mathcal{PSH}(D)$ (Proposition 3.4.17) and hence, by the maximum principle (Proposition 3.4.16) $v(z) < 1$ for any $z \in D$.

Fix a $t_0 > 0$ such that $u \leq -t_0$ on K and put $h := (1/t_0)u + 1$. Then $h \in \mathcal{PSH}(D)$, $h \leq 1$, and $h \leq 0$ on K. Hence $h \leq h_{D,K} \leq v$. Consequently, $v - 1$ is a negative psh function with (*). We will show that v is continuous (then $v_0 := v - 1$ satisfies all the required conditions).

By the Oka theorem (Proposition 3.2.31), for any point $b \in \partial\mathbb{B}(a, r)$ we get

$$v(b) = \lim_{[0,1)\ni t\to 1} v(a + t(b - a)) = 0.$$

Thus $v = 0$ on K.

For $\alpha \in (0, 1)$ let $D_\alpha := \{z \in D : v(z) < \alpha\}$. Notice that $K \subset D_\alpha \subset\subset D$ and $D_\alpha \nearrow D$ when $\alpha \nearrow 1$. The same proof as above shows that $h^*_{D_\alpha,K} = 0$ on K. Observe that

$$\alpha h^*_{D_\alpha,K} \leq v \text{ on } D_\alpha.$$

Indeed, define

$$h := \begin{cases} \max\{\alpha h^*_{D_\alpha,K}, v\} & \text{on } D_\alpha \\ v & \text{on } D \setminus D_\alpha \end{cases}.$$

Then

$$\limsup_{D_\alpha\ni z\to\zeta} \alpha h^*_{D,K}(z) \leq \alpha \leq v(\zeta), \quad \zeta \in \partial D_\alpha.$$

Hence, by Proposition 3.2.11, $h \in \mathcal{PSH}(D)$. Obviously $h \leq 1$ on D and $h = 0$ on K. Thus $h \leq h_{D,K} \leq v$. In particular, $\alpha h^*_{D_\alpha,K} \leq h \leq v$ on D_α.

Fix a point $z_0 \in D$. We want to prove that v is continuous at z_0. Let $\beta(\alpha) := \max_{\overline{D}_\alpha} v$. Observe that $\alpha \leq \beta(\alpha) < 1$. In particular, $\beta(\alpha) \longrightarrow 1$ when $\alpha \longrightarrow 1$. Fix $\eta > 0$ and $\alpha \in (0, 1)$ such that $z_0 \in D_\alpha$ and $\beta/\alpha - 1 \leq \eta$, where $\beta := \beta(\alpha)$.

Let $(v_\varepsilon)_{0<\varepsilon\leq\varepsilon_0}$ be a family of $\mathcal{C}^\infty$ psh functions defined in a neighborhood Ω of $\overline{D}_\alpha$, $\Omega\subset D$, such that $v_\varepsilon \searrow v$ on Ω when $\varepsilon\searrow 0$ (Proposition 3.4.24). By the Hartogs lemma (Proposition 3.4.20), one can find a function $w\in\mathcal{PSH}(\Omega)\cap\mathcal{C}^\infty(\Omega)$ such that $w\geq v$ on Ω, $w\leq\eta$ on K, and $w\leq\beta+\eta$ on $\overline{D}_\alpha$. Consequently,

$$(w-\eta)/\beta\leq h^*_{D_\alpha,K} \text{ on } D_\alpha.$$

Hence,

$$0\leq w-v\leq\beta h^*_{D_\alpha,K}+\eta-v\leq(\beta/\alpha-1)v+\eta\leq\beta/\alpha-1+\eta\leq 2\eta \text{ on } D_\alpha.$$

Now, by the continuity of w, there exists a neighborhood U od z_0, $U\subset D_\alpha$, such that $|w(z)-w(z_0)|\leq\eta$ for $z\in U$. Finally, $|v(z)-v(z_0)|\leq 5\eta$ for $z\in U$. □

Exercises

3.1. Let $D\subset\mathbb{C}$ be a domain, $h\in\mathcal{H}(D)$, $h\not\equiv$ const. Prove that $h^2\notin\mathcal{H}(D)$.

3.2. Let $h\in\mathcal{H}(\mathbb{D})$, $h\geq 0$. Prove that

$$\frac{1-|z|}{1+|z|}h(0)\leq h(z)\leq\frac{1+|z|}{1-|z|}h(0),\quad z\in\mathbb{D}.$$

3.3. Determine the set $\{h(1/2): h\in\mathcal{H}(\mathbb{D}),\ h\geq 0,\ h(0)=1\}$.

3.4. Let $\Omega\subset\mathbb{C}$ be open and let $(h_\iota)_{\iota\in I}\subset\mathcal{H}(\Omega)$ be locally uniformly bounded. Show that the function $\sup_{\iota\in I}h_\iota$ is continuous.

3.5. Let $\Omega\subset\mathbb{C}$ be open. Prove that $-\log d_\Omega\in\mathcal{SH}(\Omega)$.

3.6. Given a domain $D\subset\mathbb{C}$, find a continuous subharmonic function $u: D\longrightarrow\mathbb{R}$ such that $\{z\in D: u(z)\leq t\}\subset\subset D$ for any $t\in\mathbb{R}$.

3.7. Let $\Omega\subset\mathbb{C}^n$, $u:\Omega\longrightarrow\mathbb{R}_+$. Prove that $\log u\in\mathcal{PSH}(\Omega)$ iff for any $a\in\mathbb{C}^n$ the function

$$\Omega\ni z\longmapsto|e^{\langle z,a\rangle}|u(z)$$

is psh (cf. Proposition 3.2.34) ([22]).

3.8. Construct a function $u\in\mathcal{PSH}(\mathbb{C}^n)$, $u\not\equiv 0$, such that $u=0$ on dense subset of $\mathbb{C}^n$.

3.9. Let $D:=\{(z_1,z_2)\in\mathbb{C}^2: |z_1z_2|<1\}$, $u\in\mathcal{PSH}(D)$, $u\leq 0$. Prove that there exists a $v\in\mathcal{SH}(\mathbb{D})$ such that $u(z_1,z_2)=v(z_1z_2)$, $(z_1,z_2)\in D$.

3.10. Let $u\in\mathcal{C}^2(\Omega,\mathbb{R})$. Prove that

$$\sum_{j,k=1}^{2n}\frac{\partial^2 u}{\partial x_j\partial x_k}(z)Y_jY_k=2\operatorname{Re}\Big(\sum_{j,k=1}^{n}\frac{\partial^2 u}{\partial z_j\partial z_k}(z)X_jX_k\Big)+2\mathcal{L}u(z;X),$$

where

$$z=(z_1,\dots,z_n)=(x_1+ix_2,\dots,x_{2n-1}+ix_{2n})\in\Omega,$$
$$X=(X_1,\dots,X_n)=(Y_1+iY_2,\dots,Y_{2n-1}+iY_{2n})\in\mathbb{C}^n.$$

3.11. (Convex functions.) Let $D\subset\mathbb{R}^N$ be a convex domain and let $\mathcal{CV}(D)$ denote the set of all convex functions $u: D\longrightarrow[-\infty,\infty)$.

(a) Let $u\in\mathcal{CV}(D)$. Prove that either $u\equiv-\infty$ or $u\in\mathcal{C}(D,\mathbb{R})$.

(b) Let $(u_\iota)_{\iota\in I}\subset\mathcal{CV}(D)$ be locally bounded from above in D. Prove that $u:=\sup_{\iota\in I}u_\iota\in\mathcal{CV}(D)$ (cf. Propositions 3.2.10, 3.4.17).

(c) Let $(u_\nu)_{\nu=1}^\infty\subset\mathcal{CV}(D)$ be locally bounded from above in D. Prove that $u:=\limsup_{\nu\to+\infty}u_\nu\in\mathcal{CV}(D)$ (cf. Propositions 3.2.14, 3.4.18).

(d) Let $u\in\mathcal{C}^2(D,\mathbb{R})$. Prove that $u\in\mathcal{CV}(D)$ iff $\mathcal{H}u(x;X)\geq 0$ for any $x\in D$ and $X\in\mathbb{R}^N$, where $\mathcal{H}u$ denotes the *real Hessian* of u,

$$\mathcal{H}u(x;X):=\sum_{j,k=1}^{N}\frac{\partial^2 u}{\partial x_j\partial x_k}(x)X_jX_k,\quad x\in D,\ X=(X_1,\dots,X_N)\in\mathbb{R}^N.$$

([22]) $\langle z,a\rangle=\sum_{j=1}^n z_j\bar{a}_j$.

(e) Let ϱ_D denote the distance to the boundary of D with respect to the Euclidean norm. Prove that $-\log \varrho_D \in \mathcal{CV}(D)$.

(f) Let $\Phi \in \mathcal{C}_0^\infty(\mathbb{R}^N, \mathbb{R}_+)$ be such that
$\operatorname{supp} \Phi = \overline{\mathbb{B}}_N =$ the unit Euclidean ball in $\mathbb{R}^N$,
$\Phi(x) = \Phi(|x_1|, \dots, |x_N|)$, $x = (x_1, \dots, x_N) \in \mathbb{R}^N$,
$\int_{\mathbb{R}^N} \Phi \, d\mathcal{L}^N = 1$.
Define

$$D_\varepsilon := \{x \in D : \varrho_D(x) > \varepsilon\}, \qquad u_\varepsilon(x) := \int_{\mathbb{B}_N} u(x + \varepsilon y)\Phi(y) \, d\mathcal{L}^N(y), \quad x \in D_\varepsilon, \ \varepsilon > 0.$$

Prove that if $u \in \mathcal{CV}(D)$, $u \not\equiv 0$, then $u_\varepsilon \in \mathcal{CV}(D_\varepsilon) \cap \mathcal{C}^\infty(D_\varepsilon)$, $\varepsilon > 0$, and $u_\varepsilon \searrow u$ when $\varepsilon \searrow 0$ (cf. Propositions 3.2.25 and 3.4.24).

(g) Let $u \in \mathcal{C}(D, \mathbb{R})$. Prove that $u \in \mathcal{CV}(D)$ iff $\mathcal{H}u \geq 0$ in the sense of distributions, i.e. for any function $\varphi \in \mathcal{C}_0^\infty(D, \mathbb{R}_+)$ we have

$$\int_D u(x)\mathcal{H}\varphi(x; X) \, d\mathcal{L}^N(x) \geq 0, \quad X \in \mathbb{R}^N$$

(cf. Propositions 3.2.28, 3.4.28).

(h) Let $u : D \longrightarrow [-\infty, \infty)$. Put

$$\widetilde{D} := D + i\mathbb{R}^N \subset \mathbb{C}^N, \quad \widetilde{u}(x + iy) := u(x), \ x + iy \in \widetilde{D}.$$

Prove that $u \in \mathcal{CV}(D)$ iff $\widetilde{u} \in \mathcal{PSH}(\widetilde{D})$.

(i) Let $u : D \longrightarrow \mathbb{R}_+$. Prove that $\log u \in \mathcal{CV}(D)$ iff for any $a \in \mathbb{R}^N$ the function

$$D \ni x \longmapsto e^{\langle x, a \rangle} u(x)$$

is convex (cf. Exercise 3.7).

CHAPTER 4

Pseudoconvexity and the $\overline{\partial}$-problem

4.1. Pseudoconvexity

Definition 4.1.1. An open set $\Omega \subset \mathbb{C}^n$ is called *pseudoconvex* if

$$-\log d_\Omega \in \mathcal{PSH}(\Omega).$$

Notice that $\mathbb{C}^n$ is pseudoconvex (because $-\log d_{\mathbb{C}^n} \equiv -\infty$). The empty set $\varnothing$ is pseudoconvex by definition.

Observe that Ω is pseudoconvex iff each connected component of Ω is pseudoconvex. We will see (Corollary 4.1.6(a)) that any convex domain is pseudoconvex, which partially justifies the terminology.

Proposition 4.1.2. (a) *Every open set $\Omega \subset \mathbb{C}$ is pseudoconvex.*
(b) *If $(\Omega_\iota)_{\iota\in I}$ is a family of pseudoconvex open subsets of $\mathbb{C}^n$, then*

$$\Omega := \operatorname{int} \bigcap_{\iota\in I} \Omega_\iota$$

is pseudoconvex.
(c) *If $(\Omega_j)_{j=1}^\infty$ is a sequence of pseudoconvex subsets of $\mathbb{C}^n$ such that $\Omega_j \subset \Omega_{j+1}$, $j \geq 1$, then*

$$\Omega := \bigcup_{j=1}^\infty \Omega_j$$

is pseudoconvex.
(d) *If Ω_j is a pseudoconvex subset of $\mathbb{C}^{n_j}$, $j = 1, \dots, N$, then*

$$\Omega := \Omega_1 \times \dots \times \Omega_N$$

is pseudoconvex in $\mathbb{C}^{n_1+\dots+n_N}$ (cf. Corollary 4.1.10).

In particular, for any open sets $\Omega_1, \dots, \Omega_n \subset \mathbb{C}$, the set $\Omega := \Omega_1 \times \dots \times \Omega_n$ is pseudoconvex in $\mathbb{C}^n$.

Proof. (a) If $\Omega \subsetneq \mathbb{C}$, then

$$d_\Omega(z) = \inf\{|z - \zeta| : \zeta \notin \Omega\}, \quad z \in \Omega.$$

Hence, by Propositions 3.2.8 and 3.2.10, $-\log d_\Omega \in \mathcal{SH}(\Omega)$.

(b) One can prove that

$$d_\Omega = \inf\{d_{\Omega_\iota} : \iota \in I\}.$$

Hence, by Proposition 3.4.17, $-\log d_\Omega \in \mathcal{PSH}(\Omega)$.

(c) Since $-\log d_{\Omega_j} \searrow -\log d_\Omega$, we use Proposition 3.4.8.

(d) We have

$$d_\Omega(z_1, \dots, z_n) = \min\{d_{\Omega_j}(z_j) : j = 1, \dots, N\}, \quad (z_1, \dots, z_n) \in \Omega.$$

Hence, by Proposition 3.4.17, $-\log d_\Omega \in \mathcal{PSH}(\Omega)$. □

For an open set $\Omega \subset \mathbb{C}^n$, put

$$\delta_{\Omega,X}(a) := \sup\{r > 0 : a + K(r) \cdot X \subset \Omega\}, \quad X \in \mathbb{C}^n,\ a \in \Omega.$$

Obviously, if $n = 1$, then $\delta_{\Omega,X} = d_\Omega/|X|$.

Note that
$$\delta_{\Omega,X}(a+\lambda X)=d_{\Omega_{a,X}}(\lambda),\quad \lambda\in\Omega_{a,X}.$$

Lemma 4.1.3. *The function*
$$\Omega\times\mathbb{C}^n\ni(a,X)\longrightarrow\delta_{\Omega,X}(a)\in(0,+\infty]$$
is lower semicontinuous.

Proof. Fix $(a_0,X_0)\in\Omega\times\mathbb{C}^n$ and $0<r_0<\delta_{\Omega,X_0}(a_0)$. Since the set $a_0+\overline{K}(r_0)\cdot X_0$ is compact, there exists an $\varepsilon>0$ such that $a+\overline{K}(r_0)\cdot X\subset\Omega$ for any $(a,X)\in U:=\mathbb{B}(a_0,\varepsilon)\times\mathbb{B}(X_0,\varepsilon)\subset\Omega\times\mathbb{C}^n$. Consequently, $\delta_{\Omega,X}(a)>r_0$ for any $(a,X)\in U$. □

Given a $\mathbb{C}$-norm $\mathfrak{q}:\mathbb{C}^n\longrightarrow\mathbb{R}_+$, define
$$d_{\Omega,\mathfrak{q}}(a)=\sup\{r>0: B_{\mathfrak{q}}(a,r)\subset\Omega\},\quad a\in\Omega,$$
where $B_{\mathfrak{q}}(a,r):=\{z\in\mathbb{C}^n:\mathfrak{q}(z-a)<r\}$. Obviously, $d_{\Omega,|\,|}=d_\Omega$. Notice that the function $d_{\Omega,\mathfrak{q}}$ is continuous.

Remark 4.1.4. $d_{\Omega,\mathfrak{q}}=\inf\{\delta_{\Omega,X}: X\in\mathbb{C}^n,\ \mathfrak{q}(X)=1\}$.

For a compact $K\subset\Omega$ and a family $\mathcal{S}\subset\mathcal{PSH}(\Omega)$ let
$$\widetilde{K}_{\mathcal{S}}:=\{z\in\Omega:\forall_{u\in\mathcal{S}}:\ u(z)\le\max_K u\}.$$

By Proposition 3.4.5 $\widetilde{K}_{\mathcal{PSH}(\Omega)}\subset\widehat{K}_{\mathcal{O}(\Omega)}$.

Moreover, $\widetilde{K}_{\mathcal{PSH}(\Omega)}\subset\widetilde{K}_{\mathcal{PSH}(\Omega)\cap\mathcal{C}(\Omega)}$ and the set $\widetilde{K}_{\mathcal{PSH}(\Omega)\cap\mathcal{C}(\Omega)}$ is relatively closed in Ω.

A function $u:\Omega\longrightarrow\mathbb{R}$ is called an *exhaustion function* if for any $t\in\mathbb{R}$ the set $\{z\in\Omega: u(z)\le t\}$ is relatively compact in Ω.

Theorem 4.1.5. *Let Ω be an open subset of $\mathbb{C}^n$. Then the following conditions are equivalent:*
(PC1) $-\log\delta_{\Omega,X}\in\mathcal{PSH}(\Omega)$ *for every* $X\in\mathbb{C}^n$*;*
(PC2) $-\log d_{\Omega,\mathfrak{q}}\in\mathcal{PSH}(\Omega)$ *for every $\mathbb{C}$-norm* $\mathfrak{q}$*;*
(PC3) *Ω is pseudoconvex;*
(PC4) *there exists an exhaustion function* $u\in\mathcal{PSH}(\Omega)\cap\mathcal{C}(\Omega)$*;*
(PC5) *there exists an exhaustion function* $u\in\mathcal{PSH}(\Omega)$*;*
(PC6) $\widetilde{K}_{\mathcal{PSH}(\Omega)\cap\mathcal{C}(\Omega)}$ *is compact in Ω for every compact $K\subset\Omega$;*
(PC7) $\widetilde{K}_{\mathcal{PSH}(\Omega)}$ *is relatively compact in Ω for every compact $K\subset\Omega$;*
(PC8) *every point $a\in\partial\Omega$ has an open neighborhood U_a such that $U_a\cap\Omega$ is pseudoconvex.* ([1])

Proof.

$$\begin{tikzcd}
 & & & & (6) \arrow[dr] & & \\
(1) \arrow[r] & (2) \arrow[r] & (3) \arrow[r] \arrow[dr] & (4) \arrow[r] \arrow[ur] & (5) \arrow[r] & (7) \arrow[r] & (1) \\
 & & & (8) \arrow[u] & & &
\end{tikzcd}$$

The case $\Omega=\mathbb{C}^n$ is obvious (in (PC4) we can take for instance $u(z):=\|z\|$, $z\in\mathbb{C}^n$ (cf. Proposition 3.4.26)). Thus we may assume that $\Omega\subsetneq\mathbb{C}^n$.

(PC1) $\Longrightarrow$ (PC2) follows from Remark 4.1.4 and Proposition 3.4.17.

The implication (PC2) $\Longrightarrow$ (PC3) is trivial.

For the proof of (PC3) $\Longrightarrow$ (PC4) we can take $u(z):=\max\{-\log d_\Omega(z),\|z\|\}$, $z\in\Omega$.

The implications (PC4) $\Longrightarrow$ (PC5) and (PC6) $\Longrightarrow$ (PC7) are trivial.

([1]) That is, Ω is *locally pseudoconvex*.

For the proof of (PC5) $\Longrightarrow$ (PC7) observe that if u is as in (PC5), then

$$\widetilde{K}_{\mathcal{PSH}(\Omega)} \subset \{z \in \Omega : u(z) \le \max_K u\} \subset\subset \Omega.$$

In the same way we check that (PC4) $\Longrightarrow$ (PC6).

(PC7) $\Longrightarrow$ (PC1) (This is the main part of the proof.) Fix $a \in \Omega$, $X, Y \in \mathbb{C}^n \setminus \{0\}$. We want to show that the function

$$\Omega_{a,Y} \ni \lambda \longmapsto -\log \delta_{\Omega,X}(a + \lambda Y)$$

is subharmonic.

First consider the case where X and Y are linearly dependent. We may assume that $X = Y$. Since $\delta_{\Omega,X}(a + \lambda X) = d_{\Omega_{a,X}}(\lambda)$, $\lambda \in \Omega_{a,X}$, we can use Proposition 4.1.2(a).

Now assume that X, Y are linearly independent. It is sufficient to prove (cf. Proposition 3.2.7) that if $\overline{K}(r) \subset \Omega_{a,Y}$, and if $p \in \mathcal{P}(\mathbb{C})$ is such that

$$-\log \delta_{\Omega,X}(a + \lambda Y) \le \operatorname{Re} p(\lambda), \quad \lambda \in \partial K(r),$$

then the same inequality holds for all $\lambda \in K(r)$. In other words, if

$$\delta_{\Omega,X}(a + \lambda Y) \ge e^{-\operatorname{Re} p(\lambda)}, \quad \lambda \in \partial K(r),$$

then the same is true for all $\lambda \in K(r)$. Thus we have to show that if

$$a + \lambda Y + K(|e^{-p(\lambda)}|) \cdot X \subset \Omega, \quad \lambda \in \partial K(r),$$

then the same inclusion holds for all $\lambda \in K(r)$.

For $0 \le \theta < 1$ let

$$K_\theta := \{a + \lambda Y + \overline{K}(\theta |e^{-p(\lambda)}|) \cdot X : \ \lambda \in \partial K(r)\},$$
$$M_\theta := \{a + \lambda Y + \overline{K}(\theta |e^{-p(\lambda)}|) \cdot X : \ \lambda \in \overline{K}(r)\}.$$

Observe that K_θ and M_θ are compact. Our problem is to show that if $K_\theta \subset \Omega$ for all $0 \le \theta < 1$, then $M_\theta \subset \Omega$ for all $0 \le \theta < 1$. Thus assume that $K_\theta \subset \Omega$ for all $0 \le \theta < 1$ and let $I_0 := \{\theta \in [0,1) : M_\theta \subset \Omega\}$.

Notice that $M_0 = a + \overline{K}(r)Y \subset \Omega$. Hence $I_0 \ne \varnothing$. Suppose that $\theta_0 \in I_0$. Since M_{θ_0} is compact, there exists a $\theta \in (\theta_0, 1)$ such that $M_\theta \subset \Omega$. Consequently, I_0 is open. It remains to prove that I_0 is closed in $[0,1)$, i.e. if $M_\theta \subset \Omega$ for $0 < \theta < \theta_0 < 1$, then $M_{\theta_0} \subset \Omega$.

Fix $0 < \theta < \theta_0$. Observe that

$$M_\theta = \{a + \lambda Y + \zeta e^{-p(\lambda)} X : \ |\lambda| \le r, \ |\zeta| \le \theta\} \subset\subset \Omega.$$

Take a $u \in \mathcal{PSH}(\Omega)$ and define

$$v_\zeta(\lambda) := u(a + \lambda Y + \zeta e^{-p(\lambda)} X), \quad \zeta \in \overline{K}(\theta), \ \lambda \in \overline{K}(r).$$

Then v_ζ is subharmonic and, therefore, the maximum principle gives

$$v_\zeta(\lambda) \le \max_{\partial K(r)} v_\zeta \le \max_{K_\theta} u \le \max_{K_{\theta_0}} u.$$

Consequently, $M_\theta \subset (\widetilde{K_{\theta_0}})_{\mathcal{PSH}(\Omega)} \subset\subset \Omega$ for any $0 < \theta < \theta_0$ and hence $M_{\theta_0} \subset \Omega$.

The implication (PC3) $\Longrightarrow$ (PC8) is trivial.

(PC8) $\Longrightarrow$ (PC4). For $a \in \partial\Omega$ let U_a be a neighborhood of a such that $U_a \cap \Omega$ is pseudoconvex. Clearly, there exists a smaller neighborhood $V_a \subset U_a$ such that $d_\Omega = d_{U_a \cap \Omega}$ in $V_a \cap \Omega$ (Exercise). In particular, $-\log d_\Omega \in \mathcal{PSH}(V_a \cap \Omega)$. Consequently, there exists a closed set $F \subset \mathbb{C}^n$ such that $F \subset \Omega$ and $-\log d_\Omega \in \mathcal{PSH}(\Omega \setminus F)$. Let

$$\varphi_0(t) := \max\{-\log d_\Omega(z) : z \in F, \ \|z\| \le t\}, \quad t \in \mathbb{R}$$

(with $\max \varnothing = -\infty$). One can easily prove (Exercise) that there exists an increasing convex function $\varphi : \mathbb{R} \longrightarrow \mathbb{R}_+$ such that $\varphi(t) > \max\{t, \varphi_0(t)\}$, $t \in \mathbb{R}$. Put

$$u(z) := \max\{-\log d_\Omega(z), \varphi(\|z\|)\}, \quad z \in \Omega.$$

The function u is obviously continuous. Since $\varphi(\|z\|) > -\log d_\Omega(z)$ for z in a neighborhood of F, the function u is plurisubharmonic in Ω (cf. Proposition 3.4.6). Moreover,

$$\{z \in \Omega : u(z) \le t\} \subset \{z \in \Omega : d_\Omega(z) \ge e^{-t},\ \|z\| \le t\} \subset\subset \Omega, \quad t \in \mathbb{R}. \qquad \square$$

Corollary 4.1.6. (a) *Any holomorphically convex open set $\Omega \subset \mathbb{C}^n$ is pseudoconvex.* (2) *In particular, any convex domain is pseudoconvex.*
(b) *Any hyperconvex domain $D \subset \mathbb{C}^n$ (Definition 3.4.32) is pseudoconvex (cf. Exercise 4.2).*

Proof. (a) $\widetilde{K}_{\mathcal{PSH}(\Omega)} \subset \widehat{K}_{\mathcal{O}(\Omega)}$

(b) Let $u : D \longrightarrow (-\infty, 0)$ be a continuous psh function such that

$$\{z \in D : u(z) \le t\} \subset\subset D$$

for any $t < 0$. Then for any compact $K \subset D$ we get

$$\widetilde{K}_{\mathcal{PSH}(D)} \subset \{z \in D : u(z) \le \max_K u\} \subset\subset D.$$

$\square$

Proposition 4.1.7. *If $\Omega_1 \subset \mathbb{C}^n$, $\Omega_2 \subset \mathbb{C}^m$ are pseudoconvex and $f \in \mathcal{O}(\Omega_1, \mathbb{C}^m)$, then $\Omega := f^{-1}(\Omega_2)$ is pseudoconvex (cf. Proposition 2.7.13).*

Proof. (Cf. the proof of Proposition 2.7.13.) Let $K \subset \Omega$ be compact. Then

$$\widetilde{K}_{\mathcal{PSH}(\Omega)} \subset \Omega \cap \widetilde{K}_{\mathcal{PSH}(\Omega_1)} \subset\subset \Omega_1.$$

Suppose that there exists a sequence $(z_\nu)_{\nu=1}^\infty \subset \widetilde{K}_{\mathcal{PSH}(\Omega)}$ such that $z_\nu \longrightarrow z_0 \in \Omega_1 \cap \partial\Omega$. Observe that for any $z \in \widetilde{K}_{\mathcal{PSH}(\Omega)}$ and $v \in \mathcal{PSH}(\Omega_2)$ we get

$$v(f(z)) \le \max_K v \circ f = \max_{f(K)} v.$$

Hence

$$f(\widetilde{K}_{\mathcal{PSH}(\Omega)}) \subset \widetilde{f(K)}_{\mathcal{PSH}(\Omega_2)} \subset\subset \Omega_2.$$

In particular,

$$f(z_\nu) \in \widetilde{f(K)}_{\mathcal{PSH}(\Omega_2)} \subset\subset \Omega_2, \quad \nu \ge 1,$$

and so $f(z_0) \in \Omega_2$; contradiction. $\square$

Proposition 4.1.8. *Ω is pseudoconvex iff the function*

$$\Omega \times \mathbb{C}^n \ni (z, X) \longmapsto -\log \delta_{\Omega,X}(z)$$

is psh.

Proof. The implication $\Longleftarrow$ is obvious. To prove $\Longrightarrow$ let

$$\Omega \times \mathbb{C}^n \ni (z, X) \overset{g}{\longmapsto} (z, X, 0) \in \Omega \times \mathbb{C}^n \times \mathbb{C} \ni (z, X, \lambda) \overset{f}{\longmapsto} z + \lambda X \in \mathbb{C}^n.$$

Put $G := f^{-1}(\Omega)$. Note that $g(\Omega \times \mathbb{C}^n) \subset G$. By Proposition 4.1.7, G is pseudoconvex. In particular, by (PC1) the function $-\log \delta_{G,Y_0}$ is psh in G, where $Y_0 := (0, 0, 1) \in \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}$. Now we only need to prove that

$$\delta_{\Omega,X}(z) = \delta_{G,Y_0}(g(z, X)), \quad (z, X) \in \Omega \times \mathbb{C}^n.$$

Indeed, we have

$$\begin{aligned}\delta_{\Omega,X}(z) = \sup\{r > 0 : z + K(r) \cdot X \subset \Omega\} &= \sup\{r > 0 : \{z\} \times \{X\} \times K(r) \subset G\} \\ &= \sup\{r > 0 : (z, X, 0) + K(r) \cdot Y_0 \subset G\} = \delta_{G,Y_0}(g(z, X)).\end{aligned} \qquad \square$$

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(2) It is natural to ask whether the converse implication is also true. This is the famous *Levi Problem*, which will be solved in Chapter 5.

Proposition 4.1.9. *Let $\Omega \subset \mathbb{C}^n$ be pseudoconvex. Then for any complex affine subspace $V \subset \mathbb{C}^n$ the set $\Omega \cap V$ is pseudoconvex in V, i.e. for any $a \in \mathbb{C}^n$ and linearly independent vectors $v_1, \dots, v_k$ the set*

$$G := \{(\lambda_1, \dots, \lambda_k) \in \mathbb{C}^k : a + \lambda_1 v_1 + \dots + \lambda_k v_k \in \Omega\}$$

is pseudoconvex in $\mathbb{C}^k$.

Proof. If u is a psh exhaustion function for Ω (cf. (PC5)), then the function

$$v(\lambda) := u(a + \lambda_1 v_1 + \dots + \lambda_k v_k), \quad \lambda = (\lambda_1, \dots, \lambda_k) \in G,$$

is a psh exhaustion function for G. (3) $\square$

Corollary 4.1.10. *Let Ω_j be open in $\mathbb{C}^{n_j}$, $j = 1, \dots, N$. Then $\Omega_1 \times \dots \times \Omega_N$ is pseudoconvex iff each Ω_j is pseudoconvex, $j = 1, \dots, N$.*

Proposition 4.1.11. *Let $\Omega \subset \mathbb{C}^n$ be pseudoconvex. Then for any biholomorphic mapping $\Phi : \Omega \longrightarrow \Phi(\Omega)$ the set $\Phi(\Omega)$ is pseudoconvex.*

Proof. If u is a psh exhaustion function for Ω, then $u \circ \Phi^{-1}$ is a psh exhaustion function for $\Phi(\Omega)$. $\square$

Proposition 4.1.12. *Let $\Omega \subset \mathbb{C}^n$ be pseudoconvex and let $u \in \mathcal{PSH}(\Omega)$. Then*

$$G := \{z \in \Omega : u(z) < 0\}$$

is pseudoconvex.

Proof. First assume additionally that u is continuous. Take an arbitrary compact $K \subset G$. Then

$$\widetilde{K}_{\mathcal{PSH}(\Omega)} \subset \{z \in \Omega : u(z) \le \max_K u\} \subset G.$$

Consequently, $\widetilde{K}_{\mathcal{PSH}(G)} \subset \widetilde{K}_{\mathcal{PSH}(\Omega)} \subset\subset G$.

Now, let u be arbitrary. Put

$$\Omega_\varepsilon := \{z \in \Omega : d_\Omega(z) > \varepsilon\}, \quad \varepsilon > 0.$$

By the first part of the proof Ω_ε is pseudoconvex for any $\varepsilon > 0$. Let $u_\varepsilon \in \mathcal{PSH}(\Omega_\varepsilon) \cap \mathcal{C}^\infty(\Omega_\varepsilon)$ be the ε-regularization of u (cf. Proposition 3.4.24). Define

$$G_\varepsilon := \{z \in \Omega_\varepsilon : u_\varepsilon(z) < 0\}, \quad \varepsilon > 0.$$

By the first part of the proof we know that G_ε is pseudoconvex for any $\varepsilon > 0$. It remains to observe that $G_\varepsilon \nearrow G$ as $\varepsilon \searrow 0$ and use Proposition 4.1.2(c). $\square$

Corollary 4.1.13. *An open set $\Omega \subset \mathbb{C}^n$ is pseudoconvex iff for arbitrary $\varepsilon > 0$ the set $\{z \in \Omega : d_\Omega(z) > \varepsilon\}$ is pseudoconvex.*

Proposition 4.1.14. *Let*

$$D = \{(z, w) \in G \times \mathbb{C}^k : H(z, w) < 1\}$$

be a Hartogs domain over G with k-dimensional balanced fibers (Definition 1.6.3), where H is as in Remark 1.6.4(a). Then D is pseudoconvex iff G is pseudoconvex and $\log H \in \mathcal{PSH}(G \times \mathbb{C}^k)$.

In particular, we get the following results:

(a) *A balanced domain*

$$D = \{z \in \mathbb{C}^n : h(z) < 1\},$$

where h is the Minkowski functional of D, is pseudoconvex iff $\log h$ is psh on $\mathbb{C}^n$ (cf. Proposition 3.4.27).

(b) *A complete Hartogs domain*

$$D := \{(z, w) \in G \times \mathbb{C} : |w| < e^{-u(z)}\},$$

(3) Observe that the result follows also from Proposition 4.1.7 with

$$\mathbb{C}^k \ni (\lambda_1, \dots, \lambda_k) \overset{f}{\longmapsto} a + \lambda_1 v_1 + \dots + \lambda_k v_k \in \mathbb{C}^n.$$

where $G \subset \mathbb{C}^{n-1}$ is a domain and $u : G \longrightarrow [-\infty, +\infty)$ is upper semicontinuous (cf. Remark 1.6.4(c)), is pseudoconvex iff G is pseudoconvex and $u \in \mathcal{PSH}(G)$.

Proof. $\Longleftarrow$. Observe that $G \times \mathbb{C}^k$ is pseudoconvex and

$$D = \{(z, w) \in G \times \mathbb{C}^k : \log H(z, w) < 0\}.$$

It remains to use Proposition 4.1.12.

$\Longrightarrow$. We have $G \times \{0\} = D \cap \{w = 0\}$. Hence, by Proposition 4.1.9, G is pseudoconvex. Moreover,

$$\delta_{D,(0,w)}(z, 0) = \frac{1}{H(z, w)}, \quad z \in G,\ w \in \mathbb{C}^k.$$

Hence, by Proposition 4.1.8, $\log H \in \mathcal{PSH}(G \times \mathbb{C}^k)$. $\square$

Proposition 4.1.15. *Let $\Omega \subset \mathbb{C}^n$ be pseudoconvex open set, let $K \subset \Omega$ be a compact set, and let $U \subset \Omega$ be an open neighborhood of $\widetilde{K}_{\mathcal{PSH}(\Omega)}$. Then there exists a strictly psh exhaustion function $u \in \mathcal{C}^\infty(\Omega)$ such that*

$$u < 0 \text{ on } K, \quad u > 0 \text{ on } \Omega \setminus U.$$

Proof. Let u_0 be a continuous psh exhaustion function on Ω (cf. (PC4)). We may assume that $u_0 < 0$ on K. Define

$$L := \{z \in \Omega : u_0(z) \le 2\}, \quad M := \{z \in \Omega \setminus U : u_0(z) \le 0\}.$$

The sets L, M are compact, $M \subset L$.

Suppose that $M \neq \varnothing$. Since $M \cap \widetilde{K}_{\mathcal{PSH}(\Omega)} = \varnothing$, for any point $a \in M$ there exists a function $u_a \in \mathcal{PSH}(\Omega)$ such that $u_a < 0$ on K and $u_a(a) > 0$. Let $\Omega_\varepsilon := \{z \in \Omega : d_\Omega(z) > \varepsilon\}$ and let $(u_a)_\varepsilon \in \mathcal{PSH}(\Omega_\varepsilon) \cap \mathcal{C}^\infty(\Omega_\varepsilon)$ denote the ε-regularization of u_a. It is clear that there exists an $\varepsilon(a) > 0$ such that $L \subset \Omega_{\varepsilon(a)}$, $(u_a)_{\varepsilon(a)} < 0$ on K and $(u_a)_{\varepsilon(a)}(a) > 0$. Put $v_a := (u_a)_{\varepsilon(a)}$. Since v_a is continuous, there exists a neighborhood $V_a \subset \Omega_{\varepsilon(a)}$ of a such that $v_a > 0$ in V_a. Since M is compact, there exist points $a_1, \dots, a_N \in M$ such that $M \subset V_{a_1} \cup \dots \cup V_{a_N}$. Define $\varepsilon := \max\{\varepsilon(a_1), \dots, \varepsilon(a_N)\}$, $w := \max\{v_{a_1}, \dots, v_{a_N}\} \in \mathcal{PSH}(\Omega_\varepsilon) \cap \mathcal{C}(\Omega_\varepsilon)$ and observe that $L \subset \Omega_\varepsilon$, $w < 0$ on K and $w > 0$ on M. Let $c := \max\{1, \max_L w\}$ and put

$$v(z) := \begin{cases} \max\{w(z), cu_0(z)\} & \text{if } u_0(z) < 2 \\ cu_0(z) & \text{if } u_0(z) > 1 \end{cases}.$$

Then v is a well-defined continuous psh function on Ω. Observe that $\{v < t\} \subset \{u_0 < 2\} \cup \{u_0 < t/c\} \subset\subset \Omega$ for any $t \in \mathbb{R}$, so v is an exhaustion function. Clearly, $v < 0$ on K. Moreover, if $z \in \Omega \setminus U$ and $u_0(z) > 0$, then $v(z) \ge cu_0(z) > 0$; if $z \in \Omega \setminus U$ and $u_0(z) \le 0$, then $z \in M$ and therefore $v(z) \ge w(z) > 0$. Thus $v > 0$ on $\Omega \setminus U$.

If $M = \varnothing$ we put $v := u_0$.

It remains to smooth v. Put

$$G_\nu := \{z \in \Omega : v(z) < \nu\}, \quad \nu \in \mathbb{Z}.$$

Let $\widetilde{v}_\varepsilon \in \mathcal{PSH}(\Omega_\varepsilon) \cap \mathcal{C}^\infty(\Omega_\varepsilon)$ be the ε-regularization of v and let $v_\varepsilon := \widetilde{v}_\varepsilon + \varepsilon\|z\|^2 + \varepsilon$. Then v_ε is strictly psh and $v_\varepsilon \searrow v$ as $\varepsilon \searrow 0$ on Ω. Since v is continuous, $v_\varepsilon \longrightarrow v$ locally uniformly on Ω (use the Dini theorem). For each $\nu \in \mathbb{N}_0$ let $\varepsilon(\nu) > 0$ be such that $G_\nu \subset\subset \Omega_{\varepsilon(\nu)}$, $v < v_{\varepsilon(\nu)} < v + 1$ on $\overline{G}_\nu$ and $v_{\varepsilon(\nu)} < 0$ on K. Let $\varphi_\nu \in \mathcal{C}_0^\infty(\Omega, [0, 1])$ be such that $\operatorname{supp} \varphi_\nu \subset \Omega_{\varepsilon(\nu)}$ and $\varphi_\nu = 1$ in a neighborhood of $\overline{G}_\nu$. Define $\psi_\nu := \varphi_\nu \cdot v_{\varepsilon(\nu)}$ on G_ν and $\psi_\nu = 0$ on $\Omega \setminus \operatorname{supp} \varphi_\nu$. Observe that ψ_ν is a well-defined $\mathcal{C}^\infty$ function on Ω and $\psi_\nu = v_{\varepsilon(\nu)}$ in a neighborhood of $\overline{G}_\nu$, $\nu \in \mathbb{N}_0$.

Let $\chi : \mathbb{R} \longrightarrow \mathbb{R}_+$ be a $\mathcal{C}^\infty$ increasing convex function such that $\chi(t) = 0$ for $t \le 0$ and $\chi'(t) > 0$ for $t > 0$. Define $V_\nu := \chi(\psi_\nu + 1 - \nu) \in \mathcal{C}^\infty(\Omega, \mathbb{R}_+)$. Observe that:

(a) V_ν is psh in a neighborhood of $\overline{G}_\nu$.

(b) $V_\nu = 0$ on $\overline{G}_{\nu-2}$. Indeed, if $z \in \overline{G}_{\nu-2}$, then $\psi_\nu(z) + 1 - \nu < v(z) + 1 + 1 - \nu \le 0$.

(c) V_ν is strictly psh and > 0 in a neighborhood of $\overline{G}_\nu \setminus G_{\nu-1}$. Indeed, if $z \in \overline{G}_\nu \setminus G_{\nu-1}$, then $\psi_\nu(z) + 1 - \nu > v(z) + 1 - \nu \geq 0$. Hence $V_\nu(z) > 0$ and for $X \in (\mathbb{C}^n)_*$ we get

$$\mathcal{L}V_\nu(z;X) = \chi''(\psi_\nu(z)+1-\nu)\Big|\sum_{j=1}^n \frac{\partial\psi_\nu}{\partial z_j}(z)X_j\Big|^2 + \chi'(\psi_\nu(z)+1-\nu)\mathcal{L}\psi_\nu(z;X) \geq \chi'(\psi_\nu(z)+1-\nu)\mathcal{L}\psi_\nu(z;X) > 0$$

We are going to construct a sequence $(c_\nu)_{\nu=1}^\infty \subset \mathbb{R}_{>0}$ such that for each $\nu \in \mathbb{N}$ the function $W_\nu := \psi_0 + c_1V_1 + \dots + c_\nu V_\nu$ is $> v$ and strictly psh in a neighborhood of $\overline{G}_\nu$. We proceed by induction over ν. Put $W_0 := \psi_0$ and suppose that $c_1, \dots, c_\nu$ are already constructed for some $\nu \geq 0$ (this condition is empty for $\nu = 0$). By (a), for any $c_{\nu+1} > 0$, the function $W_{\nu+1} = W_\nu + c_{\nu+1}V_{\nu+1}$ is strictly psh and $> v$ in a neighborhood of $\overline{G}_\nu$. We have to find $c_{\nu+1}$ such that $W_{\nu+1}$ is strictly psh and $> v$ in a neighborhood of $H := \overline{G}_{\nu+1} \setminus G_\nu$. Fix an $A > 0$ such that $\mathcal{L}W_\nu(z;X) \geq -A\|X\|^2$, $z \in H$, $X \in \mathbb{C}^n$. In virtue of (c) there exists a constant $B > 0$ such that $\mathcal{L}V_{\nu+1}(z;X) \geq B\|X\|^2$, $z \in H$, $X \in \mathbb{C}^n$. Hence $\mathcal{L}W_{\nu+1}(z;X) = \mathcal{L}W_\nu(z;X) + c_{\nu+1}\mathcal{L}V_{\nu+1}(z;X) \geq (-A + c_{\nu+1}B)\|X\|^2$, $z \in H$, $X \in \mathbb{C}^n$, which shows that with $c_{\nu+1} \gg 0$ the function $W_{\nu+1}$ is strictly psh on $\overline{G}_{\nu+1}$. Recall that $V_{\nu+1} > 0$ on H (cf. (c)). Hence, if $c_{\nu+1} \gg 0$, then $W_{\nu+1} > v$ on H.

Observe that $W_\nu = W_\mu$ on $G_{\mu-1}$ for any $\nu > \mu$ (use (b)). Thus $u := \lim_{\nu\to+\infty} W_\nu$ is a well-defined $\mathcal{C}^\infty$ strictly psh function on Ω. If $z \in K \subset G_0$, then we get $u(z) = W_1(z) = \psi_0(z) + c_1\chi(\psi_1(z)) = \psi_0(z) < 0$. Moreover, $u > v$ and therefore u is an exhaustion function. $\square$

Corollary 4.1.16. *Let $\Omega \subset \mathbb{C}^n$ be a pseudoconvex open set and let $K \subset \Omega$ be a compact set. Then $\widetilde{K}_{\mathcal{PSH}(\Omega)} = \widetilde{K}_{\mathcal{PSH}(\Omega)\cap\mathcal{C}^\infty(\Omega)}$. In particular, the set $\widetilde{K}_{\mathcal{PSH}(\Omega)}$ is closed.*

Proof. Obviously, $\widetilde{K}_{\mathcal{PSH}(\Omega)} \subset \widetilde{K}_{\mathcal{PSH}(\Omega)\cap\mathcal{C}^\infty(\Omega)}$. Take an $a \notin \widetilde{K}_{\mathcal{PSH}(\Omega)}$ and let $U := \Omega \setminus \{a\}$. Then by Proposition 4.1.15 there exists a function $u \in \mathcal{PSH}(\Omega) \cap \mathcal{C}^\infty(\Omega)$ such that $u < 0$ on K and $u(a) > 0$. Consequently, $a \notin \widetilde{K}_{\mathcal{PSH}(\Omega)\cap\mathcal{C}^\infty(\Omega)}$. $\square$

Definition 4.1.17. We say that an open bounded set $\Omega \subset \mathbb{C}^n$ is *strongly pseudoconvex* if for any point $a \in \partial\Omega$ there exist a polydisc $P \subset \mathbb{C}^n$ and a strictly psh function $u \in \mathcal{C}^2(P, \mathbb{R})$ (cf. § 3.4) such that

- $a \in P$,
- $\Omega \cap P = \{z \in P : u(z) < 0\}$,
- $P \setminus \overline{\Omega} = \{z \in P : u(z) > 0\}$,
- $\operatorname{grad} u(z) \neq 0$ for any $z \in P \cap \partial\Omega$.

Observe that, by (PC8) and Proposition 4.1.12, any strongly pseudoconvex open set is pseudoconvex.

Proposition 4.1.18. *Assume that Ω is holomorphically convex open set.*
(a) *Let $K \subset \Omega$ be compact and let U be an open neighborhood of $\widehat{K}_{\mathcal{O}(\Omega)}$. Then there exists a strictly psh real analytic exhaustion function $u : \Omega \longrightarrow \mathbb{R}$ such that $u < 0$ on K and $u > 0$ on $\Omega \setminus U$.*
(b) *$\Omega = \bigcup_{k=1}^\infty \Omega_k$, where*
Ω_k is a relatively compact open subset of Ω with $\Omega_k \subset \Omega_{k+1}$, and
Ω_k is strongly pseudoconvex with real analytic boundary, $k \geq 1$.

Proof. (a) There exists a sequence of holomorphically convex compact sets $(K_j)_{j=1}^\infty$ such that $K_1 = \widehat{K}_{\mathcal{O}(\Omega)}$, $K_j \subset \operatorname{int} K_{j+1}$, $j \geq 1$, and $\Omega = \bigcup_{j=1}^\infty K_j$; cf. Remark 2.7.11(g). Fix open sets U_j, $j \geq 1$, such that $U_1 \subset U$, $K_j \subset U_j \subset K_{j+1}$, $j \geq 1$. Fix a $j \geq 1$. For any point $z \in K_{j+2} \setminus U_j$ there exists an $f_z \in \mathcal{O}(\Omega)$ such that $|f_z(z)| > 1 > \|f_z\|_{K_j}$. Let V_z be a neighborhood of z such that $|f_z(w)| > 1$ for $w \in \overline{V}_z$. There exist points $z_1, \dots, z_{k(j)}$ such that $K_{j+2} \setminus U_j \subset V_{z_1} \cup \dots \cup V_{z_{k(j)}}$. Define $f_{j,\nu} := f_{z_\nu}^\ell$, $\nu = 1, \dots, k(j)$, where $\ell = \ell(j)$ is such that

$$\sum_{\nu=1}^{k(j)} |f_{j,\nu}(z)|^2 < \frac{1}{2^{j+1}},\ z \in K_j, \quad \sum_{\nu=1}^{k(j)} |f_{j,\nu}(z)|^2 > j,\ z \in K_{j+2} \setminus U_j.$$

Put
$$v := -1 + \sum_{j=1}^{\infty}\sum_{\nu=1}^{k(j)} |f_{j,\nu}|^2.$$
Observe that the series converges locally normally in Ω. It is clear that $v \in \mathcal{PSH}(\Omega)$, $v < 0$ on K, and that $v > j-1$ on $\Omega \setminus U_j$, $j \geq 1$. In particular, $v > 0$ on $\Omega \setminus U$ and v is an exhaustion function.

To prove that v is real analytic we proceed as follows.

Define $\Omega^* := \{\overline{z} : z \in \Omega\}$ and $\varphi : \Omega \times \Omega^* \longrightarrow \mathbb{C}$,
$$\varphi(z,w) := -1 + \sum_{j=1}^{\infty}\sum_{\nu=1}^{k(j)} f_{j,\nu}(z)\overline{f_{j,\nu}(\overline{w})}.$$
The series converges locally normally in $\Omega \times \Omega^*$ and therefore $\varphi \in \mathcal{O}(\Omega \times \Omega^*)$. Since $v(z) = \varphi(z, \overline{z})$, $z \in \Omega$, the function v is real analytic.

Finally, we put $u := v + \varepsilon\|z\|^2$, where $\varepsilon > 0$ is so small that $u < 0$ on K. It is clear that u satisfies all the required conditions.

(b) Let u be as in (a). By the Sard theorem (cf. [6], Th. 3.4.3) ([4]) there exists a sequence $\mathbb{R} \ni t_k \nearrow +\infty$ such that $t_k \notin u(\{z \in \Omega : \operatorname{grad} u(z) = 0\})$. We put $\Omega_k := \{z \in \Omega : u(z) < t_k\}$, $k \geq 1$. □

Let $(A_\nu)_{\nu=0}^{\infty}$ be an arbitrary sequence of subsets of $\mathbb{C}^n$. Define
$$A_0 = \lim_{\nu\to+\infty} A_\nu \overset{\text{def}}{\Longleftrightarrow} \forall_{\varepsilon>0}\ \exists_{\nu_0} : \forall_{\nu\geq\nu_0} :\ A_\nu \subset (A_0)^{(\varepsilon)},\ A_0 \subset (A_\nu)^{(\varepsilon)}. \quad ([5])$$

One can easily check that if $A_\nu \longrightarrow A_0$, A_0 is bounded, and $u \in \mathcal{C}(\mathbb{C}^n, \mathbb{R})$, then $\inf_{A_\nu} u \longrightarrow \inf_{A_0} u$.

Theorem 4.1.19 (Kontinuitätssatz)**.** *Let Ω be an open subset of $\mathbb{C}^n$. Then the following conditions are equivalent:*

(PC3) *Ω is pseudoconvex;*

(PC9) *for any $k \in \mathbb{N}$, a bounded domain $D \subset \mathbb{C}^k$, and a sequence $(\gamma_\nu)_{\nu=1}^{\infty} \subset \mathcal{O}(D,\Omega)\cap\mathcal{C}(\overline{D},\Omega)$, the following implication is true:*
if $\gamma_\nu(\overline{D}) \longrightarrow A$ and $\gamma_\nu(\partial D) \longrightarrow A_0$, where A is bounded and $A_0 \subset\subset \Omega$, then $A \subset\subset \Omega$;

(PC10) *for every sequence of injective holomorphic mappings $(\gamma_\nu)_{\nu=1}^{\infty} \subset \mathcal{O}(\mathbb{C}, \mathbb{C}^n)$ such that $\bigcup_{\nu=1}^{\infty} \gamma_\nu(\mathbb{D}) \subset \Omega$, the following implication is true:*
if $\gamma_\nu(\mathbb{D}) \longrightarrow A$ and $\gamma_\nu(\mathbb{T}) \longrightarrow A_0$, where A is bounded and $A_0 \subset\subset \Omega$, then $A \subset\subset \Omega$.

Proof. (PC3) $\Longrightarrow$ (PC9). Let $u_\nu := -\log d_\Omega \circ \gamma_\nu$. Then $u_\nu \in \mathcal{PSH}(D) \cap \mathcal{C}(\overline{D})$ (cf. Proposition 3.4.25). In particular, by the maximum principle we have
$$\max_{\overline{D}} u_\nu = \max_{\partial D} u_\nu, \quad \nu \geq 1.$$
This means that
$$\inf_{\gamma_\nu(\overline{D})} d_\Omega = \inf_{\gamma_\nu(\partial D)} d_\Omega, \quad \nu \geq 1.$$
Put $d_\Omega := 0$ on $\mathbb{C}^n \setminus \Omega$. Then $d_\Omega \in \mathcal{C}(\mathbb{C}^n)$. We get
$$\inf_{\gamma_\nu(\overline{D})} d_\Omega \longrightarrow \inf_A d_\Omega, \qquad \inf_{\gamma_\nu(\partial D)} d_\Omega \longrightarrow \inf_{A_0} d_\Omega,$$
which proves that $\inf_A d_\Omega = \inf_{A_0} d_\Omega > 0$, and so $A \subset\subset \Omega$.

The implication (PC9) $\Longrightarrow$ (PC10) is obvious.

(PC10) $\Longrightarrow$ (PC3). We keep all the notations from the proof of the implication (PC7) $\Longrightarrow$ (PC1) in Theorem 4.1.5. Recall that the only problem is to show that the set I_0 is closed in $[0,1)$. Take an $I_0 \ni \theta_\nu \longrightarrow \theta_0 \in [0,1)$ and fix a $\zeta \in \overline{\mathbb{D}}$. Define
$$\gamma_\nu(\lambda) := a + r\lambda Y + \theta_\nu \zeta e^{-p(r\lambda)} X, \quad \lambda \in \mathbb{C},\ \nu \geq 1.$$

([4]) For any $u \in \mathcal{C}^{2n}(\Omega, \mathbb{R})$, the set $u(\{z \in \Omega : \operatorname{grad} u(z) = 0\}) \subset \mathbb{R}$ is of Lebesgue measure zero.
([5]) Recall that $A^{(\varepsilon)} := \bigcup_{a\in A} \mathbb{P}(a,\varepsilon)$.

Then γ_ν is injective (because X and Y are linearly independent). Obviously,

$$\Omega \supset \gamma_\nu(\overline{\mathbb{D}}) \longrightarrow A, \quad \gamma_\nu(\mathbb{T}) \longrightarrow A_0,$$

where

$$A := \{a + r\lambda Y + \theta_0\zeta e^{-p(r\lambda)}X : |\lambda| \le 1\} \subset M_{\theta_0},$$
$$A_0 := \{a + r\lambda Y + \theta_0\zeta e^{-p(r\lambda)}X : |\lambda| = 1\} \subset K_{\theta_0} \subset\subset \Omega.$$

Consequently, $A \subset \Omega$ and hence (since ζ is arbitrary) $\theta_0 \in I_0$. $\square$

4.2. The $\overline{\partial}$-problem

Let $\Omega \subset \mathbb{C}^n$ be open, $p, q \in \mathbb{N}_0$, and let u be a differential form on Ω of order $p+q$. We say that u is of *type* (p,q) ($u \in \mathcal{F}_{(p,q)}(\Omega)$) if

$$u = \sum_{I \in \Xi_p^n, J \in \Xi_q^n} u_{I,J} dz_I \wedge d\overline{z}_J, \tag{4.2.1}$$

where
$\Xi_p^n := \{I = (i_1, \dots, i_p) \in \mathbb{N}^p : 1 \le i_1 < \dots < i_p \le n\}$,
$u_{I,J} : \Omega \longrightarrow \mathbb{C}$,
$dz_I := dz_{i_1} \wedge \dots \wedge dz_{i_p}$, $d\overline{z}_J := d\overline{z}_{j_1} \wedge \dots \wedge d\overline{z}_{j_q}$.
To simplify notation, we will write

$$\sideset{}{'}\sum_{|I|=p, |J|=q} \dots$$

instead of

$$\sum_{I \in \Xi_p^n, J \in \Xi_q^n} \dots .$$

The representation (4.2.1) is uniquely determined; it is called the *canonical representation* of u.
Notice that $\mathcal{F}_{(p,q)}(\Omega) = \{0\}$ for $p > n$ or $q > n$.
Let

$$\mathcal{C}^k_{(p,q)}(\Omega) := \{u \in \mathcal{F}_{(p,q)}(\Omega) : \forall_{I \in \Xi_p^n,\ J \in \Xi_q^n} :\ u_{I,J} \in \mathcal{C}^k(\Omega)\}, \quad 0 \le k \le \infty.$$

For $u \in \mathcal{C}^1_{(p,q)}(\Omega)$ in the form (4.2.1) define

$$\partial u := \sideset{}{'}\sum_{|I|=p, |J|=q} \sum_{j=1}^n \frac{\partial u_{I,J}}{\partial z_j} dz_j \wedge dz_I \wedge d\overline{z}_J, \tag{4.2.2}$$

$$\overline{\partial} u := \sideset{}{'}\sum_{|I|=p, |J|=q} \sum_{j=1}^n \frac{\partial u_{I,J}}{\partial \overline{z}_j} d\overline{z}_j \wedge dz_I \wedge d\overline{z}_J. \tag{4.2.3}$$

Note that the right-hand sides of (4.2.2) and (4.2.3) are not in the canonical form. For instance, the canonical form of (4.2.3) is the following one

$$\overline{\partial} u = \sideset{}{'}\sum_{|I|=p, |K|=q+1} \Big(\sum_{J \in \Xi_q^n,\ j \in \{1, \dots, n\}} \varepsilon(I, K, J, j) \frac{\partial u_{I,J}}{\partial \overline{z}_j} \Big) dz_I \wedge d\overline{z}_K,$$

where for $K = (k_1, \dots, k_{q+1})$ and $J = (j_1, \dots, j_q)$,

$$\varepsilon(I, K, J, j) := 0$$

if $\{k_1, \dots, k_{q+1}\} \ne \{j, j_1, \dots, j_q\}$, and

$$\varepsilon(I, K, J, j) \in \{-1, +1\} \text{ is such that } d\overline{z}_j \wedge dz_I \wedge d\overline{z}_J = \varepsilon(I, K, J, j) dz_I \wedge d\overline{z}_K$$

if $\{k_1, \dots, k_{q+1}\} = \{j, j_1, \dots, j_q\}$.

Observe that

$$\partial : \mathcal{C}^k_{(p,q)}(\Omega) \longrightarrow \mathcal{C}^{k-1}_{(p+1,q)}(\Omega), \quad \overline{\partial} : \mathcal{C}^k_{(p,q)}(\Omega) \longrightarrow \mathcal{C}^{k-1}_{(p,q+1)}(\Omega), \quad \partial + \overline{\partial} = d,$$

where d denotes the standard exterior differentiation operator.

A form $u \in \mathcal{C}^1_{(p,q)}(\Omega)$ is called *$\overline{\partial}$-closed* if $\overline{\partial}u = 0$.

A form $v \in \mathcal{C}_{(p,q+1)}(\Omega)$ is called *$\overline{\partial}$-exact* if there exists a $u \in \mathcal{C}^1_{(p,q)}(\Omega)$ such that $\overline{\partial}u = v$.

The equation $\overline{\partial}u = v$ is called the *inhomogeneous Cauchy–Riemann equation* or the *$\overline{\partial}$-equation* or the *$\overline{\partial}$-problem*.

Remark 4.2.1. (a) Since $\partial + \overline{\partial} = d$, we get

$$(\partial \circ \partial)(u) = 0, \quad (\overline{\partial} \circ \overline{\partial})(u) = 0, \quad (\partial \circ \overline{\partial})(u) = -(\overline{\partial} \circ \partial)(u), \quad u \in C^2_{(p,q)}(\Omega).$$

In particular, if $v \in \mathcal{C}^1_{(p,q+1)}(\Omega)$ and $v = \overline{\partial}u$ for a form $u \in \mathcal{C}^2_{(p,q)}(\Omega)$, then $\overline{\partial}v = 0$, i.e. v must be $\overline{\partial}$-closed.

(b) For $f \in \mathcal{C}^1(\Omega)$ we have

$$f \in \mathcal{O}(\Omega) \Longleftrightarrow \overline{\partial}f = 0.$$

More generally, if $u = \sum'_{|I|=p} u_I dz_I \in \mathcal{C}^1_{(p,0)}(\Omega)$, then

$$\overline{\partial}u = 0 \Longleftrightarrow \forall_{I \in \Xi^n_p} : \ u_I \in \mathcal{O}(\Omega).$$

(c) If $u \in \mathcal{F}_{(p,q)}(\Omega)$, $v \in \mathcal{F}_{(r,s)}(\Omega)$, then $u \wedge v \in \mathcal{F}_{(p+r,q+s)}(\Omega)$.

(d) Let $\Phi = (\Phi_1, \dots, \Phi_n) : \Omega' \longrightarrow \Omega$ be a holomorphic mapping, where Ω' is an open set in $\mathbb{C}^m$. Then

$$\Phi^*(\mathcal{F}_{(p,q)}(\Omega)) \subset \mathcal{F}_{(p,q)}(\Omega'),$$

where for $u = \sum'_{|I|=p,|J|=q} u_{I,J} dz_I \wedge d\overline{z}_J$ we put

$$\begin{aligned}\Phi^*(u) &:= \sideset{}{'}\sum_{|I|=p,|J|=q} (u_{I,J} \circ \Phi)\, d\Phi_{i_1} \wedge \dots \wedge d\Phi_{i_p} \wedge d\overline{\Phi}_{j_1} \wedge \dots \wedge d\overline{\Phi}_{j_q} \\ &= \sideset{}{'}\sum_{|I|=p,|J|=q} (u_{I,J} \circ \Phi)\, \partial\Phi_{i_1} \wedge \dots \wedge \partial\Phi_{i_p} \wedge \overline{\partial\Phi}_{j_1} \wedge \dots \wedge \overline{\partial\Phi}_{j_q}.\end{aligned}$$

Moreover,

$$\Phi^*(\mathcal{C}^k_{(p,q)}(\Omega)) \subset \mathcal{C}^k_{(p,q)}(\Omega'), \qquad (\Phi^* \circ \overline{\partial})(u) = (\overline{\partial} \circ \Phi^*)(u), \quad u \in \mathcal{C}^1_{(p,q)}(\Omega).$$

$$\begin{array}{ccc} \mathcal{C}^k_{(p,q)}(\Omega) & \xrightarrow{\Phi^*} & \mathcal{C}^k_{(p,q)}(\Omega') \\ {\scriptstyle\overline{\partial}}\big\downarrow & & \big\downarrow{\scriptstyle\overline{\partial}} \\ \mathcal{C}^{k-1}_{(p,q+1)}(\Omega) & \xrightarrow{\Phi^*} & \mathcal{C}^{k-1}_{(p,q+1)}(\Omega') \end{array}$$

In particular, if $\overline{\partial}u = v$, then $\overline{\partial}(\Phi^*(u)) = \Phi^*(v)$. Consequently, if $u \in \mathcal{C}^1_{(p,q)}(\Omega)$ is $\overline{\partial}$-closed, then $\Phi^*(u)$ is also $\overline{\partial}$-closed.

(e)

$$\overline{\partial}(u \wedge v) = (\overline{\partial}u) \wedge v + (-1)^{p+q} u \wedge (\overline{\partial}v), \quad u \in \mathcal{C}^1_{(p,q)}(\Omega),\ v \in \mathcal{C}^1_{(r,s)}(\Omega).$$

For u as in (4.2.1) we define the *support* of u by the formula

$$\operatorname{supp} u := \bigcup_{I \in \Xi^n_p,\ J \in \Xi^n_q} \operatorname{supp} u_{I,J}.$$

Proposition 4.2.2. *Let $v \in \mathcal{C}^k_{(0,1)}(\mathbb{C}^n)$ be a $\overline{\partial}$-closed form with $k \geq 1$, $n \geq 2$, and $\operatorname{supp} v \subset\subset \mathbb{C}^n$. Then there exists a function $u \in \mathcal{C}^k_0(\mathbb{C}^n)$ such that $\overline{\partial}u = v$ and $u = 0$ in the unbounded connected component of the set $\mathbb{C}^n \setminus \operatorname{supp} v$.*

Remark 4.2.3. The above result is not true for $n = 1$.

Indeed, let $v = v_0 d\overline{z} \in \mathcal{C}^\infty_{(0,1)}(\mathbb{C})$ be such that $\operatorname{supp} v_0 \subset\subset \mathbb{C}$ and $\int_{\mathbb{C}} v_0 \, d\mathcal{L}^2 \neq 0$. Suppose that $v_0 = \frac{\partial u}{\partial \overline{z}}$ for a function $u \in \mathcal{C}^1_0(\mathbb{C})$. Then, by the Stokes theorem, for sufficiently large $R > 0$ we have

$$0 = \int_{\partial K(R)} u dz = \int_{K(R)} \frac{\partial u}{\partial \overline{z}}(z) d\overline{z} \wedge dz = 2i \int_{K(R)} v_0 d\mathcal{L}^2 \neq 0;$$

contradiction.

Lemma 4.2.4 (The Cauchy–Green formula). *Let $D \subset \mathbb{C}$ be a regular domain (cf. Definition 1.2.4) and let $f \in \mathcal{C}^1(\overline{D})$. Then*

$$f(z) = \frac{1}{2\pi i}\Big(\int_{\partial D} \frac{f(\zeta)}{\zeta - z} d\zeta + \int_D \frac{\frac{\partial f}{\partial \overline{\zeta}}(\zeta)}{\zeta - z} d\zeta \wedge d\overline{\zeta}\Big), \quad z \in D.$$

Proof. Fix an $a \in D$ and take a disc $K(a, \varepsilon) \subset\subset D$. Then, by the Stokes theorem (applied to the domain $D_\varepsilon := D \setminus \overline{K}(a, \varepsilon)$), we have

$$\int_{\partial D} \frac{f(\zeta)}{\zeta - a} d\zeta - \int_{C(a,\varepsilon)} \frac{f(\zeta)}{\zeta - a} d\zeta = \int_{\partial D_\varepsilon} \frac{f(\zeta)}{\zeta - a} d\zeta = \int_{D_\varepsilon} d\Big(\frac{f(\zeta)}{\zeta - a} d\zeta \Big)$$
$$= - \int_{D_\varepsilon} \frac{\frac{\partial f}{\partial \overline{\zeta}}(\zeta)}{\zeta - a} d\zeta \wedge d\overline{\zeta} \underset{\varepsilon \longrightarrow 0}{\longrightarrow} - \int_D \frac{\frac{\partial f}{\partial \overline{\zeta}}(\zeta)}{\zeta - a} d\zeta \wedge d\overline{\zeta}.$$

On the other hand,

$$\lim_{\varepsilon \longrightarrow 0} \Big| \frac{1}{2\pi i} \int_{C(a,\varepsilon)} \frac{f(\zeta)}{\zeta - a} d\zeta - f(a) \Big| \leq \lim_{\varepsilon \longrightarrow 0} \Big(\max\{|f(\zeta) - f(a)| : \zeta \in C(a, \varepsilon)\} \Big) = 0.$$

□

Proof of Proposition 4.2.2. Let $v = \sum_{j=1}^n v_j d\overline{z}_j$. Note that the condition $\overline{\partial} v = 0$ means that

$$\frac{\partial v_j}{\partial \overline{z}_k} = \frac{\partial v_k}{\partial \overline{z}_j}, \quad j, k = 1, \dots, n. \tag{4.2.4}$$

Suppose that $\operatorname{supp} v \subset \mathbb{P}(R)$. For $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ define

$$u(z) := \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{v_1(\zeta, z_2, \dots, z_n)}{\zeta - z_1} d\zeta \wedge d\overline{\zeta} = - \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{v_1(z_1 - \zeta, z_2, \dots, z_n)}{\zeta} d\zeta \wedge d\overline{\zeta}.$$

Observe that $u(z) = 0$ for $(z_2, \dots, z_n) \notin \mathbb{P}_{n-1}(R)$. It is clear that $u \in \mathcal{C}^k(\mathbb{C}^n)$. Moreover, by (4.2.4), we get

$$\frac{\partial u}{\partial \overline{z}_j}(z) = - \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\frac{\partial v_1}{\partial \overline{z}_j}(z_1 - \zeta, z_2, \dots, z_n)}{\zeta} d\zeta \wedge d\overline{\zeta} = - \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\frac{\partial v_j}{\partial \overline{z}_1}(z_1 - \zeta, z_2, \dots, z_n)}{\zeta} d\zeta \wedge d\overline{\zeta}$$
$$= \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\frac{\partial v_j}{\partial \overline{z}_1}(\zeta, z_2, \dots, z_n)}{\zeta - z_1} d\zeta \wedge d\overline{\zeta} = \frac{1}{2\pi i} \int_{K(R)} \frac{\frac{\partial v_j}{\partial \overline{\zeta}}(\zeta, z_2, \dots, z_n)}{\zeta - z_1} d\zeta \wedge d\overline{\zeta}, \quad z \in \mathbb{C}^n, \ j = 1, \dots, n.$$

Now we apply the Cauchy–Green formula (with $D := K(R)$) and we get

$$\frac{\partial u}{\partial \overline{z}_j}(z) = v_j(z) - \frac{1}{2\pi i} \int_{C(R)} \frac{v_j(\zeta, z_2, \dots, z_n)}{\zeta - z_1} \, d\zeta = v_j(z), \quad z \in \mathbb{C}^n, \ j = 1, \dots, n.$$

Hence $\overline{\partial} u = v$. In particular, $\overline{\partial} u = 0$ outside $\operatorname{supp} v$, i.e. $u \in \mathcal{O}(\mathbb{C}^n \setminus \operatorname{supp} v)$. Consequently, by the identity principle, $u = 0$ in the unbounded connected component of the set $\mathbb{C}^n \setminus \operatorname{supp} v$. □

Proposition 4.2.2 permits us to give a new elegant proof of the Hartogs extension theorem (cf. Theorem 2.1.2).

Theorem 4.2.5. *Let D be a domain in $\mathbb{C}^n$, $n \geq 2$, and let K be a compact subset of D such that $D \setminus K$ is connected. Then $\mathcal{O}(D \setminus K) = \mathcal{O}(D)|_{D \setminus K}$.*

Proof. Fix an $f \in \mathcal{O}(D \setminus K)$. Let $\varphi \in \mathcal{C}_0^\infty(D)$ be such that $\varphi = 1$ in a neighborhood U of K. Put

$$F := \begin{cases} (1-\varphi)\cdot f & \text{in } D \setminus K \\ 0 & \text{in } U \end{cases}.$$

Then clearly F is well defined and $F \in \mathcal{C}^\infty(D)$. Define

$$v := \begin{cases} \overline{\partial} F & \text{in } D \\ 0 & \text{in } \mathbb{C}^n \setminus \operatorname{supp}\varphi \end{cases}.$$

Then v is well defined, $v \in \mathcal{C}^\infty_{(0,1)}(D)$, $\overline{\partial} v = 0$, and $\operatorname{supp} v \subset \operatorname{supp}\varphi \subset\subset D$. By Proposition 4.2.2 there exists a function $u \in \mathcal{C}_0^\infty(\mathbb{C}^n)$ such that $\overline{\partial} u = v$ and $u = 0$ in the unbounded connected component Ω_∞ of the set $\mathbb{C}^n \setminus \operatorname{supp} v$.

Put $\widetilde{f} := F - u$. Then $\widetilde{f} \in \mathcal{C}^\infty(D)$ and $\overline{\partial}\widetilde{f} = 0$. Thus $\widetilde{f} \in \mathcal{O}(D)$. Moreover $\widetilde{f} = F = f$ in $(D \setminus \operatorname{supp}\varphi) \cap \Omega_\infty \neq \varnothing$. Hence, by the identity principle, $\widetilde{f} = f$ in $D \setminus K$. $\square$

Definition 4.2.6. Let $S_{p,q} = S_{p,q}(\mathbb{C}^n)$ denote the family of all sets $A \subset \mathbb{C}^n$ such that for every open neighborhood G of A and for every $\overline{\partial}$-closed form $v \in \mathcal{C}^\infty_{(p,q+1)}(G)$ there exist an open neighborhood $\widetilde{G}$ of A (with $\widetilde{G} \subset G$) and a form $u \in \mathcal{C}^\infty_{(p,q)}(\widetilde{G})$ such that $\overline{\partial} u = v$ in $\widetilde{G}$.

Remark 4.2.7. (a) If $\Omega \subset \mathbb{C}^n$ is open, then $\Omega \in S_{p,q}$ iff for any $\overline{\partial}$-closed form $v \in \mathcal{C}^\infty_{(p,q+1)}(\Omega)$ there exists a $u \in \mathcal{C}^\infty_{(p,q)}(\Omega)$ such that $\overline{\partial} u = v$.
(b) Let $\Phi : \Omega \longrightarrow \Omega'$ be biholomorphic, where $\Omega, \Omega' \subset \mathbb{C}^n$ are open. Then for any $A \subset \Omega$ we have: $A \in S_{p,q} \Longleftrightarrow \Phi(A) \in S_{p,q}$.

Indeed, let $v' \in \mathcal{C}^\infty_{(p,q+1)}(G')$ be $\overline{\partial}$-closed, where G' is an open neighborhood of $\Phi(A)$. Then $v := \Phi^*(v') \in \mathcal{C}^\infty_{(p,q+1)}(G)$ and $\overline{\partial} v = 0$, where $G := \Phi^{-1}(G') \supset A$ (cf. Remark 4.2.1(d)). Since $A \in S_{p,q}$, there exist an open neighborhood $\widetilde{G}$ of A ($\widetilde{G} \subset G$) and $u \in \mathcal{C}^\infty_{(p,q)}(\widetilde{G})$ such that $\overline{\partial} u = v$ in $\widetilde{G}$. Put $u' := (\Phi^{-1})^*(u)$. Then $u' \in \mathcal{C}^\infty_{(p,q)}(\widetilde{G}')$ with $\widetilde{G}' := \Phi(\widetilde{G}) \subset G'$ and

$$\overline{\partial} u' = (\Phi^{-1})^*(\overline{\partial} u) = (\Phi^{-1})^*(v) = (\Phi^{-1})^*(\Phi^*(v')) = v'.$$

Proposition 4.2.8. *Let $\varphi_1, \dots, \varphi_m \in \mathcal{O}(\mathbb{C}^{n-m})$ ($1 \le m \le n-1$) and let*

$$\mathbb{C}^{n-m} \ni z' \overset{\mu}{\longmapsto} (z', \varphi_1(z'), \dots, \varphi_m(z')) \in \mathbb{C}^n.$$

Take a set $A \subset \mathbb{C}^n$ and put $A' := \mu^{-1}(A) \subset \mathbb{C}^{n-m}$.
(a) *If*

$$A \in S_{p,q} \cap \dots \cap S_{p,q+m-1},$$

then for any $\overline{\partial}$-closed form $w' \in \mathcal{C}^\infty_{(p,q)}(G')$, where G' is an open neighborhood of A', there exist an open neighborhood G of A and a $\overline{\partial}$-closed form $w \in \mathcal{C}^\infty_{(p,q)}(G)$ such that $w' = \mu^*(w)$ in $\mu^{-1}(G) \subset G'$.

In particular, if $\Omega \subset \mathbb{C}^n$ is open and $\Omega \in S_{0,0} \cap \dots \cap S_{0,m-1}$, then for any $f' \in \mathcal{O}(\Omega')$ there exists an $f \in \mathcal{O}(\Omega)$ such that $f' = f \circ \mu$.
(b) *If*

$$A \in S_{p,q} \cap \dots \cap S_{p,q+m},$$

then $A' \in S_{p,q}(\mathbb{C}^{n-m})$.

Proof. We will use finite induction on m (with an arbitrary n).

$m = 1$.

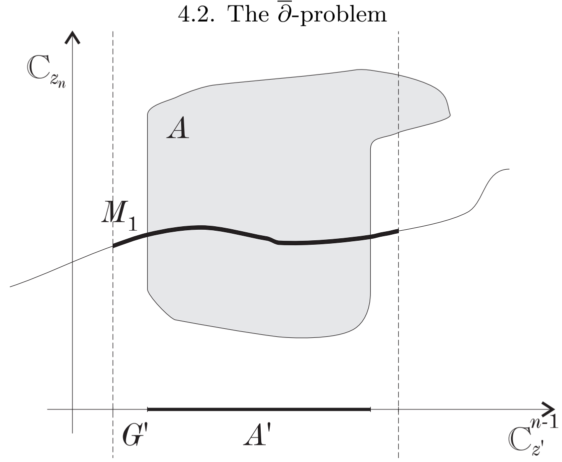

Figure 4.2.1

(a) Define $\pi : \mathbb{C}^n \longrightarrow \mathbb{C}^{n-1}$, $\pi(z', z_n) := z'$. Fix a $\overline{\partial}$-closed form $w' \in \mathcal{C}^\infty_{(p,q)}(G')$, where G' is an open neighborhood of A'. Put $\Omega := (G' \times \mathbb{C}) \cup (\mathbb{C}^n \setminus \mu(\mathbb{C}^{n-1}))$. Observe that Ω is an open neighborhood of A. Put

$$M_0 := \{z \in \Omega : \pi(z) \notin G'\}, \quad M_1 := \mu(G') = \Omega \cap \mu(\mathbb{C}^{n-1}).$$

Note that the sets M_0, M_1 are relatively closed in Ω and disjoint. In particular, there exists a function $\chi \in \mathcal{C}^\infty(\Omega)$ such that $\chi = j$ in an open neighborhood U_j of M_j, $j = 0, 1$ (cf. [32]). Let

$$\widetilde{w} := \begin{cases} \chi \cdot \pi^*(w') & \text{in } G' \times \mathbb{C} \\ 0 & \text{in } U_0 \end{cases}.$$

Obviously, $\widetilde{w}$ is well defined and $\widetilde{w} \in \mathcal{C}^\infty_{(p,q)}(\Omega)$. We have

$$\mu^*(\widetilde{w}) = \mu^*(\chi \cdot \pi^*(w')) = (\chi \circ \mu) \cdot (\pi \circ \mu)^*(w') = (\chi \circ \mu) \cdot w' = w' \ \text{ in } G'.$$

The form $\widetilde{w}$ need not be $\overline{\partial}$-closed. Therefore, we take $w := \widetilde{w} - Q \cdot u$ with $Q(z) := \varphi_1(z') - z_n$, where $u \in \mathcal{C}^\infty_{(p,q)}(G)$ will be chosen below. Independently of a choice of u, we have

$$\mu^*(w) = \mu^*(\widetilde{w}) - \mu^*(Q \cdot u) = \mu^*(\widetilde{w}) = w'.$$

Observe that

$$\overline{\partial} w = \begin{cases} (\overline{\partial}\chi) \wedge \pi^*(w') - Q\overline{\partial}u & \text{in } G' \times \mathbb{C} \\ -Q\overline{\partial}u & \text{in } U_0 \end{cases}.$$

To get $\overline{\partial} w = 0$ we only need to find u such that

$$v := \overline{\partial} u = \begin{cases} (1/Q) \cdot (\overline{\partial}\chi) \wedge \pi^*(w') & \text{in } (G' \times \mathbb{C}) \setminus M_1 \\ 0 & \text{in } U_0 \cup U_1 \end{cases}.$$

It is easy to check that $v \in \mathcal{C}^\infty_{(p,q+1)}(\Omega)$ and $\overline{\partial} v = 0$. Since $A \in S_{p,q}$, there exist an open neighborhood G of A (with $G \subset \Omega$) and $u \in \mathcal{C}^\infty_{(p,q)}(G)$ such that $\overline{\partial} u = v$ in G. Consequently, w is $\overline{\partial}$-closed in G and $w' = \mu^*(w)$ in $\mu^{-1}(G) \subset G'$.

(b) Take a $\overline{\partial}$-closed form $v' \in \mathcal{C}^\infty_{(p,q+1)}(G')$, where G' is an open neighborhood of A'. Since $A \in S_{p,q+1}$, assertion (a) implies that there exist an open neighborhood G of A and a $\overline{\partial}$-closed form $v \in \mathcal{C}^\infty_{(p,q+1)}(G)$ such that $v' = \mu^*(v)$ in $\mu^{-1}(G)$. Since $A \in S_{p,q}$, there exist an open neighborhood $\widetilde{G}$ of A (with $\widetilde{G} \subset G$) and $u \in \mathcal{C}^\infty_{(p,q)}(\widetilde{G})$ such that $\overline{\partial} u = v$ in $\widetilde{G}$. Put $u' := \mu^*(u)$ in $\widetilde{G}' := \mu^{-1}(\widetilde{G})$. Then $u' \in \mathcal{C}^\infty_{(p,q)}(\widetilde{G}')$ and $\overline{\partial} u' = \mu^*(\overline{\partial} u) = \mu^*(v) = v'$ in $\widetilde{G}'$.

$m - 1 \rightsquigarrow m$.

Observe that $\mu = \mu_2 \circ \mu_1$, where

$$\mathbb{C}^{n-m} \ni z' \xrightarrow{\mu_1} (z', \varphi_1(z')) \in \mathbb{C}^{n-m} \times \mathbb{C},$$
$$\mathbb{C}^{n-m} \times \mathbb{C} \ni (z', z'') \xrightarrow{\mu_2} (z', z'', \varphi_2(z'), \dots, \varphi_m(z')) \in \mathbb{C}^n.$$

(a) Take a $\overline{\partial}$-closed form $w' \in \mathcal{C}^\infty_{(p,q)}(G')$, where G' is an open neighborhood of A'. Since (b) is true for $m-1$, the set $A_1 := \mu_2^{-1}(A)$ belongs to $S_{p,q}(\mathbb{C}^{n-m+1})$. Note that $A' = \mu_1^{-1}(A_1)$. Since (a) is true for $m = 1$, there exist an open neighborhood G_1 of A_1 and a $\overline{\partial}$-closed form $w_1 \in \mathcal{C}^\infty_{(p,q)}(G_1)$ such that $w' = \mu_1^*(w_1)$ in $\mu_1^{-1}(G_1) \subset G'$. Now, since (a) is true for $m-1$, there exist a neighborhood G of A and a $\overline{\partial}$-closed form $w \in \mathcal{C}^\infty_{(p,q)}(G)$ such that $w_1 = \mu_2^*(w)$ in $\mu_2^{-1}(G) \subset G_1$. Then $w' = \mu_1^*(\mu_2^*(w)) = \mu^*(w)$ in $\mu^{-1}(G) \subset G'$.

(b) Since (b) is true for $m-1$, the set $A_1 := \mu_2^{-1}(A)$ belongs to $S_{p,q} \cap S_{p,q+1}$. Now we apply the case $m = 1$ to A_1 and we conclude that $A' = \mu_1^{-1}(A_1) \in S_{p,q}$.

□

Theorem 4.2.9. *Let $\Omega \subset \mathbb{C}^n$ be open, $n \geq 2$. If*

$$\Omega \in S_{0,0} \cap \dots \cap S_{0,n-2},$$

then Ω is holomorphically convex.

Proof. It is sufficient to show that for any $a \in \Omega$ there exists an $f \in \mathcal{O}(\Omega)$ such that $d(T_a f) = d_\Omega(a)$ (Proposition 2.7.5). Fix an $a = (a_1, \dots, a_n) \in \Omega$, let $P := \mathbb{P}(a, d_\Omega(a))$, and let $b \in \partial\Omega \cap \overline{P}$. Using a complex affine isomorphism of $\mathbb{C}^n$, we may assume that $b = 0$ and $a_2 = \dots = a_n = 0$ (cf. Remark 4.2.7(b)). Let

$$\Omega' := \{z' \in \mathbb{C} : (z', 0, \dots, 0) \in \Omega\}.$$

Note that $0 \in \partial\Omega$. By Proposition 4.2.8 (with $f'(z') := 1/z'$) there exists an $f \in \mathcal{O}(\Omega)$ such that $f(z', 0) = 1/z'$. Obviously, f cannot be extended across b. □

Remark 4.2.10. To solve the Levi problem it suffices to show that any pseudoconvex domain belongs to $S_{0,0} \cap \dots \cap S_{0,n-2}$.

4.3. Runge domains

Definition 4.3.1. A region of holomorphy $\Omega \subset \mathbb{C}^n$ is called a *Runge region* if every function $f \in \mathcal{O}(\Omega)$ can be approximated uniformly on every compact subset of Ω by polynomials of n complex variables.

If Ω is a connected Runge region, then we say that Ω is a *Runge domain*.

Obviously, if Ω is a Runge region, then each connected component of Ω is a Runge domain.

Note that in the above definition the space of polynomials can be replaced by the space of entire functions.

Let K be a compact subset of $\mathbb{C}^n$. Recall (Remark 2.7.11(e)) that

$$\widehat{K} := \widehat{K}_{\mathcal{O}(\mathbb{C}^n)} = \widehat{K}_{\mathcal{P}(\mathbb{C}^n)} = \{z \in \mathbb{C}^n : \forall_{P \in \mathcal{P}(\mathbb{C}^n)} : |\mathbb{P}(z)| \leq \|P\|_K\}.$$

If $K = \widehat{K}$, then we say that K is *polynomially convex.*

Remark 4.3.2. (a) If $\Omega \subset \mathbb{C}$, then Ω is a Runge region iff any connected component of Ω is simply connected (cf. [4], VIII.1).

(b) By Proposition 1.6.2 any balanced domain of holomorphy is a Runge domain.

(c) Let G be a Runge domain in $\mathbb{C}^{n-k}$ and let $D \subset \mathbb{C}^n$ be a Hartogs domain of holomorphy over G (cf. Definition 1.6.3) such that one of the following conditions is satisfied:

$\forall_{z \in G}$ D_z is connected, k-circled, and $D \cap (G \times \{0\}^k) \neq \varnothing$.

$\forall_{z \in G}$ D_z is balanced.

Then, by Propositions 2.6.3 and 1.6.5(b) D is a Runge domain

(d) Let $F : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be a polynomial mapping and let $\Omega' \subset \mathbb{C}^n$ be a Runge region. Put $\Omega := F^{-1}(\Omega')$. Assume that $F|_\Omega : \Omega \longrightarrow \Omega'$ is biholomorphic. Then Ω is also a Runge region (cf. Example 4.3.8).

Indeed, let $g \in \mathcal{O}(\Omega)$, $K \subset\subset \Omega$, and let $(p_\nu)_{\nu=1}^\infty$ be a sequence of polynomials such that $p_\nu \longrightarrow g \circ (F|_\Omega)^{-1}$ uniformly on $F(K)$. Then $p_\nu \circ F \longrightarrow g$ uniformly on K.

Theorem 4.3.3. *Let $\varOmega$ be a region of holomorphy in $\mathbb{C}^n$. The following conditions are equivalent:*
(i) *$\varOmega$ is a Runge region;*
(ii) *$\widehat{K} = \widehat{K}_{\mathcal{O}(\varOmega)}$ for any compact $K \subset \varOmega$;*
(iii) *$\widehat{K} \cap \varOmega = \widehat{K}_{\mathcal{O}(\varOmega)}$ for any compact $K \subset \varOmega$;*
(iv) *$\widehat{K} \cap \varOmega \subset\subset \varOmega$ for any compact $K \subset \varOmega$.*

$$\begin{tikzcd}
& \text{(ii)} \arrow[dl] \arrow[d] & \\
\text{(i)} \arrow[r] & \text{(iii)} \arrow[r] & \text{(iv)} \arrow[ul]
\end{tikzcd}$$

The implication (i) $\Longrightarrow$ (iii) follows immediately from the fact that polynomials are dense in $\mathcal{O}(\varOmega)$.

The implication (ii) $\Longrightarrow$ (iii) is obvious.

The implication (iii) $\Longrightarrow$ (iv) follows from Theorem 2.7.12.

In the sequel (after Proposition 4.3.6) we will prove that (iv) $\Longrightarrow$ (ii) $\Longrightarrow$ (i).

Lemma 4.3.4. *Let G be a neighborhood of a polynomially convex compact set $K \subset \mathbb{C}^n$. Then there exist $m \in \mathbb{N}$ and $P_1, \dots, P_m \in \mathcal{P}(\mathbb{C}^n)$ such that*

$$K \subset \{z \in \mathbb{C}^n : |P_j(z)| \le 1,\ j = 1, \dots, m\} =: L \subset\subset G.$$

The set L is called a *polynomial polyhedron*. Note that L is polynomially convex.

Proof. Let $\varepsilon > 0$ be such that $K \subset \overline{\mathbb{P}}(1/\varepsilon)$. Put $P_j(z) := \varepsilon z_j$, $j = 1, \dots, n$. Let $M := \overline{\mathbb{P}}(1/\varepsilon) \setminus G$. Since K is polynomially convex, for every point $z \in M$ there exists a polynomial P such that $|\mathbb{P}(z)| > 1$ and $\|P\|_K \le 1$. Consequently, there exists a finite number of polynomials $P_{n+1}, \dots, P_m$ such that $\|P_j\|_K \le 1$, $j = n+1, \dots, m$, and $\max\{|P_j| : j = n+1, \dots, m\} > 1$ on M. The polynomials $P_1, \dots, P_m$ satisfy all the required conditions. $\square$

Proposition 4.3.5. *Let $v \in \mathcal{C}^\infty_{(p,q+1)}(\mathbb{P}_n(a, \boldsymbol{r}))$ be $\overline{\partial}$-closed. Then for every polydisc $\mathbb{P}_n(a, \boldsymbol{r}') \subset\subset \mathbb{P}_n(a, \boldsymbol{r})$ there exists a $u \in \mathcal{C}^\infty_{(p,q)}(\mathbb{P}_n(a, \boldsymbol{r}'))$ such that $\overline{\partial} u = v$ in $\mathbb{P}_n(a, \boldsymbol{r}')$.*

In particular, every closed polydisc Π belongs to $S_{p,q}$ for any $p, q \in \mathbb{N}_0$.

Proof. We use induction on k, where k is such that v is independent of $d\overline{z}_{k+1}, \dots, d\overline{z}_n$ (the case $k = n$ will give the required result). If $k = 0$, then $v = 0$ and therefore the situation is trivial.

Suppose that the result has been proved for $k - 1$, and let v be independent of $d\overline{z}_{k+1}, \dots, d\overline{z}_n$. Write

$$v = d\overline{z}_k \wedge g + h, \tag{4.3.1}$$

where $g \in \mathcal{C}^\infty_{(p,q)}(\mathbb{P}_n(a, r))$, $h \in \mathcal{C}^\infty_{(p,q+1)}(\mathbb{P}_n(a, r))$, and g and h are independent of $d\overline{z}_k, \dots, d\overline{z}_n$. Write g in the canonical form

$$g = {\sum_{|I|=p, |J|=q}}' g_{I,J} dz_I \wedge d\overline{z}_J.$$

Since $\overline{\partial} v = 0$, we easily conclude that for any I, J we have

$$\frac{\partial g_{I,J}}{\partial \overline{z}_j} = 0, \quad j > k.$$

In other words, the functions $g_{I,J}$ are holomorphic with respect to $z_{k+1}, \dots, z_n$.

Choose a function $\psi \in \mathcal{C}^\infty_0(K(a_k, r_k))$ such that $\psi(z_k) = 1$ on $\overline{K}(a_k, r''_k)$ with $r'_k < r''_k < r_k$. Let

$$\begin{aligned} G_{I,J}(z) : &= \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\psi(\tau)}{\tau - z_k} g_{I,J}(z_1, \dots, z_{k-1}, \tau, z_{k+1}, \dots, z_n) d\tau \wedge d\overline{\tau} \\ &= -\frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\psi(z_k - \tau)}{\tau} g_{I,J}(z_1, \dots, z_{k-1}, z_k - \tau, z_{k+1}, \dots, z_n) d\tau \wedge d\overline{\tau}. \end{aligned}$$

The function $G_{I,J}$ is well defined for $z \in \mathbb{P}(a,r)$ and $G_{I,J} \in \mathcal{C}^\infty(\mathbb{P}(a,r))$. Observe that

$$\frac{\partial G_{I,J}}{\partial \overline{z}_j}(z) = 0, \quad j > k,\ z \in \mathbb{P}(a,r).$$

Moreover, by the Cauchy–Green formula (cf. Lemma 4.2.4), we obtain

$$\frac{\partial G_{I,J}}{\partial \overline{z}_k}(z) = \frac{1}{2\pi i}\int\limits_{K(a_k,r_k)} \frac{\partial}{\partial \overline{\tau}}(\psi(\tau) g_{I,J}(z_1,\dots,z_{k-1},\tau,z_{k+1},\dots,z_n))\frac{d\tau\wedge d\overline{\tau}}{\tau - z_k} = g_{I,J}(z), \quad z \in \mathbb{P}(a,r'').$$

Define

$$G := \sideset{}{'}\sum_{|I|=p,|J|=q} G_{I,J} dz_I \wedge d\overline{z}_J.$$

Then

$$\overline{\partial} G = \sideset{}{'}\sum_{|I|=p,|J|=q}\sum_{j=1}^n \frac{\partial G_{I,J}}{\partial \overline{z}_j} d\overline{z}_j \wedge dz_I \wedge d\overline{z}_J = d\overline{z}_k \wedge g + h_1 \text{ in } \mathbb{P}(a,r''), \tag{4.3.2}$$

where h_1 is independent of $d\overline{z}_k,\dots,d\overline{z}_n$. Consider in $\mathbb{P}(a,r'')$ the form $v - \overline{\partial}G$. In virtue of (4.3.1) and (4.3.2) we have $v - \overline{\partial}G = h - h_1$. Hence $v - \overline{\partial}G$ is independent of $d\overline{z}_k,\dots,d\overline{z}_n$. Moreover, $\overline{\partial}(v - \overline{\partial}G) = \overline{\partial}v = 0$. Therefore, there exists a $\widetilde{u} \in \mathcal{C}^\infty_{(p,q)}(\mathbb{P}(a,r'))$ such that $\overline{\partial}\widetilde{u} = v - \overline{\partial}G$. Define $u := \widetilde{u} + G$. Then $\overline{\partial}u = \overline{\partial}\widetilde{u} + \overline{\partial}G = v$. □

Proposition 4.3.6. *Let $K \subset\subset \mathbb{C}^n$ be polynomially convex and let f be holomorphic in a neighborhood of K. Then there exists a sequence $(f_j)_{j=1}^\infty \subset \mathcal{P}(\mathbb{C}^n)$ such that $f_j \longrightarrow f$ uniformly on K.*

Proof. By Lemma 4.3.4 there exists a polynomial polyhedron L such that $K \subset L$ and f is holomorphic in a neighborhood of L. Choose a closed polydisc $\Pi \supset L$ so that

$$L = \{z \in \Pi : |P_j(z)| \le 1,\ j = 1,\dots,m\}.$$

By Proposition 4.3.5 $\Pi \times \overline{\mathbb{D}}^m \in S_{0,0} \cap \dots \cap S_{0,m-1}$. Hence, by Proposition 4.2.8 with $A := \Pi \times \overline{\mathbb{D}}^m$, there exists a function F holomorphic in a neighborhood of $\Pi \times \overline{\mathbb{D}}^m \subset \mathbb{C}^n \times \mathbb{C}^m$ such that

$$f(z) = F(z, P_1(z),\dots,P_m(z))$$

in a neighborhood of L. Let F_k be the k-th partial sum of the Taylor expansion of F in a neighborhood of $\Pi \times \overline{\mathbb{D}}^m$, $k \ge 0$. Then $F_k \longrightarrow F$ uniformly on $\Pi \times \overline{\mathbb{D}}^m$. Therefore

$$F_k(z, P_1,\dots,P_m) \longrightarrow F(z,P_1,\dots,P_m) = f \text{ uniformly on } L.$$

□

The end of the proof of Theorem 4.3.3. The implication (ii) $\Longrightarrow$ (i) can be obtained by an immediate application of Proposition 4.3.6.

Now we prove that (iv) $\Longrightarrow$ (ii). Define

$$K_1 := \widehat{K} \cap \varOmega, \quad K_2 := \widehat{K} \setminus \varOmega.$$

By (iv), the set K_1 is compact. Since K_2 is a closed subset of $\widehat{K}$, K_2 is also compact. Moreover, $K_1 \cap K_2 = \varnothing$. Let $f := 0$ in a neighborhood of K_1 and $f := 1$ in a neighborhood of K_2. Then f is holomorphic in a neighborhood of $\widehat{K}$. Since $\widehat{K}$ is polynomially convex, Proposition 4.3.6 implies that there exists a polynomial g such that $|g - f| < 1/2$ on $\widehat{K}$. Hence $|g| < 1/2$ on K_1 and $|g| > 1/2$ on K_2. Since $K \subset K_1$ and $K_2 \subset \widehat{K}$, it must hold $K_2 = \varnothing$. Therefore $\widehat{K} = K_1 \subset \varOmega$. Let $f \in \mathcal{O}(\varOmega)$. By Proposition 4.3.6, the function f can be approximated uniformly on $\widehat{K} = K_1$ (in particular, on K) by polynomials of n complex variables. This means that $\varOmega$ is a Runge domain, i.e. condition (i) of Theorem 4.3.3 is fulfilled. We already know that (i) $\Longrightarrow$ (iii). Hence $\widehat{K} \cap \varOmega = \widehat{K}_{\mathcal{O}(\varOmega)}$. Finally, $\widehat{K} = \widehat{K}_{\mathcal{O}(\varOmega)}$. □

Theorem 4.3.3 and Remark 2.7.11(f) give

Corollary 4.3.7. *Ω_j is a Runge region in $\mathbb{C}^{n_j}$, $j = 1, 2$, iff $\Omega_1 \times \Omega_2$ is a Runge region in $\mathbb{C}^{n_1+n_2}$.*

In particular, if $D_j \subset \mathbb{C}$ is simply connected, $j = 1, \dots, n$, then $D_1 \times \dots \times D_n$ is a Runge domain in $\mathbb{C}^n$.

Example 4.3.8 (Wermer)**.** Recall (Remark 4.3.2(a)) that for $n = 1$ Runge regions are characterized in a purely topological way. This is no longer true for $n \geq 2$. We will show that Runge regions in $\mathbb{C}^n$ with $n \geq 2$ are not invariant under biholomorphic mappings. Namely, for $n \geq 2$ we will find a domain $D_n \subset \mathbb{C}^n$ biholomorphic to $\mathbb{D}^n$ and such that D_n is not a Runge domain.

First, following [8], we will construct a domain $D \subset \mathbb{C}^2$ biholomorphic to $\mathbb{D}^2$ such that $\mathbb{T} \times \{0\} \subset D$ but $(1/2, 0) \notin D$.

Assume for a moment that such a domain is already constructed. Then for any $n \geq 2$ put $D_n := D \times \mathbb{D}^{n-2}$. The domain D_n is obviously biholomorphic to $\mathbb{D}^n$. Moreover, $K_n := \mathbb{T} \times \{(0, \dots, 0)\} \subset D_n$, $(1/2, 0, \dots, 0) \notin D_n$. Suppose that D_n is a Runge domain. Then, by Theorem 4.3.3, $\widehat{K}_n \subset D_n$. On the other hand, by the maximum principle, $\overline{\mathbb{D}} \times \{(0, \dots, 0)\} \subset \widehat{K}_n$. In particular, $(1/2, 0, \dots, 0) \in D_n$; contradiction.

We pass to the construction. Let

$$A_0 := ([-1, 1] + i0) \times ([-1, 1] + i0) \subset \mathbb{C}^2, \qquad A := \{(x + iy, x - iy) \in \mathbb{C}^2 : x, y \in [-1, 1]\}.$$

Let U be an arbitrary open neighborhood of A_0. Then there exist open rectangles $R_1, R_2 \subset \mathbb{C}$ such that $A_0 \subset R_1 \times R_2 \subset U$. By the Riemann mapping theorem, the domain $R_1 \times R_2$ is biholomorphic to $\mathbb{D}^2$. Thus A_0 has a neighborhood basis consisting of domains biholomorphic to $\mathbb{D}^2$. Observe that the mapping

$$\mathbb{C}^2 \ni (z_1, z_2) \longmapsto (z_1 + iz_2, z_1 - iz_2) \in \mathbb{C}^2$$

maps biholomorphically A_0 onto A. Consequently, A has a neighborhood basis consisting of domains biholomorphic to $\mathbb{D}^2$.

Put

$$F(z, w) := (z, P(z, w)), \quad P(z, w) := (1 + i)w - izw^2 - z^2w^3.$$

Observe that:

- For any $\zeta \in \mathbb{T}$ we have $(\zeta, \overline{\zeta}) \in A$ and $F(\zeta, \overline{\zeta}) = (\zeta, 0)$. Hence $(\mathbb{T}) \times \{0\} \subset F(A)$.
- $(1/2, 0) \notin F(A)$. Indeed, $P(1/2, 1/2) = (1 + i)/2 - i/8 - 1/32 \neq 0$.
- $F|_A$ is injective.
- $J_{\mathbb{C}}F \neq 0$ on A. Indeed, we have $J_{\mathbb{C}}F(z, w) = \frac{\partial P}{\partial w}(z, w) = 1 + i - 2izw - 3z^2w^2$ and hence $J_{\mathbb{C}}F(z, \overline{z}) = 1 + i - 2i|z|^2 - 3|z|^4 \neq 0$.

In particular, there exists an open neighborhood U_0 of A such that $F|_{U_0}$ is biholomorphic.

Now, since A has a neighborhood basis consisting of domains biholomorphic to $\mathbb{D}^2$, we find a domain U biholomorphic to $\mathbb{D}^2$ such that $A \subset U \subset U_0$ and $(1/2, 0) \notin F(U)$. Finally, we put $D := F(U)$.

Notice that for $n \geq 3$ the construction of the required domain D_n may be essentially simplified. Instead of $D_n = D \times \mathbb{D}^{n-2}$ (as above) we will construct (in a simpler way) a domain $G \subset \mathbb{C}^3$ biholomorphic to a polydisc such that $(\mathbb{T}) \times \{(1, 0)\} \subset G$ but $(0, 1, 0) \notin G$. Next, for $n \geq 3$ we take $D_n := G \times \mathbb{D}^{n-3}$ and we repeat the above argument showing that D_n is not Runge.

The example is also due to Wermer (cf. [13]). Let

$$F : \mathbb{C}^3 \longrightarrow \mathbb{C}^3, \quad F(x, y, z) = (x, xy + z, xy^2 - y + 2yz).$$

Then $J_{\mathbb{C}}F(x, y, z) = 1 - 2z$. In particular, F is locally biholomorphic on $\mathbb{C}^3 \setminus \{z = 1/2\}$. We will show that F is injective on $U := \mathbb{C} \times \mathbb{C} \times K(1/2)$.

Let $(x_1, y_1, z_1), (x_2, y_2, z_2) \in U$ be such that $F(x_1, y_1, z_1) = F(x_2, y_2, z_2)$. Then obviously $x_1 = x_2 =: x$ and

$$xy_1 + z_1 = xy_2 + z_2, \qquad xy_1^2 - y_1 + 2y_1z_1 = xy_2^2 - y_2 + 2y_2z_2.$$

In other words,

$$x(y_1 - y_2) = z_2 - z_1, \qquad x(y_1 - y_2)(y_1 + y_2) - y_1 + y_2 + 2y_1z_1 - 2y_2z_2 = 0.$$

Thus $(z_2 - z_1)(y_1 + y_2) - y_1 + y_2 + 2y_1z_1 - 2y_2z_2 = 0$ and hence $(y_1 - y_2)(z_1 + z_2 - 1) = 0$. Now, since $|z_1| + |z_2| < 1$, we conclude that $y_1 = y_2$ and $z_1 = z_2$.

In particular, $F|_{\mathbb{P}(0,(2,2,1/2))}$ is injective. Let $G := F(\mathbb{P}(0,(2,2,1/2)))$. Observe that $F(x,1/x,0) = (x,1,0)$. Hence $(\mathbb{T}) \times \{(1,0)\} \subset G$. Obviously, $(0,1,0) \notin G$.

4.4. Hefer's theorem

Let $\Omega \subset \mathbb{C}^n$, $n \geq 2$, be open. Assume that $0 \in \Omega$ and let

$$\Omega_k := \{z' \in \mathbb{C}^k : (z',0,\dots,0) \in \Omega\}, \quad M_k := \Omega \cap (\mathbb{C}^k \times \{0\}^{n-k}), \quad k = 1,\dots,n-1.$$

Let, moreover, $M_0 := \{0\}$. Observe that if

$$f(z) := \sum_{j=k+1}^{n} z_j f_j(z), \quad z = (z_1,\dots,z_n) \in \Omega,$$

where $f_j \in \mathcal{O}(\Omega)$, $j = k+1,\dots,n$, then $f = 0$ on M_k $(k = 0,\dots,n-1)$.

Our aim is to prove a converse theorem.

Remark 4.4.1. The case $k = n-1$ is elementary.

Indeed, let $f \in \mathcal{O}(\Omega)$ be such that $f = 0$ on M_{n-1}, i.e. $f(z',0) = 0$ for any $z' \in \Omega_{n-1}$. Define

$$f_n(z) := f(z)/z_n, \quad z \in \Omega \setminus M_{n-1}.$$

It remains to observe that f_n extends holomorphically to Ω.

Remark 4.4.2. Assume that Ω is a star-shaped domain with respect to 0 and $k = 0$. Let $f \in \mathcal{O}(\Omega)$ be such that $f(0) = 0$. Then we have

$$f(z) = \int_0^1 \frac{d}{dt} f(tz)dt = \int_0^1 \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(tz)dt =: \sum_{j=1}^{n} z_j f_j(z), \quad z \in \Omega.$$

It is clear that $f_1,\dots,f_n \in \mathcal{O}(\Omega)$.

Proposition 4.4.3. *Assume that*

$$\Omega \in S_{0,0} \cap \dots \cap S_{0,n-k-2}$$

for some $k \in \{0,\dots,n-1\}$ [6]. *Then for any* $f \in \mathcal{O}(\Omega)$ *such that* $f = 0$ *on* M_k *there exist* $f_{k+1},\dots,f_n \in \mathcal{O}(\Omega)$ *such that*

$$f = \sum_{j=k+1}^{n} z_j f_j.$$

Proof. We apply finite induction on $n-k$. The case $n-k = 1$ has been solved in Remark 4.4.1.

$n-k \rightsquigarrow n-k+1$.

Assume that $\Omega \in S_{0,0} \cap \dots \cap S_{0,n-(k-1)-2}(\mathbb{C}^n)$. By Proposition 4.2.8,

$$\Omega_{n-1} \in S_{0,0} \cap \dots \cap S_{0,(n-1)-(k-1)-2}(\mathbb{C}^{n-1}).$$

Consequently, there exist $F_k,\dots,F_{n-1} \in \mathcal{O}(\Omega_{n-1})$ such that

$$f(z',0) = \sum_{j=k}^{n-1} z_j F_j(z'), \quad z' = (z_1,\dots,z_{n-1}) \in \Omega_{n-1}.$$

By Proposition 4.2.8 there exist $f_k,\dots,f_{n-1} \in \mathcal{O}(\Omega)$ such that $F_j(z') = f_j(z',0)$, $j = k,\dots,n-1$. Put

$$\widetilde{f} := f - \sum_{j=k}^{n-1} z_j f_j.$$

---

[6] If $k = n-1$, then Ω is arbitrary.

Then $\widetilde{f} \in \mathcal{O}(\Omega)$ and $\widetilde{f} = 0$ on M_{n-1}. Hence $\widetilde{f} = z_n f_n$ for a function $f_n \in \mathcal{O}(\Omega)$. Finally,

$$f = \sum_{j=k}^{n} z_j f_j \quad \text{on } \Omega. \qquad \square$$

Theorem 4.4.4. *Assume that $\Omega \subset \mathbb{C}^n$ is an open set such that*

$$\Omega \times \Omega \in S_{0,0} \cap \dots \cap S_{0,n-2}(\mathbb{C}^{2n}).$$

(a) *For any $f \in \mathcal{O}(\Omega \times \Omega)$ with*

$$f(z,z) = 0, \quad z \in \Omega,$$

there exist $f_1, \dots, f_n \in \mathcal{O}(\Omega \times \Omega)$ such that

$$f(z,w) = \sum_{j=1}^{n} (z_j - w_j) f_j(z,w), \quad z = (z_1, \dots, z_n),\ w = (w_1, \dots, w_n) \in \Omega.$$

(b) (Hefer's theorem) *For any $f_0 \in \mathcal{O}(\Omega)$ there exist $f_1, \dots,\ f_n \in \mathcal{O}(\Omega \times \Omega)$ such that*

$$f_0(z) - f_0(w) = \sum_{j=1}^{n} (z_j - w_j) f_j(z,w), \quad z, w \in \Omega.$$

Proof. (a) Let $\Phi : \mathbb{C}^{2n} \longrightarrow \mathbb{C}^{2n}$ be given by the formula

$$\Phi(\xi_1, \dots, \xi_{2n}) := (\xi_1, \dots, \xi_n, \xi_1 - \xi_{n+1}, \dots, \xi_n - \xi_{2n}).$$

The mapping Φ is a $\mathbb{C}$-linear isomorphism. Let $\Psi := \Phi^{-1}$. Then $\Psi(z,w) = (z, z-w)$. Observe that $\Psi(\Omega \times \Omega) \in S_{0,0} \cap \dots \cap S_{0,n-2}$ (cf. Remark 4.2.7(b)). Now we apply to $\Psi(\Omega \times \Omega)$ Proposition 4.4.3 with $k = n$. Consequently, there exist $\widetilde{f}_{n+1}, \dots, \widetilde{f}_{2n} \in \mathcal{O}(\Psi(\Omega \times \Omega))$ such that

$$f \circ \Phi = \sum_{j=n+1}^{2n} \xi_j \widetilde{f}_j.$$

Let $f_j := \widetilde{f}_{n+j} \circ \Psi$, $j = 1, \dots, n$. Then

$$f = (f \circ \Phi) \circ \Psi = \Big(\sum_{j=n+1}^{2n} \xi_j \widetilde{f}_j \Big) \circ \Psi = \sum_{j=1}^{n} (z_j - w_j) f_j.$$

(b) follows directly from (a) with $f(z,w) := f_0(z) - f_0(w)$. $\square$

Exercises

4.1. Schwarz type lemma (cf. Lemma 1.4.26): Let $D_j \subset \mathbb{C}^{n_j}$ be a balanced domain and let h_j denote the Minkowski functional of D_j, $j = 1, 2$. Assume that D_2 is pseudoconvex. Let $F : D_1 \longrightarrow D_2$ be a holomorphic mapping with $F(0) = 0$. Using Propositions 3.2.35 and 4.1.14, prove the following results:

(a) $h_2 \circ F \le h_1$ on D_1 and $h_2 \circ F'(0) \le h_1$. In particular, $F'(0)$ maps D_1 into D_2.

(b) If F is biholomorphic, then $h_2 \circ F = h_1$ on D_1 and $h_2 \circ F'(0) = h_1$. In particular, $F'(0)$ is a $\mathbb{C}$-linear isomorphism which maps D_1 onto D_2.

4.2. Let T be the Hartogs triangle, $T := \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < |z_2| < 1\}$. Prove that T is pseudoconvex but not hyperconvex.

4.3. Let $F : D \longrightarrow D'$ be a proper mapping, where $D, D' \subset \mathbb{C}^n$ are domains. Suppose that D' is pseudoconvex. Show that D is pseudoconvex.

4.4. Let $\omega \subset \mathbb{R}^n$ be a domain. Define

$$T_\omega := \{z \in \mathbb{C}^n : \operatorname{Re} z \in \omega\}.$$

Prove that the following conditions are equivalent:

(i) ω is convex;

(ii) T_ω is pseudoconvex.

4.5. Give an example of a pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$, which is not biholomorphically equivalent to a convex domain.

4.6. Let $D := \{(z_1, z_2) \in \mathbb{C}^2 : |z_1 z_2| < 1\}$. Does there exist $\varphi \in \mathcal{O}(D) \cap \mathcal{C}(\overline{D})$ such that $\varphi(1, 1) = 1$, $|\varphi| < 1$ in $\overline{D} \setminus \{(1, 1)\}$?

CHAPTER 5

Hörmander's solution of the $\overline{\partial}$-problem

Summary. In this chapter, following [17], we present a detailed proof of Hörmander's solution of the Levi problem. Section 5.1 contains basic facts from the distribution theory, which are used in the sequel; details are omitted, and it is assumed that the reader who requires to check the proofs, will consult classical monographs on the subject.

The exposition of Hörmander's solution of the Levi problem begins in Section 5.2. The proof is based on the theory of (unbounded) operators in Hilbert spaces, and the solution of the $\overline{\partial}$-problem for pseudoconvex domains in appropriately chosen spaces of differential forms with coefficients which are L^2-integrable with respect to convenient weight functions. The main result of this section is Hörmander's L^2-estimate. This makes possible to solve the $\overline{\partial}$-problem in pseudoconvex domains for differential forms with coefficients which are locally square integrable.

This enables in turn to do similar, but for differential forms with coefficients in Sobolev spaces, and thus, by the Sobolev inclusion, for forms with smooth coefficients. Thus, in virtue of results in Section 4.2, we obtain the solution of the Levi problem; the details are presented in Section 5.3.

5.1. Distributions

For the reader's convenience we collect below basic facts from distribution theory (the details may be found for instance in [31]).

For $K \subset \mathbb{R}^N$ let

$$\mathcal{D}(K) := \{f \in \mathcal{C}_0^\infty(\mathbb{R}^N, \mathbb{C}) : \operatorname{supp} f \subset K\}, \qquad q_{K,k}(f) := \sum_{\alpha \in \mathbb{N}_0^N :\, |\alpha| \le k} \sup_K |D^\alpha f|, \quad k \in \mathbb{N}_0,$$

where

$$D^\alpha := \Big(\frac{\partial}{\partial x_1}\Big)^{\alpha_1} \circ \dots \circ \Big(\frac{\partial}{\partial x_N}\Big)^{\alpha_N}, \quad \alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}_0^N.$$

The seminorms $(q_{K,k})_{k \in \mathbb{N}_0}$ generate on $\mathcal{D}(K)$ a Fréchet topology; we have

$$f_\nu \overset{\mathcal{D}(K)}{\longrightarrow} f_0 \iff \forall_{\alpha \in \mathbb{N}_0^N} : D^\alpha f_\nu \longrightarrow D^\alpha f_0 \text{ uniformly on } K.$$

For an open set $\Omega \subset \mathbb{R}^N$ let

$$\mathcal{D}(\Omega) := \mathcal{C}_0^\infty(\Omega, \mathbb{C}) = \bigcup_{K \subset\subset \Omega} \mathcal{D}(K).$$

We define

$$f_\nu \overset{\mathcal{D}(\Omega)}{\longrightarrow} f_0 \overset{\text{def}}{\iff} \exists_{K \subset\subset \Omega} : (f_\nu)_{\nu=1}^\infty \subset \mathcal{D}(K),\ f_\nu \overset{\mathcal{D}(K)}{\longrightarrow} f_0.$$

Let $\mathcal{E}(\Omega) := \mathcal{C}^\infty(\Omega)$. The seminorms $(q_{K,k})_{K \subset\subset \Omega,\, k \in \mathbb{N}_0}$ generate on $\mathcal{E}(\Omega)$ a Fréchet topology; we have

$$f_\nu \overset{\mathcal{E}(\Omega)}{\longrightarrow} f_0 \iff \forall_{\alpha \in \mathbb{N}_0^N} : D^\alpha f_\nu \longrightarrow D^\alpha f_0 \text{ locally uniformly in } \Omega.$$

A linear functional $\mathfrak{T} : \mathcal{D}(\Omega) \longrightarrow \mathbb{C}$ is a *distribution* on Ω ($\mathfrak{T} \in \mathcal{D}'(\Omega)$) if for any compact $K \subset \Omega$ the mapping $\mathfrak{T}|_{\mathcal{D}(K)} : \mathcal{D}(K) \longrightarrow \mathbb{C}$ is continuous.

5.1.1. For a linear operator $\mathfrak{T} : \mathcal{D}(\Omega) \longrightarrow \mathbb{C}$ the following conditions are equivalent:
(i) $\mathfrak{T} \in \mathcal{D}'(\Omega)$;
(ii) for any compact $K \subset \Omega$ there exist $C > 0$ and $k \in \mathbb{N}_0$ such that

$$|\mathfrak{T}(f)| \le C q_{K,k}(f), \quad f \in \mathcal{D}(K);$$

(iii) if $f_\nu \overset{\mathcal{D}(\Omega)}{\longrightarrow} f_0$, then $\mathfrak{T}(f_\nu) \longrightarrow \mathfrak{T}(f_0)$.

We endow $\mathcal{D}'(\Omega)$ with the weak topology, i.e.

$$\mathfrak{T}_\nu \overset{\mathcal{D}'(\Omega)}{\longrightarrow} \mathfrak{T}_0 \Longleftrightarrow \forall_{f\in\mathcal{D}(\Omega)} : \mathfrak{T}_\nu(f) \longrightarrow \mathfrak{T}_0(f).$$

5.1.2. For $u \in L^1(\Omega, \mathrm{loc})$ let

$$[u](f) := \int_\Omega uf\, d\mathcal{L}^N, \quad f \in \mathcal{D}(\Omega).$$

Then $[u] \in \mathcal{D}'(\Omega)$. Moreover, the mapping

$$L^1(\Omega, \mathrm{loc}) \ni u \longmapsto [u] \in \mathcal{D}'(\Omega) \tag{5.1.1}$$

is injective. Consequently, we may identify $L^1(\Omega, \mathrm{loc})$ with a subspace of $\mathcal{D}'(\Omega)$ and we will frequently write u instead of $[u]$.

If we consider on $L^1(\Omega, \mathrm{loc})$ the standard topology, i.e.

$$u_\nu \overset{L^1(\Omega,\mathrm{loc})}{\longrightarrow} u_0 \Longleftrightarrow \forall_{K\subset\subset\Omega} : u_\nu|_K \overset{L^1(K)}{\longrightarrow} u_0|_K,$$

then the mapping (5.1.1) is continuous.

Let $\mathfrak{T} \in \mathcal{D}'(\Omega)$. We say that $\mathfrak{T} = 0$ on an open set $U \subset \Omega$ if $\mathfrak{T}|_{\mathcal{D}(U)} = 0$.

The *support* $\operatorname{supp}\mathfrak{T}$ of $\mathfrak{T}$ is the set of all $a \in \Omega$ such that $\mathfrak{T} \neq 0$ on any neighborhood of a; $\operatorname{supp}\mathfrak{T}$ is relatively closed in Ω; $\mathfrak{T} = 0$ on $\Omega \setminus \operatorname{supp}\mathfrak{T}$; if $f_1, f_2 \in \mathcal{D}(\Omega)$ are such that $f_1 = f_2$ in a neighborhood of $\operatorname{supp}\mathfrak{T}$, then $\mathfrak{T}(f_1) = \mathfrak{T}(f_2)$.

If $\operatorname{supp}\mathfrak{T} \subset\subset \Omega$, then $\mathfrak{T}$ extends to a distribution on $\mathbb{R}^N$.

5.1.3. For $\mathfrak{T} \in \mathcal{D}'(\Omega)$ let

$$\operatorname{Dom}(\mathfrak{T}) := \{f \in \mathcal{E}(\Omega) : \operatorname{supp}\mathfrak{T} \cap \operatorname{supp} f \subset\subset \Omega\}.$$

Then $\operatorname{Dom}(\mathfrak{T})$ is a linear subspace of $\mathcal{E}(\Omega)$. Obviously, $\mathcal{D}(\Omega) \subset \operatorname{Dom}(\mathfrak{T})$. If $\operatorname{supp}\mathfrak{T} \subset\subset \Omega$, then $\operatorname{Dom}(\mathfrak{T}) = \mathcal{E}(\Omega)$. Define

$$\widetilde{\mathfrak{T}} : \operatorname{Dom}(\mathfrak{T}) \longrightarrow \mathbb{C}, \quad \widetilde{\mathfrak{T}}(f) := \mathfrak{T}(\varphi f),$$

where $\varphi \in \mathcal{D}(\Omega)$ and $\varphi = 1$ in a neighborhood of $K_f := \operatorname{supp}\mathfrak{T} \cap \operatorname{supp} f$. Then the definition of $\widetilde{\mathfrak{T}}(f)$ is independent of φ and $\widetilde{\mathfrak{T}} = \mathfrak{T}$ on $\mathcal{D}(\Omega)$. The operator $\widetilde{\mathfrak{T}}$ is also continuous in the following sense:

$$\Big(\operatorname{Dom}(\mathfrak{T}) \ni f_\nu \overset{\mathcal{E}(\Omega)}{\longrightarrow} f_0 \in \operatorname{Dom}(\mathfrak{T}) \text{ and } \bigcup_{\nu=1}^\infty K_{f_\nu} \subset\subset \Omega\Big) \Longrightarrow \widetilde{\mathfrak{T}}(f_\nu) \longrightarrow \widetilde{\mathfrak{T}}(f_0).$$

In particular, if $\operatorname{supp}\mathfrak{T} \subset\subset \Omega$, then $\widetilde{\mathfrak{T}}$ is continuous in the standard sense.

Consequently, $\mathfrak{T}$ extends to continuous linear functional on $\mathcal{E}(\Omega)$ iff $\operatorname{supp}\mathfrak{T} \subset\subset \Omega$.

Let $\mathfrak{T} \in \mathcal{D}'(\Omega)$. For $\alpha \in \mathbb{N}_0^N$ define

$$(D^\alpha\mathfrak{T})(f) := (-1)^{|\alpha|}\mathfrak{T}(D^\alpha f), \quad f \in \mathcal{D}(\Omega).$$

The mapping $D^\alpha\mathfrak{T}$ is called the *α-th derivative* of $\mathfrak{T}$.

5.1.4. (a) The mapping $\mathcal{D}(\Omega) \ni f \longrightarrow D^\alpha f \in \mathcal{D}(\Omega)$ is continuous. Consequently, $D^\alpha\mathfrak{T} \in \mathcal{D}'(\Omega)$.

In particular, any function $u \in L^1(\Omega, \mathrm{loc})$ has all derivatives in the sense of distribution.

(b) $D^\alpha(D^\beta\mathfrak{T}) = D^{\alpha+\beta}\mathfrak{T}$.

(c) The mapping $\mathcal{D}'(\Omega) \ni \mathfrak{T} \longrightarrow D^\alpha\mathfrak{T} \in \mathcal{D}'(\Omega)$ is linear and continuous.

(d) If $u \in \mathcal{C}^k(\Omega)$, then

$$D^\alpha[u] = [D^\alpha u], \quad |\alpha| \le k.$$

For $\eta \in \mathcal{E}(\Omega)$ define

$$(\eta\mathfrak{T})(f) := \mathfrak{T}(\eta f), \quad f \in \mathcal{D}(\Omega).$$

5.1.5. (a) The mapping $\mathcal{E}(\Omega) \times \mathcal{D}(\Omega) \ni (\eta, f) \longrightarrow \eta f \in \mathcal{D}(\Omega)$ is continuous. Consequently, $\eta\mathfrak{T} \in \mathcal{D}'(\Omega)$.
(b) The mapping $\mathcal{E}(\Omega) \times \mathcal{D}'(\Omega) \ni (\eta, \mathfrak{T}) \longrightarrow \eta\mathfrak{T} \in \mathcal{D}'(\Omega)$ is bilinear and continuous.
(c) $\vartheta(\eta\mathfrak{T}) = (\vartheta\eta)\mathfrak{T}$ $(\vartheta, \eta \in \mathcal{E}(\Omega))$.
(d) If $u \in L^1(\Omega, \mathrm{loc})$, then $\eta[u] = [\eta u]$.

5.1.6. Let Ω_j be open in $\mathbb{R}^{N_j}$, $j = 1, 2$, let $\mathfrak{T} \in \mathcal{D}'(\Omega_1)$, and let $f \in \mathcal{E}(\Omega_1 \times \Omega_2)$ be such that

$$\bigcup_{x_2 \in \Omega_2} \operatorname{supp} f(\cdot, x_2) \subset\subset \Omega_1$$

(note that the last condition is satisfied if $f \in \mathcal{D}(\Omega_1 \times \Omega_2)$). Put

$$F(x_2) = F_f(x_2) := \mathfrak{T}(f(\cdot, x_2)), \quad x_2 \in \Omega_2.$$

Then $F \in \mathcal{E}(\Omega_2)$ and

$$D^\alpha F(x_2) = \mathfrak{T}(D^\alpha_{x_2} f(\cdot, x_2)), \quad x_2 \in \Omega_2,\ \alpha \in \mathbb{N}_0^{N_2}.$$

Moreover, if $f \in \mathcal{D}(\Omega_1 \times \Omega_2)$, then $F \in \mathcal{D}(\Omega_2)$ and the mapping

$$\mathcal{D}(\Omega_1 \times \Omega_2) \ni f \longmapsto F_f \in \mathcal{D}(\Omega_2)$$

is continuous.

Observe that if $u_j \in L^1(\Omega_j, \mathrm{loc})$, $j = 1, 2$, then $u_1 \otimes u_2 \in L^1(\Omega_1 \times \Omega_2, \mathrm{loc})$ [1] . Moreover, if $u_j \in \mathcal{D}(\Omega_j)$, $j = 1, 2$, then $u_1 \otimes u_2 \in \mathcal{D}(\Omega_1 \times \Omega_2)$ and $\operatorname{supp}(u_1 \otimes u_2) = (\operatorname{supp} u_1) \times (\operatorname{supp} u_2)$. Let $\mathcal{D}(\Omega_1) \otimes \mathcal{D}(\Omega_2)$ denote the subspace of $\mathcal{D}(\Omega_1 \times \Omega_2)$ generated by all functions $u_1 \otimes u_2$ with $u_j \in \mathcal{D}(\Omega_j)$, $j = 1, 2$.

5.1.7. $\mathcal{D}(\Omega_1) \otimes \mathcal{D}(\Omega_2)$ is dense in $\mathcal{D}(\Omega_1 \times \Omega_2)$.

5.1.8. (a) Let $\mathfrak{T}_j \in \mathcal{D}'(\Omega_j)$, $j = 1, 2$. For $f \in \mathcal{D}(\Omega_1 \times \Omega_2)$ define

$$F_1(x_1) := \mathfrak{T}_2(f(x_1, \cdot)),\ x_1 \in \Omega_1, \quad F_2(x_2) := \mathfrak{T}_1(f(\cdot, x_2)),\ x_2 \in \Omega_2,$$
$$\mathfrak{U}_1(f) := \mathfrak{T}_1(F_1), \quad \mathfrak{U}_2(f) := \mathfrak{T}_2(F_2)$$

(note that, by Property 5.1.6, $\mathfrak{U}_j \in \mathcal{D}'(\Omega_1 \times \Omega_2)$, $j = 1, 2$). Then

$$\mathfrak{U}_1(u_1 \otimes u_2) = \mathfrak{U}_2(u_1 \otimes u_2) = \mathfrak{T}_1(u_1)\mathfrak{T}_2(u_2), \quad u_j \in \mathcal{D}(\Omega_j),\ j = 1, 2.$$

Consequently, by Property 5.1.7, $\mathfrak{U}_1 = \mathfrak{U}_2$. Put $\mathfrak{T}_1 \otimes \mathfrak{T}_2 := \mathfrak{U}_1 (= \mathfrak{U}_2)$. The distribution $\mathfrak{T}_1 \otimes \mathfrak{T}_2$ is called the *tensor product* of $\mathfrak{T}_1$ and $\mathfrak{T}_2$. It is the only distribution on $\Omega_1 \times \Omega_2$ satisfying

$$(\mathfrak{T}_1 \otimes \mathfrak{T}_2)(u_1 \otimes u_2) = \mathfrak{T}_1(u_1)\mathfrak{T}_2(u_2), \quad u_j \in \mathcal{D}(\Omega_j),\ j = 1, 2.$$

(b) $[u_1] \otimes [u_2] = [u_1 \otimes u_2]$ for any $u_j \in L^1(\Omega_j, \mathrm{loc})$, $j = 1, 2$.
(c) The operation

$$\mathcal{D}'(\Omega_1) \times \mathcal{D}'(\Omega_2) \ni (\mathfrak{T}_1, \mathfrak{T}_2) \longmapsto \mathfrak{T}_1 \otimes \mathfrak{T}_2 \in \mathcal{D}'(\Omega_1 \times \Omega_2)$$

is bilinear.
(d) $\operatorname{supp}(\mathfrak{T}_1 \otimes \mathfrak{T}_2) = (\operatorname{supp} \mathfrak{T}_1) \times (\operatorname{supp} \mathfrak{T}_2)$.
(e) $(\mathfrak{T}_1 \otimes \mathfrak{T}_2) \otimes \mathfrak{T}_3 = \mathfrak{T}_1 \otimes (\mathfrak{T}_2 \otimes \mathfrak{T}_3)$ for any $\mathfrak{T}_j \in \mathcal{D}'(\Omega_j)$, $j = 1, 2, 3$.
(f) $D^{\alpha_1}_{x_1} D^{\alpha_2}_{x_2}(\mathfrak{T}_1 \otimes \mathfrak{T}_2) = (D^{\alpha_1}\mathfrak{T}_1) \otimes (D^{\alpha_2}\mathfrak{T}_2)$ for arbitrary $\alpha_j \in \mathbb{N}_0^{N_j}$, $j = 1, 2$.

5.1.9. Let $u \in L^p(\mathbb{R}^N)$, $v \in L^q(\mathbb{R}^N)$ with $1 \le p, q \le +\infty$, $1/p + 1/q \ge 1$. Then the function

$$(u * v)(x) := \int_{\mathbb{R}^N} u(x - y)v(y)\, d\mathcal{L}^N(y)$$

is defined for almost all $x \in \mathbb{R}^N$ and $u * v \in L^r(\mathbb{R}^N)$, where $1/r = 1/p + 1/q - 1$. The operator

$$L^p(\mathbb{R}^N) \times L^q(\mathbb{R}^N) \ni (u, v) \longmapsto u * v \in L^r(\mathbb{R}^N)$$

---

[1] $(f \otimes g)(x, y) = f(x)g(y)$.

is bilinear symmetric and continuous. Moreover,

$$\|u * v\|_{L^r(\mathbb{R}^N)} \le \|u\|_{L^p(\mathbb{R}^N)}\|v\|_{L^q(\mathbb{R}^N)}.$$

If $r = \infty$, then $u * v \in \mathcal{C}(\mathbb{R}^N)$.

The function $u * v$ is called the *convolution* of u and v. The convolution $u * v$ may be also defined under weaker assumptions on u and v (cf. Property 5.1.11).

Let $\Phi \in \mathcal{C}_0^\infty(\mathbb{R}^N, \mathbb{R}_+)$ be such that $\int_{\mathbb{R}^N} \Phi \, d\mathcal{L}^N = 1$. Put

$$\Phi_\varepsilon(x) := \varepsilon^{-N}\Phi(x/\varepsilon), \quad x \in \mathbb{R}^N, \ \varepsilon > 0.$$

For $u \in L^p(\mathbb{R}^N)$ let

$$u_\varepsilon := u * \Phi_\varepsilon.$$

The function u_ε is called the *ε-th regularization* of u (with respect to Φ); $u_\varepsilon \in L^p(\mathbb{R}^N)$ and $\|u_\varepsilon\|_{L^p(\mathbb{R}^N)} \le \|u\|_{L^p(\mathbb{R}^N)}$.

5.1.10. Let $u \in L^p(\mathbb{R}^N)$ $(1 \le p < +\infty)$ be such that $u = 0$ outside a compact $K \subset \mathbb{R}^N$. Then:
(a) $u_\varepsilon \in \mathcal{D}(\mathbb{R}^N)$, $\operatorname{supp}(u_\varepsilon) \subset K + \varepsilon(\operatorname{supp}\Phi)$;
(b) if $u \in \mathcal{C}_0(\mathbb{R}^N)$, then $u_\varepsilon \longrightarrow u$ uniformly on $\mathbb{R}^N$ when $\varepsilon \longrightarrow 0$;
(c) $u_\varepsilon \overset{L^p(\mathbb{R}^N)}{\longrightarrow} u$ when $\varepsilon \longrightarrow 0$;
(d) if $u \in \mathcal{C}_0^k(\mathbb{R}^N)$, then $D^\alpha(u_\varepsilon) = (D^\alpha u)_\varepsilon$ and $D^\alpha(u_\varepsilon) \longrightarrow D^\alpha u$ uniformly on $\mathbb{R}^N$ when $\varepsilon \longrightarrow 0$ for any $|\alpha| \le k$.

5.1.11. Let $\mathfrak{T}_j \in \mathcal{D}'(\mathbb{R}^N)$, $j = 1, 2$. Assume that

$$\forall_{K \subset\subset \mathbb{R}^N} : \{(x_1, x_2) \in (\operatorname{supp}\mathfrak{T}_1) \times (\operatorname{supp}\mathfrak{T}_2) : x_1 + x_2 \in K\} \subset\subset \mathbb{R}^{2N}. \tag{5.1.2}$$

Put

$$(\mathfrak{T}_1 * \mathfrak{T}_2)(f) := (\widetilde{\mathfrak{T}_1 \otimes \mathfrak{T}_2})(f \circ \sigma), \quad f \in \mathcal{D}(\mathbb{R}^N),$$

where $\sigma : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$, $\sigma(x_1, x_2) := x_1 + x_2$ (cf. Property 5.1.3). Then $\mathfrak{T}_1 * \mathfrak{T}_2 \in \mathcal{D}'(\mathbb{R}^N)$.

The distribution $\mathfrak{T}_1 * \mathfrak{T}_2$ is called the *convolution* of $\mathfrak{T}_1$ and $\mathfrak{T}_2$.

5.1.12. (a) The operation $(\mathfrak{T}_1, \mathfrak{T}_2) \longrightarrow \mathfrak{T}_1 * \mathfrak{T}_2$ is bilinear and symmetric.
(b) Condition (5.1.2) holds if $\operatorname{supp}\mathfrak{T}_1$ or $\operatorname{supp}\mathfrak{T}_2$ is compact.
(c) If $u_1, u_2 \in L^1(\mathbb{R}^N, \mathrm{loc})$, $\operatorname{supp} u_1 \subset\subset \mathbb{R}^N$, then

$$[u_1] * [u_2] = [u_1 * u_2].$$

(d) $\operatorname{supp}(\mathfrak{T}_1 * \mathfrak{T}_2) \subset (\operatorname{supp}\mathfrak{T}_1) + (\operatorname{supp}\mathfrak{T}_2)$.
(e) If

$$\forall_{K \subset\subset \mathbb{R}^N} : \{(x_1, x_2, x_3) \in \operatorname{supp}\mathfrak{T}_1 \times \operatorname{supp}\mathfrak{T}_2 \times \operatorname{supp}\mathfrak{T}_3 : x_1 + x_2 + x_3 \in K\} \subset\subset \mathbb{R}^{3N},$$

then $(\mathfrak{T}_1 * \mathfrak{T}_2) * \mathfrak{T}_3 = \mathfrak{T}_1 * (\mathfrak{T}_2 * \mathfrak{T}_3)$.
(f) $D^\alpha(\mathfrak{T}_1 * \mathfrak{T}_2) = (D^\alpha\mathfrak{T}_1) * \mathfrak{T}_2 = \mathfrak{T}_1 * (D^\alpha\mathfrak{T}_2)$ for any $\alpha \in \mathbb{N}_0^N$.
(g)

$$\Big(\mathfrak{T}_\nu \overset{\mathcal{D}'(\mathbb{R}^N)}{\longrightarrow} \mathfrak{T}_0 \text{ and } \bigcup_{\nu=1}^\infty \operatorname{supp}\mathfrak{T}_\nu \subset\subset \mathbb{R}^N\Big) \Longrightarrow \mathfrak{T}_\nu * \mathfrak{U} \overset{\mathcal{D}'(\mathbb{R}^N)}{\longrightarrow} \mathfrak{T}_0 * \mathfrak{U},$$

$$\Big(\mathfrak{T}_\nu \overset{\mathcal{D}'(\mathbb{R}^N)}{\longrightarrow} \mathfrak{T}_0 \text{ and } \operatorname{supp}\mathfrak{U} \subset\subset \mathbb{R}^N\Big) \Longrightarrow \mathfrak{T}_\nu * \mathfrak{U} \overset{\mathcal{D}'(\mathbb{R}^N)}{\longrightarrow} \mathfrak{T}_0 * \mathfrak{U}.$$

(h) If $\mathfrak{T} \in \mathcal{D}'(\mathbb{R}^N)$ and $u \in \mathcal{D}(\mathbb{R}^N)$, then

$$\mathfrak{T} * [u] = [v],$$

where

$$v(x) := \mathfrak{T}(u(x - \cdot)), \quad x \in \mathbb{R}^N.$$

Moreover, by Property 5.1.6, $v \in \mathcal{E}(\mathbb{R}^N)$.

Define

$$\mathfrak{T}_\varepsilon := \mathfrak{T} * \Phi_\varepsilon.$$

The distribution $\mathfrak{T}_\varepsilon$ is called the *ε-th regularization* of $\mathfrak{T}$ (with respect to Φ).

5.1.13. (a) If $u \in L^1(\mathbb{R}^N, \mathrm{loc})$, then $[u]_\varepsilon = [u_\varepsilon]$.
(b) $\operatorname{supp}(\mathfrak{T}_\varepsilon) \subset (\operatorname{supp} \mathfrak{T}) + \varepsilon(\operatorname{supp} \Phi)$.
(c) $D^\alpha \mathfrak{T}_\varepsilon = (D^\alpha \mathfrak{T})_\varepsilon$. In particular, for any differential operator L with constant coefficients we get $(L(\mathfrak{T}))_\varepsilon = L(\mathfrak{T}_\varepsilon)$.
(d) $\mathfrak{T}_\varepsilon \overset{\mathcal{D}'(\mathbb{R}^N)}{\longrightarrow} \mathfrak{T}$ when $\varepsilon \longrightarrow 0$.
(e) $\mathfrak{T}_\varepsilon \in \mathcal{E}(\mathbb{R}^N)$.
(f) If $\operatorname{supp} \mathfrak{T} \subset\subset \mathbb{R}^N$, then $\mathfrak{T}_\varepsilon \in \mathcal{D}(\mathbb{R}^N)$.

5.2. Hörmander's inequality

Let Ω be an open set in $\mathbb{C}^n$.

For $p, q \in \mathbb{N}_0$ let $\mathcal{D}'_{(p,q)}(\Omega)$ denote the space of all forms of type (p,q) with coefficients (in the canonical form) in $\mathcal{D}'(\Omega)$. If

$$\mathfrak{T} = \sum_{|I|=p, |J|=q}{}' \mathfrak{T}_{I,J} dz_I \wedge d\overline{z}_J \in \mathcal{D}'_{(p,q)}(\Omega),$$

then for any sequence $S = (s_1, \dots, s_q)$, $1 \le s_1, \dots, s_q \le n$, we define

$$\mathfrak{T}_{I,S} := \begin{cases} 0 & \text{if } \#\{s_1, \dots, s_q\} < q \\ (\operatorname{sgn} \sigma) \mathfrak{T}_{I,\sigma(S)} & \text{if } \#\{s_1, \dots, s_q\} = q \end{cases},$$

where $\sigma = \sigma_S$ is the permutation such that $\sigma(S) \in \Xi^n_q$; we define $\overrightarrow{S} := \sigma(S)$.

If $j \in \{1, \dots, n\}$, $K \in \Xi^n_{q-1}$, then we put $jK := (j, k_1, \dots, k_{q-1})$.

Observe that $\mathfrak{T}_{I,jkL} = -\mathfrak{T}_{I,kjL}$.

The operators ∂ and $\overline{\partial}$ defined in § 4.2 on $\mathcal{C}^1_{(p,q)}(\Omega)$ can be easily extended to $\mathcal{D}'_{(p,q)}(\Omega)$, namely:

$$\partial \mathfrak{T} := \sum_{|I|=p, |J|=q}{}' \sum_{j=1}^n \frac{\partial \mathfrak{T}_{I,J}}{\partial z_j} dz_j \wedge dz_I \wedge d\overline{z}_J, \qquad \overline{\partial} \mathfrak{T} := \sum_{|I|=p, |J|=q}{}' \sum_{j=1}^n \frac{\partial \mathfrak{T}_{I,J}}{\partial \overline{z}_j} d\overline{z}_j \wedge dz_I \wedge d\overline{z}_J.$$

Recall the canonical form of $\overline{\partial} \mathfrak{T}$:

$$\overline{\partial} \mathfrak{T} = \sum_{|I|=p, |K|=q+1}{}' \Big(\sum_{J \in \Xi^n_q,\ j \in \{1,\dots,n\}} \varepsilon(I,K,J,j) \frac{\partial \mathfrak{T}_{I,J}}{\partial \overline{z}_j} \Big) dz_I \wedge d\overline{z}_K,$$

where $\varepsilon = \varepsilon(I,K,J,j) \in \{-1,+1\}$ is such that

$$d\overline{z}_j \wedge dz_I \wedge d\overline{z}_J = \varepsilon dz_I \wedge d\overline{z}_K$$

if $\{k_1, \dots, k_{q+1}\} = \{j, j_1, \dots, j_q\}$ and $\varepsilon(I,K,J,j) := 0$ otherwise.

Moreover, we define

$$D^{\alpha,\beta} \mathfrak{T} := \sum_{|I|=p, |J|=q}{}' (D^{\alpha,\beta} \mathfrak{T}_{I,J}) dz_I \wedge d\overline{z}_J, \quad \alpha, \beta \in \mathbb{N}_0^n, \qquad \vartheta \mathfrak{T} := \sum_{|I|=p, |K|=q-1}{}' \Big(\sum_{j=1}^n \frac{\partial \mathfrak{T}_{I,jK}}{\partial z_j} \Big) dz_I \wedge d\overline{z}_K.$$

(Notice that ϑ is defined if $q \ge 1$.) Observe that

$$\partial : \mathcal{D}'_{(p,q)}(\Omega) \longrightarrow \mathcal{D}'_{(p+1,q)}(\Omega), \quad \overline{\partial} : \mathcal{D}'_{(p,q)}(\Omega) \longrightarrow \mathcal{D}'_{(p,q+1)}(\Omega), \quad \partial \circ \partial = 0, \quad \overline{\partial} \circ \overline{\partial} = 0, \quad \partial \circ \overline{\partial} = -\overline{\partial} \circ \partial,$$

$$D^{\alpha,\beta} : \mathcal{D}'_{(p,q)}(\Omega) \longrightarrow \mathcal{D}'_{(p,q)}(\Omega), \quad D^{\alpha,\beta} \circ \partial = \partial \circ D^{\alpha,\beta}, \quad D^{\alpha,\beta} \circ \overline{\partial} = \overline{\partial} \circ D^{\alpha,\beta},$$

$$\vartheta : \mathcal{D}'_{(p,q)}(\Omega) \longrightarrow \mathcal{D}'_{(p,q-1)}(\Omega), \quad \vartheta : \mathcal{C}^k_{(p,q)}(\Omega) \longrightarrow \mathcal{C}^{k-1}_{(p,q-1)}(\Omega), \quad \vartheta \circ \vartheta = 0, \quad \vartheta \circ D^{\alpha,\beta} = D^{\alpha,\beta} \circ \vartheta.$$

Fix a function $\Psi \in \mathcal{C}_0^\infty(\mathbb{C}, \mathbb{R}_+)$ such that

$$\operatorname{supp}\Psi = \overline{\mathbb{D}}, \quad \Psi(z) = \Psi(|z|),\ z \in \mathbb{C}, \quad \int_{\mathbb{C}} \Psi d\mathcal{L}^2 = 1,$$

and let

$$\Phi(z_1, \dots, z_n) := \Psi(z_1) \cdot \dots \cdot \Psi(z_n), \quad (z_1, \dots, z_n) \in \mathbb{C}^n, \quad \Phi_\varepsilon(z) := \frac{1}{\varepsilon^{2n}}\Phi(\frac{z}{\varepsilon}), \quad \varepsilon > 0,\ z \in \mathbb{C}^n.$$

Observe that $\operatorname{supp}\Phi_\varepsilon = \overline{\mathbb{P}}_n(\varepsilon)$ and $\int_{\mathbb{C}^n} \Phi_\varepsilon \, d\mathcal{L}^{2n} = 1$, $\varepsilon > 0$.

Let $\mathfrak{T} \in \mathcal{D}'_{(p,q)}(\Omega)$ be such that $\operatorname{supp}\mathfrak{T} \subset\subset \Omega$ ($\operatorname{supp}\mathfrak{T} = \bigcup_{I \in \Xi_p^n,\ J \in \Xi_q^n} \operatorname{supp}\mathfrak{T}_{I,J}$). One may identify $\mathfrak{T}$ with an element of $\mathcal{D}'_{(p,q)}(\mathbb{C}^n)$.

We define the *ε-regularization* of $\mathfrak{T}$ by the formula

$$\mathfrak{T}_\varepsilon := {\sum_{|I|=p, |J|=q}}' (\mathfrak{T}_{I,J})_\varepsilon dz_I \wedge d\overline{z}_J \in \mathcal{D}'_{(p,q)}(\mathbb{C}^n).$$

Recall that $\operatorname{supp}\mathfrak{T}_\varepsilon \subset (\operatorname{supp}\mathfrak{T})^{(\varepsilon)}$ (Property 5.1.13(b)). In particular, $\operatorname{supp}\mathfrak{T}_\varepsilon \subset \Omega$ provided $0 < \varepsilon \ll 1$. By Property 5.1.13(c), we get

$$\partial(\mathfrak{T}_\varepsilon) = (\partial\mathfrak{T})_\varepsilon, \quad \overline{\partial}(\mathfrak{T}_\varepsilon) = (\overline{\partial}\mathfrak{T})_\varepsilon, \quad D^{\alpha,\beta}(\mathfrak{T}_\varepsilon) = (D^{\alpha,\beta}\mathfrak{T})_\varepsilon, \quad \vartheta(\mathfrak{T}_\varepsilon) = (\vartheta\mathfrak{T})_\varepsilon. \tag{5.2.1}$$

If $\varepsilon \longrightarrow 0$, then $\mathfrak{T}_\varepsilon \overset{\mathcal{D}'_{(p,q)}(\Omega)}{\longrightarrow} \mathfrak{T}$, i.e. $(\mathfrak{T}_{I,J})_\varepsilon \overset{\mathcal{D}'(\Omega)}{\longrightarrow} \mathfrak{T}_{I,J}$ for any $I \in \Xi_p^n$ and $J \in \Xi_q^n$ (Property 5.1.13(d)).

For a continuous function $\varphi : \Omega \longrightarrow \mathbb{R}$ let

$$L^2(\Omega, \varphi) := L^2(\Omega, e^{-\varphi}\mathcal{L}^{2n}) = \{f : \Omega \longrightarrow \mathbb{C} : \int_\Omega |f|^2 e^{-\varphi} d\mathcal{L}^{2n} < +\infty\}.$$

It is clear that $L^2(\Omega, \varphi) \subset L^2(\Omega, \mathrm{loc})$. The space $L^2(\Omega, \varphi)$ with the scalar product

$$(f, g) \longmapsto \langle f, g\rangle_\varphi := \int_\Omega f\overline{g}e^{-\varphi} d\mathcal{L}^{2n}$$

is a complex Hilbert space. Let $\|\ \|_\varphi$ denote the norm generated by the above product. Observe that the space $\mathcal{D}(\Omega)$ is dense in $L^2(\Omega, \varphi)$.

Let $L^2_{(p,q)}(\Omega, \varphi)$ (resp. $L^r_{(p,q)}(\Omega, \mathrm{loc})$, $1 \le r \le +\infty$) denote the space of all forms of type (p, q) with coefficients in $L^2(\Omega, \varphi)$ (resp. $L^r(\Omega, \mathrm{loc})$). Obviously,

$$\mathcal{D}_{(p,q)}(\Omega) \subset (\mathcal{C}_0^k)_{(p,q)}(\Omega) \subset \mathcal{C}^k_{(p,q)}(\Omega) \subset L^r_{(p,q)}(\Omega, \mathrm{loc}) \subset L^1_{(p,q)}(\Omega, \mathrm{loc}) \subset \mathcal{D}'_{(p,q)}(\Omega),$$
$$\mathcal{D}_{(p,q)}(\Omega) \subset (\mathcal{C}_0^k)_{(p,q)}(\Omega) \subset L^2_{(p,q)}(\Omega, \varphi) \subset L^2_{(p,q)}(\Omega, \mathrm{loc}) \subset L^1_{(p,q)}(\Omega, \mathrm{loc}) \subset \mathcal{D}'_{(p,q)}(\Omega).$$

For $u, v \in L^1_{(p,q)}(\Omega, \mathrm{loc})$ we define $\langle u, v\rangle : \Omega \longrightarrow \mathbb{C}$ and $\|u\| : \Omega \longrightarrow \mathbb{R}_+$ by the formulae

$$\langle u, v\rangle := {\sum_{|I|=p, |J|=q}}' u_{I,J}\overline{v}_{I,J}, \quad \|u\| := \sqrt{\langle u, u\rangle}.$$

If $u, v \in L^2_{(p,q)}(\Omega, \varphi)$, then we put

$$\langle u, v\rangle_\varphi := \int_\Omega \langle u, v\rangle e^{-\varphi} d\mathcal{L}^{2n} = {\sum_{|I|=p, |J|=q}}' \langle u_{I,J}, v_{I,J}\rangle_\varphi,$$
$$\|u\|_\varphi^2 := \langle u, u\rangle_\varphi = \int_\Omega \|u\|^2 e^{-\varphi} d\mathcal{L}^{2n} = {\sum_{|I|=p, |J|=q}}' \|u_{I,J}\|_\varphi^2.$$

Note that the space $L^2_{(p,q)}(\Omega, \varphi)$ with the scalar product $\langle\ ,\ \rangle_\varphi$ is a complex Hilbert space and $\mathcal{D}_{(p,q)}(\Omega)$ is dense in $L^2_{(p,q)}(\Omega, \varphi)$.

By Property 5.1.10(d), for any $u \in L^2_{(p,q)}(\Omega, \mathrm{loc})$ with $\operatorname{supp} u \subset\subset \Omega$ we have

$$u_\varepsilon \overset{L^2_{(p,q)}(\Omega)}{\longrightarrow} u, \text{ i.e. } \forall_{I\in\Xi^n_p,\ J\in\Xi^n_q} : (u_{I,J})_\varepsilon \overset{L^2(\Omega)}{\longrightarrow} u_{I,J}. \tag{5.2.2}$$

For any $\eta \in \mathcal{C}^\infty_{(r,s)}(\Omega)$, $\mathfrak{T} \in \mathcal{D}'_{(p,q)}(\Omega)$ the wedge product $\eta \wedge \mathfrak{T}$ (calculated according to standard rules) is well defined and $\eta \wedge \mathfrak{T} \in \mathcal{D}'_{(r+p,s+q)}(\Omega)$.

Notice that the wedge product $\eta\wedge\mathfrak{T}$ is also defined in many other cases, for instance if $\eta \in L^2_{(r,s)}(\Omega, \mathrm{loc})$ and $u \in L^2_{(p,q)}(\Omega, \mathrm{loc})$, then $\eta \wedge u \in L^1_{(r+p,s+q)}(\Omega, \mathrm{loc})$. We have

$$\overline{\partial}(\eta \wedge \mathfrak{T}) = (\overline{\partial}\eta) \wedge \mathfrak{T} + (-1)^{r+s}\eta \wedge (\overline{\partial}\mathfrak{T}).$$

One can prove (cf. [6]) that

$$\|u \wedge v\| \le \Big(\binom{r+p}{r}\binom{s+q}{s}\Big)^{1/2}\|u\| \cdot \|v\|, \quad u \in L^2_{(r,s)}(\Omega, \mathrm{loc}),\ v \in L^2_{(p,q)}(\Omega, \mathrm{loc}).$$

Moreover, if $r + s \le 1$ or $p + q \le 1$, then $\|u \wedge v\| \le \|u\| \cdot \|v\|$.

Observe that

$$\vartheta(\eta\mathfrak{T}) = \eta\vartheta\mathfrak{T} + \mathcal{A}_\eta(\mathfrak{T}), \quad \eta \in \mathcal{C}^\infty(\Omega),\ \mathfrak{T} \in \mathcal{D}'_{(p,q)}(\Omega), \tag{5.2.3}$$

where $\mathcal{A}_\eta : \mathcal{D}'_{(p,q)}(\Omega) \longrightarrow \mathcal{D}'_{(p,q-1)}(\Omega)$,

$$\mathcal{A}_\eta(\mathfrak{T}) := \sideset{}{'}\sum_{|I|=p,|K|=q-1} \Big(\sum_{j=1}^n \frac{\partial\eta}{\partial z_j}\mathfrak{T}_{I,jK}\Big) dz_I \wedge d\overline{z}_K.$$

Suppose that we are given three continuous functions $\varphi_j : \Omega \longrightarrow \mathbb{R}$, $j = 1, 2, 3$. Define operators

$$L^2_{(p,q)}(\Omega, \varphi_1) \supset \mathrm{Dom}(T) \ni u \overset{T}{\longrightarrow} \overline{\partial}u \in L^2_{(p,q+1)}(\Omega, \varphi_2),$$
$$L^2_{(p,q+1)}(\Omega, \varphi_2) \supset \mathrm{Dom}(S) \ni v \overset{S}{\longrightarrow} \overline{\partial}v \in L^2_{(p,q+2)}(\Omega, \varphi_3),$$

where

$$\mathrm{Dom}(T) := \{u \in L^2_{(p,q)}(\Omega, \varphi_1) : \overline{\partial}u \in L^2_{(p,q+1)}(\Omega, \varphi_2)\},$$
$$\mathrm{Dom}(S) := \{v \in L^2_{(p,q+1)}(\Omega, \varphi_2) : \overline{\partial}v \in L^2_{(p,q+2)}(\Omega, \varphi_3)\}.$$

Observe that $\mathcal{D}_{(p,q)}(\Omega) \subset \mathrm{Dom}(T)$ and $\mathcal{D}_{(p,q+1)}(\Omega) \subset \mathrm{Dom}(S)$. Thus T and S are densely defined. Note that $S \circ T = 0$. In particular, $\mathcal{R}(T) \subset \mathrm{Ker}(S)$ ($\mathcal{R}(T) := T(\mathrm{Dom}(T))$). It is clear that the operators are closed. Consequently, $F := \mathrm{Ker}(S)$ is a closed subspace of $L^2_{(p,q+1)}(\Omega, \varphi_2)$.

The following lemma will be the main tool used to solve the $\overline{\partial}$-problem.

Lemma 5.2.1. *Assume that $\mathcal{H}_1$, $\mathcal{H}_2$ are complex Hilbert spaces. Let*

$$\mathcal{H}_1 \supset \mathrm{Dom}(T) \overset{T}{\longrightarrow} \mathcal{H}_2$$

be a linear closed densely defined operator and let F be a closed subspace of $\mathcal{H}_2$ such that $\mathcal{R}(T) \subset F$. Assume that

$$\exists_{C>0} : \|f\|_{\mathcal{H}_2} \le C\|T^*(f)\|_{\mathcal{H}_1}, \quad f \in F \cap \mathrm{Dom}(T^*). \tag{5.2.4}$$

Then

$$\forall_{v\in F}\ \exists_{u\in\mathrm{Dom}(T)\cap\mathcal{R}(T^*)} : T(u) = v,\ \|u\|_{\mathcal{H}_1} \le C\|v\|_{\mathcal{H}_2}.$$

Proof. Let $P : \mathcal{H}_2 \longrightarrow F$ denote the orthogonal projection. It is known that $\mathrm{Ker}(T^*) = \mathcal{R}(T^{**})^\perp = \mathcal{R}(T)^\perp$ (cf. [34]). Hence $F^\perp = \mathrm{Ker}(P) \subset \mathrm{Ker}(T^*)$ and therefore

$$T^* \circ P = T^*.$$

In particular, $\mathcal{R}(T^*) = T^*(F \cap \mathrm{Dom}(T^*))$. Now, condition (5.2.4) implies that $\mathcal{R}(T^*)$ is a closed subspace of $\mathcal{H}_1$.

Indeed, let $T^*(f_\nu) \longrightarrow u_0 \in \mathcal{H}_1$ for $(f_\nu)_{\nu=1}^\infty \subset F \cap \operatorname{Dom}(T^*)$. Then, by (5.2.4), $\|f_\nu - f_\mu\|_{\mathcal{H}_2} \le C\|T^*(f_\nu) - T^*(f_\mu)\|_{\mathcal{H}_1}$, $\nu, \mu \ge 1$. Consequently, $(f_\nu)_{\nu=1}^\infty$ is convergent, $f_\nu \longrightarrow f_0 \in \mathcal{H}_2$. Obviously $f_0 \in F$. Let $u \in \operatorname{Dom}(T)$. Then

$$\langle u_0, u\rangle_{\mathcal{H}_1} = \lim_{\nu\to+\infty}\langle T^*(f_\nu), u\rangle_{\mathcal{H}_1} = \lim_{\nu\to+\infty}\langle f_\nu, T(u)\rangle_{\mathcal{H}_1} = \langle f_0, T(u)\rangle_{\mathcal{H}_1}.$$

Thus $f_0 \in \operatorname{Dom}(T^*)$ and $T^*(f_0) = u_0$.

We pass to the main part of the proof. Fix a $v \in F$ and let $L : \mathcal{R}(T^*) \longrightarrow \mathbb{C}$,

$$L(T^*(f)) := \langle v, f\rangle_{\mathcal{H}_2}, \quad f \in F \cap \operatorname{Dom}(T^*).$$

By (5.2.4) L is well defined and

$$|L(T^*(f))| \le C\|v\|_{\mathcal{H}_2}\|T^*(f)\|_{\mathcal{H}_1}, \quad f \in F \cap \operatorname{Dom}(T^*),$$

which shows that L is continuous. Since $\mathcal{R}(T^*)$ is closed, the Riesz theorem implies that there exists a $u \in \mathcal{R}(T^*)$ with $\|u\|_{\mathcal{H}_1} \le C\|v\|_{\mathcal{H}_2}$ such that

$$L(T^*(f)) = \langle u, T^*(f)\rangle_{\mathcal{H}_1}, \quad f \in \operatorname{Dom}(T^*).$$

Thus $u \in \operatorname{Dom}(T^{**}) = \operatorname{Dom}(T)$ and $T^{**}(u) = T(u) = v$ (cf. [34]). $\square$

The above lemma implies that to solve the equation $\overline{\partial}u = v$ for given $v \in L^2_{(p,q+1)}(\Omega, \varphi_2)$ with $u \in L^2_{(p,q)}(\Omega, \varphi_1)$, it suffices to prove that there exists a $C > 0$ such that

$$\|f\|_{\varphi_2} \le C(\|T^*(f)\|_{\varphi_1} + \|S(f)\|_{\varphi_3}), \quad f \in \operatorname{Dom}(T^*) \cap \operatorname{Dom}(S). \tag{5.2.5}$$

In the first step we will describe a class of continuous functions φ_j, $j = 1, 2, 3$, for which the proof of (5.2.5) reduces to $f \in \mathcal{D}_{(p,q+1)}(\Omega)$.

Let a sequence $(\eta_\nu)_{\nu=1}^\infty \subset \mathcal{C}_0^\infty(\Omega, [0, 1])$ be such that

$$\forall_{K\subset\subset\Omega}\ \exists_{\nu_0} :\ \forall_{\nu\ge\nu_0} :\ \eta_\nu|_K \equiv 1, \tag{5.2.6}$$

and let $\psi \in \mathcal{C}^\infty(\Omega, \mathbb{R})$ satisfy

$$\|\overline{\partial}\eta_\nu\|^2 \le e^\psi, \quad \nu \in \mathbb{N}. \tag{5.2.7}$$

Fix a $\varphi \in \mathcal{C}^2(\Omega, \mathbb{R})$ and define $\varphi_j := \varphi - (3 - j)\psi$, $\mathcal{H}_j := L^2_{(p,q+j-1)}(\Omega, \varphi_j)$, $j = 1, 2, 3$.

Proposition 5.2.2. *$\mathcal{D}_{(p,q+1)}(\Omega)$ is dense in $\operatorname{Dom}(T^*) \cap \operatorname{Dom}(S)$ in the graph norm*

$$\operatorname{Dom}(T^*) \cap \operatorname{Dom}(S) \ni f \longmapsto \|f\|_{\varphi_2} + \|T^*(f)\|_{\varphi_1} + \|S(f)\|_{\varphi_3}.$$

Proof. We will prove that:

1^o. $\forall_{f\in\mathcal{H}_j} :\ \eta_\nu f \xrightarrow{\mathcal{H}_j} f$ when $\nu \longrightarrow +\infty$, $j = 1, 2, 3$.

2^o. $\forall_{f\in\mathcal{H}_j:\ \operatorname{supp} f\subset\subset\Omega} :\ f_\varepsilon \xrightarrow{\mathcal{H}_j} f$ when $\varepsilon \longrightarrow 0$, $j = 1, 2, 3$.

3^o. $\forall_{\eta\in\mathcal{D}(\Omega),\ f\in\operatorname{Dom}(S)} :\ \eta f \in \operatorname{Dom}(S)$.

4^o. $\forall_{f\in\operatorname{Dom}(S)} :\ S(\eta_\nu f) \xrightarrow{\mathcal{H}_3} S(f)$ when $\nu \longrightarrow +\infty$.

5^o. $\forall_{f\in\operatorname{Dom}(S):\ \operatorname{supp} f\subset\subset\Omega} :\ S(f_\varepsilon) \xrightarrow{\mathcal{H}_3} S(f)$ when $\varepsilon \longrightarrow 0$.

6^o. $\forall_{\eta\in\mathcal{D}(\Omega),\ f\in\operatorname{Dom}(T^*)} :\ \eta f \in \operatorname{Dom}(T^*)$.

7^o. $\forall_{f\in\operatorname{Dom}(T^*)} :\ T^*(\eta_\nu f) \xrightarrow{\mathcal{H}_1} T^*(f)$ when $\nu \longrightarrow +\infty$.

8^o. $\mathcal{D}_{(p,q+1)}(\Omega) \subset \operatorname{Dom}(T^*)$.

9^o. $\forall_{f\in\operatorname{Dom}(T^*)} :\ T^*(f) = (-1)^{p-1}e^{\varphi_1}\vartheta(e^{-\varphi_2}f)$.

10^o. $\forall_{f\in\operatorname{Dom}(T^*):\ \operatorname{supp} f\subset\subset\Omega} :\ T^*(f_\varepsilon) \xrightarrow{\mathcal{H}_1} T^*(f)$ when $\varepsilon \longrightarrow 0$.

It is clear that 1^o — 10^o imply the required result.

Property 1^o follows from the Lebesgue dominated convergence theorem.

Property 2^o follows from (5.2.2).

To prove 3^o it suffices to recall that $\overline{\partial}(\eta f) = (\overline{\partial}\eta) \wedge f + \eta\overline{\partial}f$.

To prove 4^o observe that, by 3^o, for $\eta = \eta_\nu$, we get

$$\|S(\eta_\nu f) - \eta_\nu S(f)\|^2 e^{-\varphi_3} = \|(\overline{\partial}\eta_\nu) \wedge f\|^2 e^{-\varphi_3} \le \|\overline{\partial}\eta_\nu\|^2 \|f\|^2 e^{-\varphi_3} \le e^{\psi}\|f\|^2 e^{-\varphi_3} = \|f\|^2 e^{-\varphi_2}.$$

Hence, by the Lebesgue theorem, $S(\eta_\nu f) - \eta_\nu S(f) \xrightarrow{\mathcal{H}_3} 0$ when $\nu \longrightarrow +\infty$. We already know by 1^o that $\eta_\nu S(f) \xrightarrow{\mathcal{H}_3} S(f)$. Thus $S(\eta_\nu f) \xrightarrow{\mathcal{H}_3} S(f)$.

Property 5^o follows from (5.2.1) and (5.2.2).

In particular, we have proved that $\mathcal{D}_{(p,q)}(\Omega)$ is dense in $\operatorname{Dom}(T)$ in the norm

$$u \longmapsto \|u\|_{\varphi_1} + \|T(u)\|_{\varphi_1}.$$

To prove 6^o first observe that by 3^o $\overline{\eta}u \in \operatorname{Dom}(T)$ for any $u \in \operatorname{Dom}(T)$. Therefore for any $u \in \operatorname{Dom}(T)$ we get

$$\langle \eta f, T(u)\rangle_{\varphi_2} = \langle f, \overline{\eta}T(u)\rangle_{\varphi_1} = \langle f, T(\overline{\eta}u)\rangle_{\varphi_2} - \langle f, \overline{\partial}\overline{\eta} \wedge u\rangle_{\varphi_2} = \langle \eta T^*(f), u\rangle_{\varphi_1} - \langle f, \overline{\partial}\overline{\eta} \wedge u\rangle_{\varphi_2}.$$

Hence, using the Hölder inequality, we obtain

$$\begin{aligned}|\langle \eta f, T(u)\rangle_{\varphi_2}| &\le \|\eta T^*(f)\|_{\varphi_1}\|u\|_{\varphi_1} + \int_\Omega \|f\| \cdot \|\overline{\partial}\overline{\eta}\| \cdot \|u\| e^{-\varphi_2} d\mathcal{L}^{2n} \\ &\le \|\eta T^*(f)\|_{\varphi_1}\|u\|_{\varphi_1} + \sup_\Omega\{\|\overline{\partial}\overline{\eta}\| e^{-\psi/2}\}\|f\|_{\varphi_2}\|u\|_{\varphi_1} = \text{const}\,\|u\|_{\varphi_1}, \quad u \in \operatorname{Dom}(T).\end{aligned}$$

Consequently, $\eta f \in \operatorname{Dom}(T^*)$ and

$$\langle T^*(\eta f), u\rangle_{\varphi_1} - \langle \eta T^*(f), u\rangle_{\varphi_1} = -\langle f, \overline{\partial}\overline{\eta} \wedge u\rangle_{\varphi_2}, \quad u \in \operatorname{Dom}(T).$$

Hence

$$\Big|\int_\Omega \langle T^*(\eta f) - \eta T^*(f), u\rangle e^{-\varphi_1} d\mathcal{L}^{2n}\Big| \le \int_\Omega \|f\| \cdot \|\overline{\partial}\overline{\eta}\| \cdot \|u\| e^{-\varphi_2} d\mathcal{L}^{2n}, \quad u \in \operatorname{Dom}(T),$$

and so

$$\Big|\int_\Omega \Big\langle (T^*(\eta f) - \eta T^*(f))e^{-\varphi_1/2}, u e^{-\varphi_1/2}\Big\rangle d\mathcal{L}^{2n}\Big| \le \int_\Omega \|f\| e^{-\varphi_2/2}\|\overline{\partial}\overline{\eta}\| e^{-\psi/2}\|u\| e^{-\varphi_1/2} d\mathcal{L}^{2n}$$

for all $u \in \operatorname{Dom}(T)$ and, consequently, for all $u \in L^2_{(p,q)}(\Omega, \varphi_1)$. Thus

$$\Big|\int_\Omega \Big\langle (T^*(\eta f) - \eta T^*(f))e^{-\varphi_1/2}, u\Big\rangle d\mathcal{L}^{2n}\Big| \le \int_\Omega \|f\| e^{-\varphi_2/2}\|\overline{\partial}\overline{\eta}\| e^{-\psi/2}\|u\| d\mathcal{L}^{2n}, \quad u \in L^2_{(p,q)}(\Omega).$$

This implies (Exercise) that

$$\|T^*(\eta f) - \eta T^*(f)\| e^{-\varphi_1/2} \le \|f\| e^{-\varphi_2/2}\|\overline{\partial}\overline{\eta}\| e^{-\psi/2}.$$

In particular, if $\eta = \eta_\nu$, then

$$\|T^*(\eta_\nu f) - \eta_\nu T^*(f)\|^2 e^{-\varphi_1} \le \|f\|^2 e^{-\varphi_2}$$

and therefore, by the Lebesgue theorem, $T^*(\eta_\nu f) - \eta_\nu T^*(f) \xrightarrow{\mathcal{H}_1} 0$ when $\nu \longrightarrow +\infty$. Thus, $T^*(\eta_\nu f) \xrightarrow{\mathcal{H}_1} T^*(f)$ and, consequently, 7^o is proved.

To prove 8^o take an $f \in \mathcal{D}_{(p,q+1)}(\Omega)$ and let

$$g := (-1)^{p-1} e^{\varphi_1}\vartheta(e^{-\varphi_2} f) \in (\mathcal{C}_0)_{(p,q)}(\Omega).$$

For any $u \in \mathcal{D}_{(p,q)}(\Omega)$ we get

$$\begin{aligned}\langle g, u\rangle_{\varphi_1} &= \sideset{}{'}\sum_{|I|=p,|K|=q} \sum_{j=1}^n (-1)^{p-1} \int_\Omega e^{\varphi_1} \frac{\partial(e^{-\varphi_2} f_{I,jK})}{\partial z_j} \overline{u}_{I,K} e^{-\varphi_1} d\mathcal{L}^{2n} \\ &= \sideset{}{'}\sum_{|I|=p,|K|=q} \sum_{j=1}^n (-1)^p \int_\Omega e^{-\varphi_2} f_{I,jK} \overline{\frac{\partial u_{I,K}}{\partial \overline{z}_j}} d\mathcal{L}^{2n} = \langle f, Tu\rangle_{\varphi_2}.\end{aligned}$$

Since the space $\mathcal{D}_{(p,q)}(\Omega)$ is dense in $\operatorname{Dom}(T)$ in the graph norm, we conclude that $f \in \operatorname{Dom}(T^*)$ and $T^*(f) = g$.

To prove 9^o fix an $f \in \operatorname{Dom}(T^*)$ and put $g := T^*(f)$. Then for any $u \in \mathcal{D}_{(p,q)}(\Omega)$ we have

$$\sideset{}{'}\sum_{|I|=p,|K|=q} \int_\Omega g_{I,K}\overline{u}_{I,K} e^{-\varphi_1} d\mathcal{L}^{2n} = \langle T^*(f), u\rangle_{\varphi_1} = \langle f, T(u)\rangle_{\varphi_2}$$

$$= \sideset{}{'}\sum_{|I|=p,|K|=q} \sum_{j=1}^n (-1)^p \int_\Omega e^{-\varphi_2} f_{I,jK} \overline{\frac{\partial u_{I,K}}{\partial \overline{z}_j}} d\mathcal{L}^{2n}.$$

Hence $g = (-1)^{p-1} e^{\varphi_1}\vartheta(e^{-\varphi_2} f)$ in the sense of distribution, which gives the required formula for T^*.

Finally, to see 10^o observe that

$$(-1)^{p-1} e^{\varphi_2-\varphi_1} T^* = \vartheta + \mathcal{A},$$

where $\mathcal{A} : \mathcal{D}'_{(p,q+1)}(\Omega) \longrightarrow \mathcal{D}'_{(p,q+1)}(\Omega)$ is a linear operator with $\mathcal{C}^1$ coefficients (cf. (5.2.3)). Consequently,

$$(-1)^{p-1} e^{\varphi_2-\varphi_1} T^*(f_\varepsilon) = (\vartheta f)_\varepsilon + \mathcal{A}(f_\varepsilon).$$

Hence $T^*(f_\varepsilon) \xrightarrow{\mathcal{H}_1} T^*(f)$ when $\varepsilon \longrightarrow 0$.

The proof of the proposition is completed. $\square$

Theorem 5.2.3 (Hörmander's $\mathbf{L^2}$-estimates)**.** *Under the above notation we have*

$$\int_\Omega \sideset{}{'}\sum_{\substack{|I|=p\\|K|=q}} \sum_{j,k=1}^n \frac{\partial^2\varphi}{\partial\overline{z}_k\partial z_j} f_{I,jK}\overline{f_{I,kK}} e^{-\varphi} d\mathcal{L}^{2n} + \int_\Omega \sideset{}{'}\sum_{\substack{|I|=p\\|J|=q+1}} \|\overline{\partial} f_{I,J}\|^2 e^{-\varphi} d\mathcal{L}^{2n}$$

$$\leq 2\|T^*(f)\|^2_{\varphi_1} + \|S(f)\|^2_{\varphi_3} + 2\int_\Omega \|f\|^2\|\partial\psi\|^2 e^{-\varphi} d\mathcal{L}^{2n}, \quad f \in \mathcal{D}_{(p,q+1)}(\Omega). \tag{5.2.8}$$

In particular, if

$$\mathcal{L}\varphi(z;X) \geq 2(\|\partial\psi(z)\|^2 + e^{\psi(z)})\|X\|^2, \quad z \in \Omega,\ X \in \mathbb{C}^n,$$

then, by Proposition 5.2.2, we get

$$\|f\|^2_{\varphi_2} \leq \|T^*(f)\|^2_{\varphi_1} + \|S(f)\|^2_{\varphi_3}, \quad f \in \operatorname{Dom}(T^*) \cap \operatorname{Dom}(S).$$

Proof. Let

$$\delta_j(g) := e^\varphi \frac{\partial(g e^{-\varphi})}{\partial z_j} = \frac{\partial g}{\partial z_j} - g\frac{\partial\varphi}{\partial z_j}, \quad j = 1, \dots, n.$$

Then

$$\int_\Omega g_1 \overline{\frac{\partial g_2}{\partial \overline{z}_k}} e^{-\varphi} d\mathcal{L}^{2n} = -\int_\Omega \delta_k(g_1)\overline{g_2} e^{-\varphi} d\mathcal{L}^{2n}, \quad g_1, g_2 \in \mathcal{D}(\Omega), \tag{5.2.9}$$

$$\delta_j\Big(\frac{\partial g}{\partial\overline{z}_k}\Big) - \frac{\partial\delta_j(g)}{\partial\overline{z}_k} = \frac{\partial^2\varphi}{\partial\overline{z}_k\partial z_j} g, \quad g \in \mathcal{D}(\Omega). \tag{5.2.10}$$

Observe that

$$e^\psi T^*(f) = (-1)^{p-1} \sideset{}{'}\sum_{|I|=p,|K|=q} \sum_{j=1}^n \delta_j(f_{I,jK}) dz_I \wedge d\overline{z}_K + (-1)^{p-1} \sideset{}{'}\sum_{|I|=p,|K|=q} \sum_{j=1}^n f_{I,jK}\frac{\partial\psi}{\partial z_j} dz_I \wedge d\overline{z}_K,$$

$$f \in \mathcal{D}_{(p,q+1)}(\Omega).$$

In particular,

$$\int_\Omega \sideset{}{'}\sum_{|I|=p,|K|=q} \sum_{j,k=1}^n \delta_j(f_{I,kK})\overline{\delta_k(f_{I,jK})} e^{-\varphi} d\mathcal{L}^{2n} \leq 2\|T^*(f)\|^2_{\varphi_1} + 2\int_\Omega \|f\|^2\|\partial\psi\|^2 e^{-\varphi} d\mathcal{L}^{2n},$$

$$f \in \mathcal{D}_{(p,q+1)}(\Omega).$$

For $f \in \mathcal{D}_{(p,q+1)}(\Omega)$ we have

$$\|S(f)\|^2 = {\sum_{|I|=p,|J|=q+1}}' \sum_{j=1}^n \Big|\frac{\partial f_{I,J}}{\partial \overline{z}_j}\Big|^2 - {\sum_{|I|=p,|K|=q}}' \sum_{j,k=1}^n \frac{\partial f_{I,jK}}{\partial \overline{z}_k}\overline{\frac{\partial f_{I,kK}}{\partial \overline{z}_j}}.$$

Indeed,

$$Sf = {\sum_{|I|=p,|Q|=q+2}}' \Big(\sum_{J\in\Xi^n_{q+1},\, j\in\{1,\dots,n\}} \varepsilon(I,Q,J,j)\frac{\partial f_{I,J}}{\partial \overline{z}_j}\Big) dz_I \wedge d\overline{z}_Q$$

(cf. § 4.2). Hence

$$\begin{aligned}\|Sf\|^2 &= {\sum_{|I|=p,|Q|=q+2}}' \sum_{j\notin J}\sum_{k\notin L} \varepsilon(I,Q,J,j)\frac{\partial f_{I,J}}{\partial \overline{z}_j}\varepsilon(I,Q,L,k)\overline{\frac{\partial f_{I,L}}{\partial \overline{z}_k}}\\
&= {\sum_{|I|=p,|J|=q+1}}' \sum_{j\notin J} \Big|\frac{\partial f_{I,J}}{\partial \overline{z}_j}\Big|^2 \quad + {\sum_{|I|=p,|K|=q}}' \sum_{j,k\notin K,\, j\neq k}\\
&\quad \varepsilon(I,\overrightarrow{jkK},\overrightarrow{kK},j)\varepsilon(I,\overrightarrow{jkK},\overrightarrow{jK},k)\operatorname{sgn}(\sigma_{kK})\operatorname{sgn}(\sigma_{jK})\frac{\partial f_{I,kK}}{\partial \overline{z}_j}\overline{\frac{\partial f_{I,jK}}{\partial \overline{z}_k}}\\
&= {\sum_{|I|=p,|J|=q+1}}' \sum_{j\notin J} \Big|\frac{\partial f_{I,J}}{\partial \overline{z}_j}\Big|^2 - {\sum_{|I|=p,|K|=q}}' \sum_{j\neq k} \frac{\partial f_{I,kK}}{\partial \overline{z}_j}\overline{\frac{\partial f_{I,jK}}{\partial \overline{z}_k}}\\
&= {\sum_{|I|=p,|J|=q+1}}' \sum_{j=1}^n \Big|\frac{\partial f_{I,J}}{\partial \overline{z}_j}\Big|^2 - {\sum_{|I|=p,|K|=q}}' \sum_{j,k=1}^n \frac{\partial f_{I,jK}}{\partial \overline{z}_k}\overline{\frac{\partial f_{I,kK}}{\partial \overline{z}_j}}.\end{aligned}$$

Thus we have proved that

$$\begin{aligned}\int_\Omega {\sum_{|I|=p,|K|=q}}' \sum_{j,k=1}^n \Big(\delta_j(f_{I,jK})\overline{\delta_k(f_{I,kK})} - \frac{\partial f_{I,jK}}{\partial \overline{z}_k}\overline{\frac{\partial f_{I,kK}}{\partial \overline{z}_j}}\Big)e^{-\varphi}d\mathcal{L}^{2n} + \int_\Omega {\sum_{|I|=p,|J|=q+1}}' \sum_{j=1}^n \Big|\frac{\partial f_{I,J}}{\partial \overline{z}_j}\Big|^2 e^{-\varphi}d\mathcal{L}^{2n}\\
\le 2\|T^*(f)\|^2_{\varphi_1} + \|S(f)\|^2_{\varphi_3} + 2\int_\Omega \|f\|^2\|\partial\psi\|^2 e^{-\varphi}d\mathcal{L}^{2n},\end{aligned}$$

which, by (5.2.9) and (5.2.10), implies the required inequality. □

Notice that (5.2.8) holds for arbitrary $\psi,\varphi \in \mathcal{C}^2(\Omega)$. In particular, it holds for $\varphi=\psi=0$.

Theorem 5.2.4. *Let $\Omega \subset \mathbb{C}^n$ be pseudoconvex. Then for any $p,q\in\mathbb{N}_0$ and for any $\overline{\partial}$-closed form $v \in L^2_{(p,q+1)}(\Omega,\mathrm{loc})$, there exist $\varphi_1,\varphi_2 \in \mathcal{C}^\infty(\Omega)$ and $u \in \operatorname{Dom}(T)\cap\mathcal{R}(T^*)$ such that $T(u)=v$, where*

$$L^2_{(p,q)}(\Omega,\varphi_1) \supset \operatorname{Dom}(T) \xrightarrow{T} L^2_{(p,q+1)}(\Omega,\varphi_2)$$

is as above.

In particular, for any $\overline{\partial}$-closed form $v \in L^2_{(p,q+1)}(\Omega,\mathrm{loc})$ there exists a form $u \in L^2_{(p,q)}(\Omega,\mathrm{loc})$ such that $\overline{\partial}u = v$.

Proof. Let w be a $\mathcal{C}^\infty$ strictly psh exhaustion function (cf. Proposition 4.1.15).

Let $\chi:\mathbb{R}\longrightarrow\mathbb{R}$ be a $\mathcal{C}^\infty$ convex increasing function. Take a $\varphi := \chi\circ w$ and define $\varphi_j := \varphi-(3-j)\psi$, $j=1,2,3$. Let T and S be as above.

We will show that the function χ can be found in such a way that:

- $\mathcal{L}\varphi(z;X) \ge 2(\|\overline{\partial}\psi\|^2+\exp(\psi))\|X\|^2$, $z\in\Omega$, $X\in\mathbb{C}^n$,
- $v \in L^2_{(p,q+1)}(\Omega,\varphi_2)$ (i.e. $v\in\operatorname{Ker} S$).

Suppose for a moment that χ is constructed. Then by Theorem 5.2.3

$$\|f\|^2_{\varphi_2} \le \|T^*(f)\|^2_{\varphi_1} + \|S(f)\|^2_{\varphi_3}, \quad f\in\operatorname{Dom}(T^*)\cap\operatorname{Dom}(S).$$

In particular,

$$\|f\|^2_{\varphi_2} \le \|T^*(f)\|^2_{\varphi_1}, \quad f \in \operatorname{Dom}(T^*) \cap \operatorname{Ker}(S).$$

Now, by Lemma 5.2.1 (with $F := \operatorname{Ker} S$), we find a $u \in \operatorname{Dom}(T) \cap \mathcal{R}(T^*)$ with $T(u) = v$

We pass to the construction of χ. Let $c_0 \in \mathcal{C}(\Omega, \mathbb{R}_{>0})$ be such that

$$\mathcal{L}w(z; X) \ge c_0\|X\|^2, \quad z \in \Omega,\ X \in \mathbb{C}^n.$$

Define

$$K_t := \{z \in \Omega : w(z) \le t\}, \quad \tau(t) := \max_{K_t}\{(2/c_0)(\|\overline{\partial}\psi\|^2 + e^{\psi})\}, \quad a_t := \int_{K_t \setminus K_{t-1}} \|v\|^2\, d\mathcal{L}^{2n}, \quad t \in \mathbb{R}.$$

Now take an increasing convex $\mathcal{C}^\infty$ function $\chi : \mathbb{R} \longrightarrow \mathbb{R}$ such that $\chi \ge \tau$ and

$$\chi(\nu - 1) \ge \sup_{K_\nu \setminus K_{\nu-1}} \psi + \nu \log 2 + \log a_\nu, \quad \nu \in \mathbb{N}.$$

Then

$$\begin{aligned}\mathcal{L}(\chi \circ w)(z; X) &= \chi''(w(z))\Big|\sum_{j=1}^n \frac{\partial w}{\partial z_j}(z) X_j\Big|^2 + \chi'(w(z))\mathcal{L}w(z; X)\\ &\ge \chi'(w(z))c_0(z)\|X\|^2 \ge \tau(w(z))c_0(z)\|X\|^2 \ge 2(\|\overline{\partial}\psi(z)\|^2 + e^{\psi(z)}), \quad z \in \Omega,\ X \in \mathbb{C}^n.\end{aligned}$$

Moreover,

$$\int_{\Omega \setminus K_0} \|v\|^2 e^{-\varphi_2} d\mathcal{L}^{2n} \le \sum_{\nu=1}^\infty a_\nu \sup_{K_\nu \setminus K_{\nu-1}} e^{\psi - \chi\circ w} \le \sum_{\nu=1}^\infty a_\nu e^{-\chi(\nu-1)} \sup_{K_\nu \setminus K_{\nu-1}} e^{\psi} \le \sum_{\nu=1}^\infty 2^{-\nu} < +\infty. \qquad \square$$

5.3. Solution of the Levi Problem

Let $\Omega \subset \mathbb{C}^n$ be open. For any $k \in \mathbb{N}_0 \cup \{\infty\}$ let $\mathcal{W}^k(\Omega)$ (resp. $\mathcal{W}^k(\Omega, \mathrm{loc})$) denote the Sobolev space of all functions $u \in L^2(\Omega)$ (resp. $u \in L^2(\Omega, \mathrm{loc})$) such that $D^{\alpha,\beta}u \in L^2(\Omega)$ (resp. $D^{\alpha,\beta}u \in L^2(\Omega, \mathrm{loc})$) for any $\alpha, \beta \in \mathbb{N}_0^n$ with $|\alpha| + |\beta| \le k$.

Let $\mathcal{W}^k_{(p,q)}(\Omega)$ (resp. $\mathcal{W}^k_{(p,q)}(\Omega, \mathrm{loc})$) be the space of all forms of type (p, q) with coefficients (in the canonical form) in $\mathcal{W}^k(\Omega)$ (resp. $\mathcal{W}^k(\Omega, \mathrm{loc})$).

Remark 5.3.1. (a) Obviously, $\mathcal{C}^k_{(p,q)}(\Omega) \subset \mathcal{W}^k_{(p,q)}(\Omega, \mathrm{loc})$. It is known that if $k > 2n$, then $\mathcal{W}^k_{(p,q)}(\Omega, \mathrm{loc}) \subset \mathcal{C}^{k-2n}_{(p,q)}(\Omega)$ (cf. [22]). In particular, $\mathcal{W}^\infty_{(p,q)}(\Omega, \mathrm{loc}) = \mathcal{C}^\infty_{(p,q)}(\Omega)$.
(b) $\overline{\partial}(\mathcal{W}^{k+1}_{(p,q)}(\Omega, \mathrm{loc})) \subset \mathcal{W}^k_{(p,q+1)}(\Omega, \mathrm{loc})$, $k \ge 0$.
(c) $\vartheta(\mathcal{W}^{k+1}_{(p,q+1)}(\Omega, \mathrm{loc})) \subset \mathcal{W}^k_{(p,q)}(\Omega, \mathrm{loc})$, $k \ge 0$.

The aim of the present section is to prove the following theorem.

Theorem 5.3.2 (Solution of the Levi Problem)**.** *Assume that Ω is pseudoconvex. Then*

$$\overline{\partial}(\mathcal{W}^{k+1}_{(p,q)}(\Omega, \mathrm{loc})) = \{v \in \mathcal{W}^k_{(p,q+1)}(\Omega, \mathrm{loc}) : \overline{\partial}v = 0\}, \quad p, q \ge 0, \quad k \in \mathbb{N}_0 \cup \{\infty\}.$$

Hence

$$\overline{\partial}(\mathcal{C}^{k+1-2n}_{(p,q)}(\Omega)) \supset \{v \in \mathcal{C}^k_{(p,q+1)}(\Omega) : \overline{\partial}v = 0\}, \quad p, q \ge 0, \quad k \in \mathbb{N}_0 \cup \{\infty\},\ k > 2n.$$

In particular, $\Omega \in S_{p,q}$ for any p, q.

Consequently, by Theorem 4.2.9, **any pseudoconvex open set is holomorphically convex**.

The proof requires some preparations.

Lemma 5.3.3. (a)

$$\mathcal{W}^{k+1}_{(p,0)}(\Omega,\mathrm{loc}) = \{u \in L^2_{(p,0)}(\Omega,\mathrm{loc}) : \overline{\partial}u \in \mathcal{W}^k_{(p,1)}(\Omega,\mathrm{loc})\}, \quad p \ge 0, \quad k \in \mathbb{N}_0.$$

(b)

$$\mathcal{W}^{k+1}_{(p,q+1)}(\Omega,\mathrm{loc}) = \{u \in L^2_{(p,q+1)}(\Omega,\mathrm{loc}) : \overline{\partial}u \in \mathcal{W}^k_{(p,q+2)}(\Omega,\mathrm{loc}),\ \vartheta u \in \mathcal{W}^k_{(p,q)}(\Omega,\mathrm{loc})\},$$
$$p,q \ge 0, \quad k \in \mathbb{N}_0.$$

Proof. (a) Take a $u = \sum'_{|I|=p} u_I dz_I \in L^2_{(p,0)}(\Omega,\mathrm{loc})$ such that $\overline{\partial}u \in \mathcal{W}^k_{(p,1)}(\Omega,\mathrm{loc})$. Observe that

$$\overline{\partial}u = \sum_{|I|=p}{}' \overline{\partial}u_I \wedge dz_I.$$

In particular, $\overline{\partial}u_I \in \mathcal{W}^k_{(0,1)}(\Omega,\mathrm{loc})$ for any $I \in \Xi^n_p$ (cf. Remark 4.2.1(b)). It suffices to prove that for each I the function u_I belongs to $\mathcal{W}^{k+1}(\Omega,\mathrm{loc})$. Thus we may assume that $p=0$.

Now we proceed by induction on k.

$k=0$.

It suffices to prove that $u|_U \in \mathcal{W}^1(U)$ for any domain $U \subset\subset \Omega$.

Fix such a domain U and let $\chi \in \mathcal{D}(\Omega)$ be such that $\chi = 1$ in a neighborhood of $\overline{U}$. Put $w := \chi u$. It is enough to show that $w \in \mathcal{W}^1(\Omega)$. Since $\overline{\partial}w = (\overline{\partial}\chi)u + \chi\overline{\partial}u$, we conclude that $\overline{\partial}w \in L^2_{(0,1)}(\Omega)$.

Thus, without loss of generality, we may assume that $K := \operatorname{supp} u \subset\subset \Omega$.

We have to show that $\partial u \in L^2_{(1,0)}(\Omega)$.

Let u_ε denote the ε-regularization of u (cf. § 5.1). Recall that $\overline{\partial}(u_\varepsilon) = (\overline{\partial}u)_\varepsilon$ (cf. (5.2.1)). Hence, since $\overline{\partial}u \in L^2_{(0,1)}(\Omega)$, we get

$$\overline{\partial}(u_\varepsilon) \overset{L^2_{(0,1)}(\Omega)}{\longrightarrow} \overline{\partial}u \text{ when } \varepsilon \longrightarrow 0.$$

(cf. (5.2.2)). Recall that $\operatorname{supp} u_\varepsilon \subset K^{(\varepsilon)}$. Fix an $\varepsilon_0 > 0$ such that $K^{(\varepsilon_0)} \subset \Omega$.

Observe that for $f \in \mathcal{D}(\Omega)$ we have

$$\|\overline{\partial}f\|_{L^2_{(0,1)}(\Omega)} = \|\partial f\|_{L^2_{(1,0)}(\Omega)}. \tag{5.3.1}$$

Indeed,

$$\begin{aligned}\|\overline{\partial}f\|^2_{L^2_{(0,1)}(\Omega)} &= \sum_{j=1}^n \int_\Omega \frac{\partial f}{\partial \overline{z}_j}\overline{\frac{\partial f}{\partial \overline{z}_j}}\, d\mathcal{L}^{2n} = -\int_\Omega \frac{\partial^2 f}{\partial z_j \partial \overline{z}_j}\overline{f}\, d\mathcal{L}^{2n} \\ &= -\sum_{j=1}^n \int_\Omega \frac{\partial^2 f}{\partial \overline{z}_j \partial z_j}\overline{f}\, d\mathcal{L}^{2n} = \sum_{j=1}^n \int_\Omega \frac{\partial f}{\partial z_j}\overline{\frac{\partial f}{\partial z_j}}\, d\mathcal{L}^{2n} = \|\partial f\|_{L^2_{(1,0)}(\Omega)}.\end{aligned}$$

Equality (5.3.1) with $f = u_{\varepsilon_\nu} - u_{\varepsilon_\mu}$, where $\varepsilon_0 > \varepsilon_\nu \searrow 0$, shows that the sequence $(\partial(u_{\varepsilon_\nu}))_{\nu=1}^\infty$ is convergent in $L^2_{(1,0)}(\Omega)$ to a form g. Clearly $g = \partial u$ in the sense of distribution.

$k-1 \rightsquigarrow k$.

We already know that $D^{\alpha,\beta}u \in L^2_{(p,0)}(\Omega,\mathrm{loc})$ for any $|\alpha|+|\beta| \le k$. Since

$$\overline{\partial}(D^{\alpha,\beta}u) = D^{\alpha,\beta}(\overline{\partial}u) \in L^2_{(p,1)}(\Omega,\mathrm{loc}),$$

we get (applying the step $k=0$ to the form $D^{\alpha,\beta}u$) that $D^{\alpha,\beta}u \in \mathcal{W}^1_{(p,0)}(\Omega,\mathrm{loc})$ for any $|\alpha|+|\beta| \le k$. Hence $u \in \mathcal{W}^{k+1}_{(p,0)}(\Omega,\mathrm{loc})$.

(b) Suppose that $u \in L^2_{(p,q+1)}(\Omega,\mathrm{loc})$ is such that $\overline{\partial}u \in \mathcal{W}^k_{(p,q+2)}(\Omega,\mathrm{loc})$ and $\vartheta u \in \mathcal{W}^k_{(p,q)}(\Omega,\mathrm{loc})$.

First consider the case $k=0$.

Observe that for any $\chi \in \mathcal{D}(\Omega)$, if $w := \chi u$, then $\overline{\partial}w = (\overline{\partial}\chi)u + \chi\overline{\partial}u \in \mathcal{W}_{(p,q+2)}(\Omega)$ and $\vartheta w = \mathcal{A}_\chi(u) + \chi\vartheta u \in \mathcal{W}^k_{(p,q)}(\Omega)$ (cf. (5.2.3)). Hence, similarly as in (a), we may assume that $\operatorname{supp} u \subset\subset \Omega$.

By Hörmander's inequality with $\psi = \varphi = 0$ we get

$$\sum_{|I|=p,|J|=q+1}{}' \|\overline{\partial} f_{I,J}\|_{L^2_{(0,1)}(\Omega)} \leq 2\|\vartheta f\|^2_{L^2_{(p,q)}(\Omega)} + \|\overline{\partial} f\|^2_{L^2_{(p,q+2)}(\Omega)}, \quad f \in \mathcal{D}_{(p,q+1)}(\Omega). \tag{5.3.2}$$

Let u_ε denote the ε-regularization of u. We have $\overline{\partial} u_\varepsilon = (\overline{\partial} u)_\varepsilon$ and $\vartheta(u_\varepsilon) = (\vartheta u)_\varepsilon$ (cf. (5.2.1)). Hence

$$\overline{\partial}(u_\varepsilon) \xrightarrow{L^2_{(p,q+2)}(\Omega)} \overline{\partial} u, \quad \vartheta(u_\varepsilon) \xrightarrow{L^2_{(p,q)}(\Omega)} \vartheta u.$$

By (5.3.2) with $f = u_\varepsilon$ $(0 < \varepsilon \ll 1)$ we get

$$\sum_{|I|=p,|J|=q+1}{}' \|\overline{\partial} u_{I,J}\|_{L^2_{(0,1)}(\Omega)} \leq 2\|\vartheta u\|^2_{L^2_{(p,q)}(\Omega)} + \|\overline{\partial} u\|^2_{L^2_{(p,q+2)}(\Omega)}.$$

Hence, by (a) (with $k = 0$), we conclude that $u_{I,J} \in \mathcal{W}^1(\Omega)$ for any I, J. Finally, $u \in \mathcal{W}^1_{(p,q+1)}(\Omega)$.

$k - 1 \rightsquigarrow k$.

We already know that for any $|\alpha| + |\beta| \leq k$ we have $D^{\alpha,\beta} u \in L^2_{(p,q+1)}(\Omega, \mathrm{loc})$. Observe that

$$\overline{\partial}(D^{\alpha,\beta} u) = D^{\alpha,\beta}(\overline{\partial} u) \in L^2_{(p,q+2)}(\Omega, \mathrm{loc}), \quad \vartheta(D^{\alpha,\beta} u) = D^{\alpha,\beta}(\vartheta u) \in L^2_{(p,q)}(\Omega, \mathrm{loc}).$$

Hence, applying the case $k = 0$ to $D^{\alpha,\beta} u$, we obtain $D^{\alpha,\beta} u \in \mathcal{W}^1_{(p,q+1)}(\Omega, \mathrm{loc})$. Consequently, $u \in \mathcal{W}^{k+1}_{(p,q+1)}(\Omega, \mathrm{loc})$. $\square$

Proof of Theorem 5.3.2. Fix a $v \in \mathcal{W}^k_{(p,q+1)}(\Omega, \mathrm{loc})$ with $\overline{\partial} v = 0$ $(k \in \mathbb{N}_0 \cup \{\infty\})$.

The case $q = 0$.

By Theorem 5.2.4 there exists a $u \in L^2_{(p,0)}(\Omega, \mathrm{loc})$ such that $\overline{\partial} u = v$. Hence, by Lemma 5.3.3(a), $u \in \mathcal{W}^{k+1}_{(p,0)}(\Omega, \mathrm{loc})$.

The case $q \geq 1$.

By Theorem 5.2.4 there exist functions $\varphi_j \in \mathcal{C}^\infty(\Omega, \mathbb{R})$, $j = 1, 2$, such that if

$$T : L^2_{(p,q)}(\Omega, \varphi_1) \supset \mathrm{Dom}(T) \xrightarrow{\overline{\partial}} L^2_{(p,q+1)}(\Omega, \varphi_2),$$

then there exists a $u \in \mathrm{Dom}(T) \cap \mathcal{R}(T^*)$ with $T(u) = v$. Recall that

$$T^*(f) = (-1)^{p-1} e^{\varphi_1} \vartheta(e^{-\varphi_2} f), \quad f \in \mathrm{Dom}(T^*).$$

Since $u \in \mathcal{R}(T^*)$ and $\vartheta^2 = 0$, we get $\vartheta(\exp(-\varphi_1) u) = 0$. Hence

$$\vartheta u = \mathcal{B} u, \tag{5.3.3}$$

where $\mathcal{B}$ is a linear operator with $\mathcal{C}^\infty$ coefficients.

In particular, $\vartheta u \in L^2_{(p,q)}(\Omega, \mathrm{loc})$. Now, by Lemma 5.3.3(b) (with $k = 0$) we conclude that $u \in \mathcal{W}^1_{(p,q+1)}(\Omega, \mathrm{loc})$. Suppose that $u \in \mathcal{W}^{\ell+1}_{(p,q+1)}(\Omega, \mathrm{loc})$ with $0 \leq \ell \leq k-1$. Then, by (5.3.3), $\vartheta u \in \mathcal{W}^{\ell+1}_{(p,q)}(\Omega, \mathrm{loc})$ and hence, by Lemma 5.3.3(b), $u \in \mathcal{W}^{\ell+2}_{(p,q+1)}(\Omega, \mathrm{loc})$. Induction on ℓ finishes the proof. $\square$

CHAPTER 6

Cousin problems

6.1. Meromorphic functions

Definition 6.1.1. Let $\Omega \subset \mathbb{C}^n$ be open and let $S \subset \Omega$ be thin (Definition 2.1.4) and relatively closed in Ω. A function $f : \Omega \setminus S \longrightarrow \mathbb{C}$ is called *meromorphic in* Ω ($f \in \mathcal{M}(\Omega)$) (1) if:

$f \in \mathcal{O}(\Omega \setminus S)$,

for every point $a \in S$ there exist a polydisc $P = \mathbb{P}(a, r) \subset \Omega$ and a function $m_a \in \mathcal{O}(P)$, $m_a \not\equiv 0$, such that the function $\ell_a := f \cdot m_a$ extends to a function $\widetilde{\ell}_a \in \mathcal{O}(P)$.

A point $a \in \Omega$ is called:

— *regular* ($a \in \mathfrak{R}(f)$) if there exist $P = \mathbb{P}(a, r) \subset \Omega$ and $m_a \in \mathcal{O}(P)$ such that $m_a(z) \neq 0$ for any $z \in P$, and the function $\ell_a := f \cdot m_a$ extends holomorphically to P (2) ,

— a *pole* ($a \in \mathfrak{P}(f)$) if there exist $P = \mathbb{P}(a, r) \subset \Omega$ and $m_a \in \mathcal{O}(P)$ such that $m_a \not\equiv 0$, $m_a(a) = 0$, and the function $\ell_a := f \cdot m_a$ extends holomorphically to P and $\widetilde{\ell}_a(a) \neq 0$ (3) ,

— a *point of indeterminacy* ($a \in \mathfrak{I}(f)$) in the remaining case, i.e. for any $P = \mathbb{P}(a, r) \subset \Omega$ and $m_a \in \mathcal{O}(P)$ such that $m_a \not\equiv 0$, and the function $\ell_a := f \cdot m_a$ extends holomorphically to P, we have $m_a(a) = \widetilde{\ell}_a(a) = 0$ (4) .

Remark 6.1.2. (a) The set $\mathfrak{R}(f)$ is open and f extends holomorphically onto $\mathfrak{R}(f)$ (5) .

Indeed, if P, m_a, ℓ_a are as in the definition of a regular point, then we put $\widetilde{f}(z) := \widetilde{\ell}_a(z)/m_a(z)$, $z \in P$. Obviously, $\widetilde{f} = f$ in $P \setminus S$.

(b) We may put $f(z) := \infty$, $z \in \mathfrak{P}(f)$. If $a \in \mathfrak{I}(f)$, then by Proposition 1.8.5, one cannot define the value of f at a.

(c) For $f(z_1, z_2) := z_1/z_2 \in \mathcal{M}(\mathbb{C}^2)$ we have

$$\mathfrak{R}(f) = \mathbb{C} \times \mathbb{C}_*, \quad \mathfrak{P}(f) = \mathbb{C}_* \times \{0\}, \quad \mathfrak{I}(f) = (0, 0).$$

(d) $\mathcal{O}(\Omega) \subset \mathcal{M}(\Omega)$.

(e) Meromorphy is a local property. If $\Phi : \Omega \longrightarrow \Omega' = \Phi(\Omega)$ is biholomorphic, then $f \in \mathcal{M}(\Omega') \iff f \circ \Phi \in \mathcal{M}(\Omega)$.

Proposition 6.1.3. *For $n = 1$ Definition 6.1.1 is equivalent to the standard definition of a meromorphic function of one complex variable (cf.* [4]*, Definition V.3.3). Moreover, if $n = 1$, then for every $f \in \mathcal{M}(\Omega)$ we have $\mathfrak{I}(f) = \varnothing$.*

Proof. Let $\Omega \subset \mathbb{C}$. Suppose that $f : \Omega \longrightarrow \widehat{\mathbb{C}}$ is meromorphic in the standard sense. Denote by S the set of all poles of f and let $k(a)$ be the rank of the pole $a \in S$. Since S consists of isolated points, S is thin and relatively closed in Ω. Moreover, for every $a \in S$, the function $f \cdot (z - a)^{k(a)}$ extends holomorphically to a neighborhood a. This means that the function f is meromorphic in the sense of Definition 6.1.1.

Now, let $f : \Omega \setminus S \longrightarrow \mathbb{C}$ be meromorphic in Ω in the sense of Definition 6.1.1. We may assume that $S := \Omega \setminus \mathfrak{R}(f)$. Since the set S is thin, it must consist of isolated points. It remains to show $S = \mathfrak{P}(f)$.

(1) Note that a function meromorphic *in* Ω need not be defined on whole of Ω.
(2) Obviously, $\Omega \setminus S \subset \mathfrak{R}(f)$.
(3) $\mathfrak{P}(f) \subset S$.
(4) $\mathfrak{I}(f) \subset S$.
(5) In particular, we may always assume that $\Omega \setminus S = \mathfrak{R}(f)$.

Indeed, if $\widetilde{\ell}_a = f \cdot m_a$ in $P = K(a,r)$, where $\widetilde{\ell}_a, m_a \in \mathcal{O}(P)$, $m_a \not\equiv 0$, $m_a(a) = 0$, then the function $g_a := f \cdot (z-a)^{k(a)}$, where $k(a) := \operatorname{ord}_a m_a - \operatorname{ord}_a \widetilde{\ell}_a$, extends holomorphically to a neighborhood of a and $\widetilde{g}_a(a) \neq 0$. $\square$

Proposition 6.1.4 (Identity principle). *Let $f, g \in \mathcal{M}(D)$, where $D \subset \mathbb{C}^n$ is a domain. Then the following conditions are equivalent:*
(i) *$\mathfrak{R}(f) = \mathfrak{R}(g)$, $\mathfrak{P}(f) = \mathfrak{P}(g)$, $\mathfrak{I}(f) = \mathfrak{I}(g)$, and $f = g$ on $\mathfrak{R}(f)$;*
(ii) *$f = g$ on $\mathfrak{R}(f) \cap \mathfrak{R}(g)$;*
(iii) *$f = g$ on a non-empty open subset of $\mathfrak{R}(f) \cap \mathfrak{R}(g)$.*

Proof. Obviously, (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii).

(iii) $\Longrightarrow$ (i). Observe that $S := D \setminus (\mathfrak{R}(f) \cap \mathfrak{R}(g)) = (D \setminus \mathfrak{R}(f)) \cup (D \setminus \mathfrak{R}(g))$ is thin. In particular, $\mathfrak{R}(f) \cap \mathfrak{R}(g)$ is a domain and, by the identity principle, $f = g$ on $\mathfrak{R}(f) \cap \mathfrak{R}(g)$. Let $a \in S$ and let $P = \mathbb{P}(a,r)$ and $m_a \in \mathcal{O}(P)$, $m_a \not\equiv 0$, be such that the function $\ell_{f,a} := f \cdot m_a$ extends holomorphically to P. Put $\ell_{g,a} := g \cdot m_a$. Then $\ell_{g,a} = \ell_{f,a}$ on $P \setminus S$. Hence $\ell_{g,a}$ extends holomorphically to P and $\widetilde{l}_{g,a} = \widetilde{\ell}_{f,a}$. Consequently, we get (i). $\square$

Proposition 6.1.5. *Let $D \subset \mathbb{C}^n$ be a domain. Then $\mathcal{M}(D)$ is a field.*

Proof. Let $f : D \setminus S_f \longrightarrow \mathbb{C}$, $g : D \setminus S_g \longrightarrow \mathbb{C}$ be meromorphic functions and let $S := S_f \cup S_g$. The set S is clearly thin and closed in D, and the functions $f + g, f \cdot g : D \setminus S \longrightarrow \mathbb{C}$ are well defined and holomorphic. For $a \in S$ let $P = \mathbb{P}(a,r) \subset D$ and $m_{f,a}, m_{g,a} \in \mathcal{O}(P)$, $m_{f,a} \not\equiv 0$, $m_{g,a} \not\equiv 0$, be such that the functions $f \cdot m_{f,a}$, $g \cdot m_{g,a}$ extend holomorphically to P. Then the functions $(f+g) \cdot m_{f,a} \cdot m_{g,a}$ and $(f \cdot g) \cdot m_{f,a} \cdot m_{g,a}$ also extend holomorphically onto P. Thus $\mathcal{M}(D)$ is a ring.

Suppose now that $f \neq 0$, i.e. $f \not\equiv 0$ in $D \setminus S_f$. Let $S := S_f \cup f^{-1}(0)$. The set S is thin and closed in D, and the function $1/f : D \setminus S \longrightarrow \mathbb{C}$ is well defined and holomorphic. Let $a \in S$.

For $a \in S_f$, let $P = \mathbb{P}(a,r) \subset D$, and $m_{f,a} \in \mathcal{O}(P)$, $m_{f,a} \not\equiv 0$, be such that the function $\ell_{f,a} := f \cdot m_{f,a}$ extends holomorphically onto P. Note that by identity principle we have $\widetilde{\ell}_{f,a} \not\equiv 0$. We can therefore take $m_{1/f,a} := \widetilde{\ell}_{f,a}$.

For $a \in D \setminus S_f$ such that $f(a) = 0$, we can choose P as an arbitrary polydisc $\mathbb{P}(a,r) \subset D \setminus S_f$ and define $m_{1/f,a} := f|_P$ (by the identity principle $m_{1/f,a} \not\equiv 0$).

Consequently, $1/f$ is a meromorphic function, and therefore $\mathcal{M}(D)$ is a field. $\square$

6.2. The Mittag–Leffler and Weierstrass theorems

Let us recall two classical theorems of one complex variable (cf. [4], Theorems VIII.3.2, V.5.15).

Theorem 6.2.1 (Mittag–Leffler). *For any open subset $\Omega \subset \mathbb{C}$, a set $B \subset \Omega$ without accumulation points in Ω, and a family of polynomials $(P_a)_{a \in B} \subset \mathcal{P}(\mathbb{C})$ with $P_a(0) = 0$, $a \in B$, there exists a function $f \in \mathcal{M}(\Omega)$ such that $f \in \mathcal{O}(\Omega \setminus B)$ and for every $a \in B$ the function*

$$f - P_a\Big(\frac{1}{z-a}\Big)$$

extends holomorphically to a neighborhood of a.

The above theorem may be reformulated as follows:

Theorem 6.2.2. *For any open subset $\Omega \subset \mathbb{C}$, an open covering $(\Omega_\alpha)_{\alpha \in A}$ of Ω, and a family $f_\alpha \in \mathcal{M}(\Omega_\alpha)$, $\alpha \in A$, such that*

$$f_\alpha - f_\beta \in \mathcal{O}(\Omega_\alpha \cap \Omega_\beta), \quad \alpha, \beta \in A, \quad (^6)$$

there exists a function $f \in \mathcal{M}(\Omega)$ such that

$$f - f_\alpha \in \mathcal{O}(\Omega_\alpha), \quad \alpha \in A.$$

(6) That is, the function $f_\alpha - f_\beta$ extends holomorphically to $\Omega_\alpha \cap \Omega_\beta$.

Proof that Theorem 6.2.2 implies Theorem 6.2.1. Let Ω, B, and $(P_a)_{a\in B}$ be as in the assumptions of Theorem 6.2.1. Choose $r_a > 0$, $a \in B$, so small that $K(a,r_a)\cap K(b,r_b) = \varnothing$ for every $a \neq b$, $a,b \in B$. Define

$$A := \{*\}\cup B, \quad \Omega_* := \Omega\setminus B, \quad \Omega_a := K(a,r_a),\ a\in B, \qquad f_* := 0, \quad f_a := P_a(1/(z-a)),\ a\in B.$$

One can easily check that all the assumptions of Theorem 6.2.2 are satisfied. Let $f \in \mathcal{M}(\Omega)$ be from the assertion of Theorem 6.2.2. Then $f = f - f_* \in \mathcal{O}(\Omega_*) = \mathcal{O}(\Omega\setminus B)$ and $f - P_a(1/(z-a)) = f - f_\alpha \in \mathcal{O}(\Omega_a) \subset \mathcal{O}(K(a,r_a))$ for every $a\in B$. $\square$

Proof that Theorem 6.2.1 implies Theorem 6.2.2. Let Ω, $(\Omega_\alpha)_{\alpha\in A}$, and $(f_\alpha)_{\alpha\in A}$ be as in the assumptions of Theorem 6.2.2. Put

$$B_\alpha := \boldsymbol{\mathcal{P}}(f_\alpha), \quad B := \bigcup_{\alpha\in A} B_\alpha.$$

Since $f_\alpha - f_\beta \in \mathcal{O}(\Omega_\alpha\cap\Omega_\beta)$, $B_\alpha\cap\Omega_\beta \subset B_\beta$ for any $\alpha,\beta\in A$. In particular, the set B has no accumulation points in Ω. For $a \in B_\alpha$, let $P_{\alpha,a} \in \mathcal{P}(\mathbb{C})$ be such that $P_{\alpha,a}(0) = 0$ and $f_\alpha - P_{\alpha,a}(1/(z-a))$ extends holomorphically to a neighborhood of a. Since $f_\alpha - f_\beta \in \mathcal{O}(\Omega_\alpha\cap\Omega_\beta)$, we conclude that $P_{\alpha,a}$ is independent of α. Put $P_a := P_{\alpha,a}$. Let $f\in\mathcal{M}(\Omega)$ be a function from the assertion of Theorem 6.2.1. Then $\boldsymbol{\mathcal{P}}(f) = B$ and for any $\alpha\in A$ and $a\in B_\alpha$, the function

$$f - f_\alpha = [f - P_a(\frac{1}{z-a})] - [f_\alpha - P_a(\frac{1}{z-a})]$$

extends holomorphically to a neighborhood of a. $\square$

Let $\mathcal{O}^*(\Omega) := \{f\in\mathcal{O}(\Omega) : f(z)\neq 0,\ z\in\Omega\}$.

Theorem 6.2.3 (Weierstrass)**.** *For any open subset $\Omega\subset\mathbb{C}$, a set $S\subset\Omega$ without accumulation points in Ω, and a function $k : S \longrightarrow \mathbb{Z}_*$, there exists a function $f\in\mathcal{M}(\Omega)$ such that $f\in\mathcal{O}^*(\Omega\setminus S)$ and for every $a\in S$ the function $\ell_a := f\cdot(z-a)^{-k(a)}$ extends holomorphically to a neighborhood of a with $\widetilde{\ell}_a(a)\neq 0$* (7) .

An equivalent formulation is following.

Theorem 6.2.4. *For any open subset $\Omega\subset\mathbb{C}$, an open covering $(\Omega_\alpha)_{\alpha\in A}$ of Ω, and a family $f_\alpha\in\mathcal{M}(\Omega_\alpha)$, $\alpha\in A$, such that*

$$f_\alpha/f_\beta \in \mathcal{O}^*(\Omega_\alpha\cap\Omega_\beta), \quad \alpha,\beta\in A,$$

there exists a function $f\in\mathcal{M}(\Omega)$ such that

$$f/f_\alpha \in \mathcal{O}^*(\Omega_\alpha), \quad \alpha\in A.$$

Proof that Theorem 6.2.4 implies Theorem 6.2.3. Let Ω, S and $k : S\longrightarrow\mathbb{Z}_*$ be as in the assumptions of Theorem 6.2.3. Choose $r_a > 0$, $a\in S$, so small that $K(a,r_a)\cap K(b,r_b) = \varnothing$ for every $a\neq b$, $a,b\in S$. Define

$$A := \{*\}\cup S, \quad \Omega_* := \Omega\setminus S, \quad \Omega_a := K(a,r_a),\ a\in S, \qquad f_* := 1, \quad f_a := (z-a)^{k(a)},\ a\in S.$$

It is easily seen that all the assumptions of Theorem 6.2.4 are satisfied. Let $f\in\mathcal{M}(\Omega)$ be the function from the assertion of Theorem 6.2.4. Then $f = f/f_* \in \mathcal{O}^*(\Omega_*) = \mathcal{O}^*(\Omega\setminus S)$ and $f\cdot(z-a)^{-k(a)} = f/f_\alpha \in \mathcal{O}^*(\Omega_a) = \mathcal{O}^*(K(a,r_a))$ for every $a\in S$. $\square$

Proof that Theorem 6.2.3 implies Theorem 6.2.4. Let Ω, $(\Omega_\alpha)_{\alpha\in A}$, and $(f_\alpha)_{\alpha\in A}$ be as in the assumptions of Theorem 6.2.4. Put

$$S_\alpha := \boldsymbol{\mathcal{P}}(f_\alpha)\cup f_\alpha^{-1}(0), \quad S := \bigcup_{\alpha\in A} S_\alpha.$$

Since $f_\alpha/f_\beta\in\mathcal{O}^*(\Omega_\alpha\cap\Omega_\beta)$, $S_\alpha\cap\Omega_\beta\subset S_\beta$ for any $\alpha,\beta\in A$. In particular, the set S has no accumulation points in Ω. For $a\in S_\alpha$ let

$$k(\alpha,a) := \begin{cases} -(\text{order of pole of } f_\alpha \text{ at } a) & \text{if } a\in\boldsymbol{\mathcal{P}}(f_\alpha) \\ \ \ (\text{order of zero of } f_\alpha \text{ at } a) & \text{if } f_\alpha(a) = 0 \end{cases}.$$

(7) That is, the function f has at a a zero of order $k(a)$, if $k(a) > 0$, and a pole of rank $-k(a)$, if $k(a) < 0$.

Since $f_\alpha/f_\beta \in \mathcal{O}^*(\Omega_\alpha \cap \Omega_\beta)$, we conclude that $k(\alpha, a)$ is independent of α. Put $k(a) := k(\alpha, a)$. Let $f \in \mathcal{M}(\Omega)$ be the function from the assertion of Theorem 6.2.3. Then f has no zeros or poles outside S, and for every $\alpha \in A$ and $a \in S_\alpha$ the function

$$\frac{f}{f_\alpha} = \frac{f \cdot (z-a)^{-k(a)}}{f_\alpha \cdot (z-a)^{-k(a)}}$$

extends to a non-vanishing holomorphic function in a neighborhood of a. □

6.3. First Cousin Problems

Definition 6.3.1. Let $\Omega \subset \mathbb{C}^n$ be open and let $\mathfrak{U} = (\Omega_\alpha)_{\alpha \in A}$ be an open covering of Ω. We say that the *first holomorphic Cousin problem has a solution for* $\mathfrak{U}$ if for any family of functions

$$\varphi_{\alpha,\beta} \in \mathcal{O}(\Omega_\alpha \cap \Omega_\beta), \quad \alpha, \beta \in A,$$

such that

$$\varphi_{\beta,\alpha} = -\varphi_{\alpha,\beta}, \quad \alpha, \beta \in A, \ (^8) \qquad \varphi_{\alpha,\beta} + \varphi_{\beta,\gamma} + \varphi_{\gamma,\alpha} = 0 \text{ in } \Omega_\alpha \cap \Omega_\beta \cap \Omega_\gamma, \quad \alpha, \beta, \gamma \in A,$$

there exist functions

$$\varphi_\alpha \in \mathcal{O}(\Omega_\alpha), \quad \alpha \in A,$$

such that

$$\varphi_{\alpha,\beta} = \varphi_\beta - \varphi_\alpha \text{ in } \Omega_\alpha \cap \Omega_\beta, \quad \alpha, \beta \in A.$$

The family $(\varphi_{\alpha,\beta})_{\alpha,\beta \in A}$ is called the *data for the first holomorphic Cousin problem for* $\mathfrak{U}$.

We say that the *first holomorphic Cousin problem has a solution for* Ω if it has a solution for any open covering. We write $\Omega \in \mathfrak{CP}^1(\mathcal{O})$.

In the above definitions the class $\mathcal{O}$ of holomorphic functions may be substituted by another class $\mathcal{F}$ (i.e. we assume that $\varphi_{\alpha,\beta} \in \mathcal{F}(\Omega_\alpha \cap \Omega_\beta)$ and we require that $\varphi_\alpha \in \mathcal{F}(\Omega_\alpha)$). For example, $\mathcal{F} = \mathcal{C}^k$. Then we can define the *first $\mathcal{F}$-Cousin problem for* $\mathfrak{U}$ (resp. the *first $\mathcal{F}$-Cousin problem for Ω*). We write $\Omega \in \mathfrak{CP}^1(\mathcal{F})$.

We say that the *first meromorphic Cousin problem has a solution for* $\mathfrak{U}$ if for every family

$$f_\alpha \in \mathcal{M}(\Omega_\alpha), \quad \alpha \in A,$$

such that

$$f_\alpha - f_\beta \in \mathcal{O}(\Omega_\alpha \cap \Omega_\beta), \quad \alpha, \beta \in A,$$

there exists a function $f \in \mathcal{M}(\Omega)$ such that

$$f - f_\alpha \in \mathcal{O}(\Omega_\alpha), \quad \alpha \in A.$$

The family $(f_\alpha)_{\alpha \in A}$ is called the *data for the first meromorphic Cousin problem for* $\mathfrak{U}$.

If the first meromorphic Cousin problem has a solution for any open covering $\mathfrak{U}$, then we say that the *first meromorphic Cousin problem has a solution for* Ω. We write $\Omega \in \mathfrak{CP}^1[\mathcal{M}]$.

Remark 6.3.2. (a) The first holomorphic and meromorphic Cousin problems are invariant under biholomorphic mappings, i.e. for every biholomorphic mapping $\Phi : \Omega \longrightarrow \Omega' = \Phi(\Omega)$ we have

$$\Omega \in \mathfrak{CP}^1(\mathcal{O}) \iff \Omega' \in \mathfrak{CP}^1(\mathcal{O}), \quad \Omega \in \mathfrak{CP}^1[\mathcal{M}] \iff \Omega' \in \mathfrak{CP}^1[\mathcal{M}].$$

(b) If $n = 1$, then by Theorem 6.2.2, $\Omega \in \mathfrak{CP}^1(\mathcal{O})$ for every $\Omega \subset \mathbb{C}$.

Proposition 6.3.3. $\Omega \in \mathfrak{CP}^1(\mathcal{O}) \Longrightarrow \Omega \in \mathfrak{CP}^1[\mathcal{M}]$.

---

(8) In particular, $\varphi_{\alpha,\alpha} = 0$, $\alpha \in A$.

Proof. Let $(f_\alpha)_{\alpha\in A}$ be data for the first meromorphic Cousin problem for an open covering $\mathfrak{U} = (\Omega_\alpha)_{\alpha\in A}$ of Ω. Define

$$\varphi_{\alpha,\beta} := f_\alpha - f_\beta \text{ in } \Omega_\alpha \cap \Omega_\beta, \quad \alpha, \beta \in A.$$

It is clear that we have defined data for the first holomorphic Cousin problem for $\mathfrak{U}$. Let $\varphi_\alpha \in \mathcal{O}(\Omega_\alpha)$, $\alpha \in A$, be a solution of this problem. Put $f := f_\alpha + \varphi_\alpha$ in Ω_α, $\alpha \in A$. We have $(f_\alpha + \varphi_\alpha) - (f_\beta + \varphi_\beta) = \varphi_{\alpha,\beta} - (\varphi_\beta - \varphi_\alpha) = 0$ in $\Omega_\alpha \cap \Omega_\beta$. Thus the function f is well defined. Moreover, $f - f_\alpha = \varphi_\alpha \in \mathcal{O}(\Omega_\alpha)$. $\square$

Proposition 6.3.4 (Cartan)**.** *For $\Omega \subset \mathbb{C}^2$, if $\Omega \in \mathfrak{CP}^1[\mathcal{M}]$, then Ω is a region of holomorphy.*

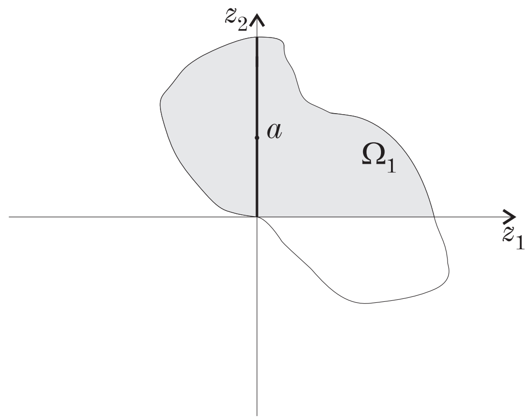

Figure 6.3.1

Proof. Suppose that for some $a = (a_1, a_2) \in \Omega$ we have $d(T_a g) \geq r > d_\Omega(a)$ for any $g \in \mathcal{O}(\Omega)$. Choose $b \in \partial\Omega$ such that $|a - b| = d_\Omega(a)$. Applying a suitable affine biholomorphism we may assume that $b = 0$ and $a_1 = 0$ (cf. Remark 6.3.2(a)). Put $U := \Omega_{0,e_2}$. Note that $0 \in \partial U$. Let

$$\Omega_1 := \Omega \cap (\mathbb{C} \times U), \quad \Omega_2 := \Omega \setminus (\{0\} \times U),$$
$$f_1(z) := (1/z_1)\exp(1/z_2),\ z = (z_1, z_2) \in \Omega_1, \quad f_2(z) := 0,\ z \in \Omega_2.$$

It is easy to see that $\Omega_1 \cup \Omega_2 = \Omega$ and $f_1 - f_2 \in \mathcal{O}(\Omega_1 \cap \Omega_2)$. Therefore we have got data for the first meromorphic Cousin problem. Let $f \in \mathcal{M}(\Omega)$ be a solution of this problem. In particular, $f = f - f_2 \in \mathcal{O}(\Omega_2)$ and hence $g := z_1 f \in \mathcal{O}(\Omega_2)$. On the other hand, $g = z_1(f - f_1) + \exp(1/z_2) \in \mathcal{O}(\Omega_1)$. Therefore $g \in \mathcal{O}(\Omega)$ and hence g extends holomorphically onto some neighborhood of $0 \in \mathbb{C}^2$. In particular, the function $g(0, z_2) = \exp(1/z_2)$ extends holomorphically to some neighborhood of $0 \in \mathbb{C}$; contradiction. $\square$

Proposition 6.3.5. *$\Omega \in \mathfrak{CP}^1(\mathcal{C}^\infty)$ for any open set $\Omega \subset \mathbb{C}^n$.*

Proof. Fix an open covering $\mathfrak{U} = (\Omega_\alpha)_{\alpha\in A}$ and data $\varphi_{\alpha,\beta} \in \mathcal{C}^\infty(\Omega_\alpha \cap \Omega_\beta)$, $\alpha, \beta \in A$. Let $(\sigma_j)_{j\in J}$ be a $\mathcal{C}^\infty$-partition of unity subordinated to $\mathfrak{U}$ and let $\varrho : J \longrightarrow A$ be such that $\operatorname{supp}\sigma_j \subset \Omega_{\varrho(j)}$, $j \in J$. Define

$$\varphi_\alpha := \sum_{j\in J} \sigma_j \varphi_{\varrho(j),\alpha} \text{ in } \Omega_\alpha,$$

where $\sigma_j \varphi_{\varrho(j),\alpha} := 0$ in $\Omega_\alpha \setminus \operatorname{supp}\sigma_j$. Then $\varphi_\alpha \in \mathcal{C}^\infty(\Omega_\alpha)$ and in $\Omega_\alpha \cap \Omega_\beta$ we have

$$\varphi_\beta - \varphi_\alpha = \sum_{j\in J} \sigma_j(\varphi_{\varrho(j),\beta} - \varphi_{\varrho(j),\alpha}) = \sum_{j\in J} \sigma_j \varphi_{\alpha,\beta} = \varphi_{\alpha,\beta}. \qquad \square$$

Theorem 6.3.6. *If $\Omega \in S_{0,0}$ (cf. § 4.4), then $\Omega \in \mathfrak{CP}^1(\mathcal{O})$.*

In particular, if Ω is a region of holomorphy, then $\Omega \in \mathfrak{CP}^1(\mathcal{O})$ (cf. Theorem 5.3.2).

Proof. Fix an open covering $\mathfrak{U} = (\Omega_\alpha)_{\alpha\in A}$ and data $\varphi_{\alpha,\beta} \in \mathcal{O}(\Omega_\alpha \cap \Omega_\beta)$, $\alpha, \beta \in A$. By Proposition 6.3.5 there exist $\psi_\alpha \in \mathcal{C}^\infty(\Omega_\alpha)$, $\alpha \in A$, such that $\varphi_{\alpha,\beta} = \psi_\beta - \psi_\alpha$. In particular, $v := -\overline{\partial}\psi_\alpha$ in Ω_α, is a well-defined differential form of class $\mathcal{C}^\infty_{(0,1)}(\Omega)$. Since $\Omega \in S_{0,0}$, there exists a function $u \in \mathcal{C}^\infty(\Omega)$ such that $\overline{\partial}u = v$. Put $\varphi_\alpha := \psi_\alpha + u$ in Ω_α, $\alpha \in A$. Then $\overline{\partial}\varphi_\alpha = 0$ and so $\varphi_\alpha \in \mathcal{O}(\Omega_\alpha)$. Moreover, $\varphi_\beta - \varphi_\alpha = \psi_\beta - \psi_\alpha = \varphi_{\alpha,\beta}$ in $\Omega_\alpha \cap \Omega_\beta$ for any $\alpha, \beta \in A$. □

6.4. Second Cousin Problems

Definition 6.4.1. Let $\Omega \subset \mathbb{C}^n$ be open and let $\mathfrak{U} = (\Omega_\alpha)_{\alpha\in A}$ be an open covering of Ω. We say that the *second holomorphic Cousin problem has a solution for* $\mathfrak{U}$ if for any family of functions

$$\varphi_{\alpha,\beta} \in \mathcal{O}^*(\Omega_\alpha \cap \Omega_\beta), \quad \alpha, \beta \in A,$$

such that

$$\varphi_{\beta,\alpha} \cdot \varphi_{\alpha,\beta} = 1, \quad \alpha, \beta \in A, \qquad \varphi_{\alpha,\beta} \cdot \varphi_{\beta,\gamma} \cdot \varphi_{\gamma,\alpha} = 1 \text{ in } \Omega_\alpha \cap \Omega_\beta \cap \Omega_\gamma, \quad \alpha, \beta, \gamma \in A, \quad (^9)$$

there exist functions

$$\varphi_\alpha \in \mathcal{O}^*(\Omega_\alpha), \quad \alpha \in A,$$

such that

$$\varphi_{\alpha,\beta} = \varphi_\beta/\varphi_\alpha \text{ in } \Omega_\alpha \cap \Omega_\beta, \quad \alpha, \beta \in A.$$

The family $(\varphi_{\alpha,\beta})_{\alpha,\beta\in A}$ is called the *data for the second holomorphic Cousin problem for* $\mathfrak{U}$.

We say that the *second holomorphic Cousin problem has a solution for* Ω if it has a solution for any open covering. We write $\Omega \in \mathfrak{CP}^2(\mathcal{O})$.

In the above definitions the class $\mathcal{O}^*$ of non-vanishing holomorphic functions may be substituted by another class $\mathcal{F}^*$ (we assume that $\varphi_{\alpha,\beta} \in \mathcal{F}^*(\Omega_\alpha \cap \Omega_\beta)$ and we require that $\varphi_\alpha \in \mathcal{F}^*(\Omega_\alpha)$ (10)). Then we can define the *second $\mathcal{F}$-Cousin problem for* $\mathfrak{U}$ (resp. the *second $\mathcal{F}$-Cousin problem for* Ω). We write $\Omega \in \mathfrak{CP}^2(\mathcal{F})$.

We say that the *second meromorphic Cousin problem has a solution for* $\mathfrak{U}$ if for every family

$$f_\alpha \in \mathcal{M}(\Omega_\alpha), \quad \alpha \in A,$$

such that

$$f_\alpha/f_\beta \in \mathcal{O}^*(\Omega_\alpha \cap \Omega_\beta), \quad \alpha, \beta \in A,$$

there exists a function $f \in \mathcal{M}(\Omega)$ such that

$$f/f_\alpha \in \mathcal{O}^*(\Omega_\alpha), \quad \alpha \in A.$$

The family $(f_\alpha)_{\alpha\in A}$ is called the *data for the second meromorphic Cousin problem for* $\mathfrak{U}$.

If the second meromorphic Cousin problem has a solution for any open covering $\mathfrak{U}$, then we say that the *second meromorphic Cousin problem has a solution for* Ω. We write $\Omega \in \mathfrak{CP}^2[\mathcal{M}]$.

Remark 6.4.2. (a) The second holomorphic and meromorphic Cousin problems are invariant under biholomorphic mappings (cf. Remark 6.3.2(a)).
(b) If $n = 1$, then by Theorem 6.2.4, $\Omega \in \mathfrak{CP}^2[\mathcal{M}]$ for every $\Omega \subset \mathbb{C}$.

Proposition 6.4.3. $\Omega \in \mathfrak{CP}^2(\mathcal{O}) \Longrightarrow \Omega \in \mathfrak{CP}^2[\mathcal{M}]$ *(cf. Proposition 6.3.3).* (11)

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(9) In particular, $\varphi^2_{\alpha,\alpha} = \varphi^3_{\alpha,\alpha} = 1$ in Ω_α, and hence $\varphi_{\alpha,\alpha} \equiv 1$.

(10) $\mathcal{F}^*(U) := \{\varphi \in \mathcal{F}(U) : \forall_{z\in U} : f(z) \neq 0\}$.

(11) One can also prove (but it is much more difficult) that if Ω is a region of holomorphy, then the converse implication is true, namely, $\Omega \in \mathfrak{CP}^2[\mathcal{M}] \Longrightarrow \Omega \in \mathfrak{CP}^2(\mathcal{O})$; we have the following

Theorem ([17]). *If Ω is a region of holomorphy, then for any data $\varphi_{\alpha,\beta} \in \mathcal{O}^*(\Omega_\alpha \cap \Omega_\beta)$, $\alpha, \beta \in A$, for the second holomorphic Cousin problem, there exist functions $f_\alpha \in \mathcal{O}(\Omega_\alpha)$, $\alpha \in A$, such that $f_\alpha = \varphi_{\alpha,\beta} \cdot f_\beta$ in $\Omega_\alpha \cap \Omega_\beta$ for any $\alpha, \beta \in A$.*

Consequently, the functions f_α, $\alpha \in A$, form data for the second meromorphic Cousin problem. If $f \in \mathcal{M}(\Omega)$ is such that $f/f_\alpha \in \mathcal{O}^*(\Omega_\alpha)$, then the functions $\varphi_\alpha := f/f_\alpha$, $\alpha \in A$, give a solution of the initial problem.

Proof. Let $(f_\alpha)_{\alpha\in A}$ be data for the second meromorphic Cousin problem for $\mathfrak{U} = (\Omega_\alpha)_{\alpha\in A}$. Define

$$\varphi_{\alpha,\beta} := f_\alpha/f_\beta \text{ in } \Omega_\alpha \cap \Omega_\beta, \quad \alpha, \beta \in A.$$

We have defined data for the second holomorphic Cousin problem for $\mathfrak{U}$. Let $\varphi_\alpha \in \mathcal{O}^*(\Omega_\alpha)$, $\alpha \in A$, be a solution of this problem. Put $f := f_\alpha \cdot \varphi_\alpha$ in Ω_α, $\alpha \in A$. Then $(f_\alpha \cdot \varphi_\alpha)/(f_\beta \cdot \varphi_\beta) = \varphi_{\alpha,\beta} \cdot (\varphi_\alpha/\varphi_\beta) = 1$ in $\Omega_\alpha \cap \Omega_\beta$ (thus f is well defined) and $f/f_\alpha = \varphi_\alpha \in \mathcal{O}^*(\Omega_\alpha)$. $\square$

Remark 6.4.4. Let $D \subset \mathbb{C}^n$ be a simply connected domain and let $f \in \mathcal{C}^*(D)$. Then there exists a function $\widetilde{f} \in \mathcal{C}(\Omega)$ such that $f = \exp(\widetilde{f})$. Moreover, $f \in \mathcal{O}^*(D)$ iff $\widetilde{f} \in \mathcal{O}(D)$.

Theorem 6.4.5. *If $\Omega \in \mathfrak{CP}^1(\mathcal{O}) \cap \mathfrak{CP}^2(\mathcal{C})$, then $\Omega \in \mathfrak{CP}^2(\mathcal{O})$.* ([12])

Proof. Fix a covering $\mathfrak{U} = (\Omega_\alpha)_{\alpha\in A}$ and data $\varphi_{\alpha,\beta} \in \mathcal{O}^*(\Omega_\alpha \cap \Omega_\beta)$, $\alpha, \beta \in A$, for the second holomorphic Cousin problem. Consider two cases.

1^o: Ω_α is simply connected for every $\alpha \in A$.

Let $\psi_\alpha \in \mathcal{C}^*(\Omega_\alpha)$, $\alpha \in A$, be such that $\varphi_{\alpha,\beta} = \psi_\beta/\psi_\alpha$ in $\Omega_\alpha \cap \Omega_\beta$, $\alpha, \beta \in A$ (recall that $\Omega \in \mathfrak{CP}^2(\mathcal{C})$). Let $\widetilde{\psi}_\alpha \in \mathcal{C}(\Omega_\alpha)$ be such that $\psi_\alpha = \exp(\widetilde{\psi}_\alpha)$ (Remark 6.4.4), $\alpha \in A$. Define $\widetilde{\psi}_{\alpha,\beta} := \widetilde{\psi}_\beta - \widetilde{\psi}_\alpha$ in $\Omega_\alpha \cap \Omega_\beta$. Since $\exp(\widetilde{\psi}_{\alpha,\beta}) = \exp(\widetilde{\psi}_\beta)/\exp(\widetilde{\psi}_\alpha) = \psi_\beta/\psi_\alpha = \varphi_{\alpha,\beta} \in \mathcal{O}^*(\Omega_\alpha \cap \Omega_\beta)$, we get $\widetilde{\psi}_{\alpha,\beta} \in \mathcal{O}(\Omega_\alpha \cap \Omega_\beta)$. Therefore the family $\widetilde{\psi}_{\alpha,\beta}$, $\alpha, \beta \in A$, gives data for the first holomorphic Cousin problem for $\mathfrak{U}$. By our assumptions, this problem has a solution $\chi_\alpha \in \mathcal{O}(\Omega)$, $\alpha \in A$, such that $\widetilde{\psi}_{\alpha,\beta} = \chi_\beta - \chi_\alpha$. Now let $\varphi_\alpha := \exp(\chi_\alpha) \in \mathcal{O}^*(\Omega_\alpha)$, $\alpha \in A$. Then $\varphi_\beta/\varphi_\alpha = \exp(\chi_\beta - \chi_\alpha) = \exp(\widetilde{\psi}_{\alpha,\beta}) = \varphi_{\alpha,\beta}$.

2^o: the general case.

Let $(U_j)_{j\in J}$, be an open covering subordinated to $\mathfrak{U}$ such that each set U_j is simply connected. Let $\varrho : J \longrightarrow A$ be such that $U_j \subset \Omega_{\varrho(j)}$, $j \in J$. Define

$$\widehat{\varphi}_{j,k} := \varphi_{\varrho(j),\varrho(k)}|_{U_j\cap U_k}, \quad j, k \in J.$$

By 1^o there exist functions $\widehat{\varphi}_j \in \mathcal{O}^*(U_j)$, $j \in J$, such that $\widehat{\varphi}_{j,k} = \widehat{\varphi}_k/\widehat{\varphi}_j$ in $U_j \cap U_k$, $j, k \in J$. Put $\varphi_\alpha := \widehat{\varphi}_j \cdot \varphi_{\varrho(j),\alpha}$ in $\Omega_\alpha \cap U_j$. Observe that

$$\frac{\widehat{\varphi}_j \cdot \varphi_{\varrho(j),\alpha}}{\widehat{\varphi}_k \cdot \varphi_{\varrho(k),\alpha}} = \widehat{\varphi}_{k,j} \cdot \varphi_{\varrho(j),\varrho(k)} = 1 \text{ in } \Omega_\alpha \cap U_j \cap U_k.$$

Consequently, φ_α is well defined in Ω_α and $\varphi_\alpha \in \mathcal{O}^*(\Omega_\alpha)$. Moreover, in $\Omega_\alpha \cap \Omega_\beta \cap U_j$ we have

$$\frac{\varphi_\beta}{\varphi_\alpha} = \frac{\widehat{\varphi}_j \cdot \varphi_{\varrho(j),\beta}}{\widehat{\varphi}_j \cdot \varphi_{\varrho(j),\alpha}} = \varphi_{\alpha,\beta}.$$

$\square$

Example 6.4.6 (Oka)**.** Let $D := A \times A \subset \mathbb{C}^2$, where

$$A := \{z \in \mathbb{C} : 3/4 < |z| < 5/4\}.$$

Then D is a domain of holomorphy such that $D \notin \mathfrak{CP}^2[\mathcal{M}]$ ([13]) .

Indeed (cf. [19]), suppose that $D \in \mathfrak{CP}^2(\mathcal{O})$ and let

$$M := \{(z_1, z_2) \in \mathbb{C}^2 : z_1 - z_2 = 1\}.$$

Notice that $(z_1, z_2) \in M \cap D$ iff $\operatorname{Re} z_1 - \operatorname{Re} z_2 = 1$, $\operatorname{Im} z_1 = \operatorname{Im} z_2$. Put

$$M_- := M \cap D \cap \{\operatorname{Im} z_1 = \operatorname{Im} z_2 < 0\}, \quad M_+ := M \cap D \cap \{\operatorname{Im} z_1 = \operatorname{Im} z_2 > 0\}.$$

([12]) This is an example of the so-called *Oka principle* saying that ‘everything’ which can be done continuously in regions of holomorphy can be also done holomorphically.

([13]) By Theorem 6.4.5, $D \notin \mathfrak{CP}^2(\mathcal{C})$.

Observe that $M_- \cup M_+ = M \cap D$. In particular, M_- and M_+ are closed in D. Put

$$D_1 := D \cap \{\operatorname{Im} z_1 > 0,\ \operatorname{Im} z_2 > 0\}, \quad D_2 := D \setminus M_+,$$
$$f_1(z_1, z_2) := z_1 - z_2 - 1, \quad f_2(z_1, z_2) := 1.$$

Then $D_1 \cup D_2 = D$ and $f_1/f_2 \in \mathcal{O}^*(D_1 \cap D_2)$.

Suppose that there exists a function $f \in \mathcal{M}(D)$ such that $f/f_j \in \mathcal{O}^*(D_j)$, $j = 1, 2$. Consequently, $f(z_1, z_2) = (z_1 - z_2 - 1)h(z_1, z_2)$ for $(z_1, z_2) \in D_1$, where $h \in \mathcal{O}^*(D_1)$, and $f|_{D_2} \in \mathcal{O}^*(D_2)$.

Put

$$F(\alpha, \beta) := f(e^{i\alpha}, e^{i\beta}), \quad \alpha, \beta \in \mathbb{R}.$$

Observe that

$$\{(\alpha, \beta) \in [0, 2\pi] \times [0, 2\pi] : F(\alpha, \beta) = 0\} = \{(\pi/3, 2\pi/3)\} =: \{c\}.$$

Let γ denote the boundary of the square $[0, 2\pi] \times [0, 2\pi]$ (considered as a curve with positive orientation with respect to the square).

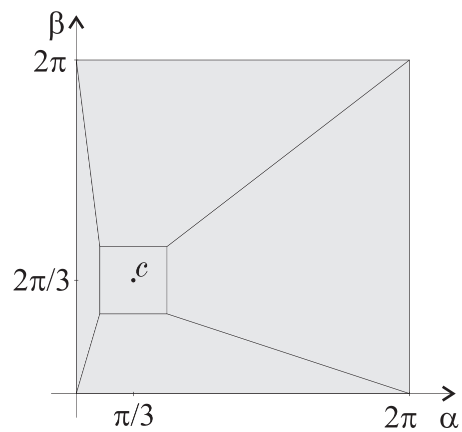

Figure 6.4.1

Define

$$\mathbb{I} := \int_\gamma \frac{dF}{F} = \int_\gamma \frac{1}{F}\frac{\partial F}{\partial \alpha} d\alpha + \frac{1}{F}\frac{\partial F}{\partial \beta} d\beta.$$

Since F is periodic (14) , we get $\mathbb{I} = 0$. Let γ_ε denote the boundary of the square $[-\varepsilon, \varepsilon] \times [-\varepsilon, \varepsilon]$ (considered as a curve). Since the form dF/F is closed in $[0, 2\pi] \times [0, 2\pi] \setminus \{c\}$, we get

$$\mathbb{I} = \int_{c+\gamma_\varepsilon} \frac{dF}{F}, \quad 0 < \varepsilon \ll 1.$$

Recall that $F = G \cdot H$ in a neighborhood of c, where

$$G(\alpha, \beta) := e^{i\alpha} - e^{i\beta} - 1, \quad H(\alpha, \beta) := h(e^{i\alpha}, e^{i\beta}).$$

Since $dF/F = dG/G + dH/H$ and dH/H is closed in a neighborhood of c, we get

$$0 = \mathbb{I} = \int_{c+\gamma_\varepsilon} \frac{dG}{G}, \quad 0 < \varepsilon \ll 1.$$

Observe that

$$J_{\mathbb{R}} G(\alpha, \beta) = \det \begin{bmatrix} -\sin\alpha, & \sin\beta \\ \cos\alpha, & -\cos\beta \end{bmatrix} = \sin(\alpha - \beta).$$

(14) $F(\alpha, \beta) = F(\alpha + 2k\pi, \beta + 2\ell\pi)$, $k, \ell \in \mathbb{Z}$.

Hence $J_{\mathbb{R}}G(c_1, c_2) = \sin(-\pi/3) \neq 0$ and, therefore, G is a diffeomorphism in a neighborhood U of c. Consequently, for $0 < \varepsilon \ll 1$, the curve $\sigma_\varepsilon := G \circ (c + \gamma_\varepsilon)$ is a Jordan curve with $(0, 0) \in \operatorname{int} \sigma_\varepsilon$ and

$$0 = \int_{c+\gamma_\varepsilon} \frac{dG}{G} = \int_{\sigma_\varepsilon} \frac{dz}{z} \neq 0;$$

contradiction.

Example 6.4.7 (Serre)**.** Let

$$D := \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1^2 + z_2^2 + z_3^2 - 1| < 1\}.$$

Then D is a homotopically simply connected domain of holomorphy such that $D \notin \mathfrak{CP}^2[\mathcal{M}]$ (cf. [11]).

Proposition 6.4.8. *Assume that $\Omega \in \mathfrak{CP}^2[\mathcal{M}]$. Then for every $(n-1)$-dimensional complex submanifold $M \subset \Omega$, there exists a function $f \in \mathcal{O}(\Omega)$ such that*
$M = f^{-1}(0)$,
for any open subset $U \subset \Omega$ and $g \in \mathcal{O}(U)$ with $M \cap U \subset g^{-1}(0)$ we have $g/f \in \mathcal{O}(U)$.

The function f with the above properties is called a *defining function for M*.

Proof. It is well known that M may be locally described (up to a biholomorphism) as $\{z_n = 0\}$. More precisely, for every point $a \in M$ there exist a neighborhood U_a and a biholomorphic mapping $\Phi_a : U_a \longrightarrow \mathbb{D}^n$ such that $\Phi_a(M \cap U_a) = \mathbb{D}^{n-1} \times \{0\}$. Moreover, if $a \in U \subset U_a$ and $g \in \mathcal{O}(U)$ are such that $M \cap U \subset g^{-1}(0)$, then the function $(1/z_n)(g \circ \Phi_a^{-1})$ extends holomorphically to $\Phi_a(U)$ (15) .

For $a \in \Omega$ let
$\Omega_a := \mathbb{P}(a, r_a) \subset \Omega \setminus M$, $f_a := 1$, if $a \notin M$,
$\Omega_a := \mathbb{P}(a, r_a) \subset U_a$, $f_a := ((\Phi_a)_n)|_{\Omega_a}$, if $a \in M$.

We have obtained data for the second meromorphic Cousin problem. Now, any solution of this problem is a defining function for M. ☐

Proposition 6.4.9. *Let $\Omega \in \mathfrak{CP}^1(\mathcal{O})$ and let M be an $(n-1)$-dimensional complex submanifold of Ω for which there exists a function $\widetilde{f} \in \mathcal{C}(\Omega)$ such that*
$M = \widetilde{f}^{-1}(0)$,
for every open subset $U \subset \Omega$ and $g \in \mathcal{O}(U)$, if g is a defining function for $M \cap U$, then $g/\widetilde{f} \in \mathcal{C}^*(U)$.
Then there exists a defining function for M. (16)

Proof. Let Ω_a, f_a, $a \in \Omega$ be as in the proof of Proposition 6.4.8. It follows from the assumptions that $f_a/\widetilde{f} \in \mathcal{C}^*(\Omega_a)$ for every $a \in \Omega$. Since Ω_a is a polydisc (and hence simply connected), Remark 6.4.4 implies that there exists a function $g_a \in \mathcal{C}(\Omega_a)$ such that $\exp(g_a) = f_a/\widetilde{f}$. We have $\exp(g_a - g_b) = f_a/f_b \in \mathcal{O}^*(\Omega_a \cap \Omega_b)$ in $\Omega_a \cap \Omega_b$ and so (Remark 6.4.4) $\varphi_{ab} := g_a - g_b \in \mathcal{O}(\Omega_a \cap \Omega_b)$. We have obtained data for the first holomorphic Cousin problem. Let $\varphi_a \in \mathcal{O}(\Omega)$, $a \in \Omega$, be such that $\varphi_{a,b} = \varphi_b - \varphi_a$ for every $a, b \in \Omega$. Define $f := f_a \exp(\varphi_a)$ w Ω_a. In $\Omega_a \cap \Omega_b$ we have

$$\frac{f_a \exp(\varphi_a)}{f_b \exp(\varphi_b)} = \exp(g_a - g_b) \exp(\varphi_a - \varphi_b) = \exp(\varphi_{a,b}) \exp(\varphi_{b,a}) = 1.$$

Hence f is a well-defined holomorphic function in Ω. Directly from the definition of f it follows that f is a defining function for M. ☐

Remark 6.4.10. Let M be a k-dimensional complex submanifold of an open set $\Omega \subset \mathbb{C}^n$ $(1 \leq k \leq n-1)$ and let $f : M \longrightarrow \mathbb{C}$. Then the following conditions are equivalent:
(i) $f \in \mathcal{O}(M)$ in the complex manifold sense, i.e. for any point $a \in M$ there exists local coordinates $\varphi : \mathbb{D}^k \longrightarrow U$ such that $f \circ \varphi \in \mathcal{O}(\mathbb{D}^k)$, where U is a neighborhood of a;
(ii) for any point $a \in M$ there exist $P = \mathbb{P}_n(a, r) \subset \Omega$ and $\widetilde{f}_a \in \mathcal{O}(P)$ such that $\widetilde{f}_a = f$ on $M \cap P$.

(15) This means that $(\Phi_a)_n$ is a defining function of $M \cap U_a$.
(16) Recall the Oka principle from Theorem 6.4.5.

Proposition 6.4.11. *Let $\varOmega \in \mathfrak{CP}^1(\mathcal{O})$ and let $M \subset \varOmega$ be an $(n-1)$-dimensional complex submanifold of $\varOmega$ for which there exists a defining function $F_0 \in \mathcal{O}(\varOmega)$. Then for any function $f \in \mathcal{O}(M)$ there exists an $\widetilde{f} \in \mathcal{O}(\varOmega)$ with $\widetilde{f} = f$ on M.* ([17])

Proof. For $a \in \varOmega$ let $\varOmega_a = \mathbb{P}(a,r)$, where the polydisc $\mathbb{P}(a,r)$ is such that

$\mathbb{P}(a,r) \cap M = \varnothing$ if $a \notin M$,

there exists a function $f_a \in \mathcal{O}(\mathbb{P}(a,r))$ such that $f_a = f$ on $M \cap \mathbb{P}(a,r)$ if $a \in M$ (cf. Remark 6.4.10).

Put $f_a := 0$ if $a \notin M$. Observe that $f_a - f_b = 0$ on $M \cap \varOmega_a \cap \varOmega_b$ (if $M \cap \varOmega_a \cap \varOmega_b \neq \varnothing$). Since F_0 is a defining function, there exists a function $\varphi_{a,b} \in \mathcal{O}(\varOmega_a \cap \varOmega_b)$ such that $f_a - f_b = F_0\varphi_{a,b}$ on $\varOmega_a \cap \varOmega_b$, $a, b \in \varOmega$.

Observe that $\varphi_{a,b}$, $a, b \in \varOmega$, are data for the first holomorphic Cousin problem.

Let $\varphi_a \in \mathcal{O}(\varOmega_a)$ be such that $\varphi_{a,b} = \varphi_b - \varphi_a$, $a, b \in \varOmega$. Then

$$(f_a + F_0\varphi_a) - (f_b + F_0\varphi_b) = f_a - f_b - F_0(\varphi_b - \varphi_a) = F_0\varphi_{a,b} - F_0\varphi_{a,b} = 0 \text{ on } \varOmega_a \cap \varOmega_b.$$

Consequently, the function $\widetilde{f} := f_a + F_0\varphi_a$ on $\varOmega_a$, $a \in \varOmega$, is the required extension. $\square$

Exercises

6.1. Let

$$D := \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1^2 + z_2^2 + z_3^2 - 1| < 1\}.$$

Prove that D is simply connected — complete the following sketch of the proof.

Let

$$Q := \{(w_1, w_2, w_3) \in \mathbb{C}^3 : w_1^2 + w_2^2 + w_3^2 = 1\}.$$

Then the mapping

$$D \ni (z_1, z_2, z_3) \longmapsto \Big(\frac{z_1}{\sqrt{z_1^2 + z_2^2 + z_3^2}}, \frac{z_2}{\sqrt{z_1^2 + z_2^2 + z_3^2}}, \frac{z_3}{\sqrt{z_1^2 + z_2^2 + z_3^2}}, z_1^2 + z_2^2 + z_3^2 - 1\Big) \in Q \times \mathbb{D},$$

is a homeomorphism ($\sqrt{\ }$ denotes the principal branch of the square root).

For $w = x + iy \in \mathbb{C}^3$ with $x, y \in \mathbb{R}^3$ we have: $w \in Q$ iff $\|x\|^2 - \|y\|^2 = 1$, $\langle x, y\rangle = 0$. Consider the mapping

$$[0,1] \times Q \ni (t, x+iy) \xrightarrow{H} \Big(\sqrt{1 + t^2\|y\|^2}\frac{x}{\|x\|}, ty\Big) \in Q$$

Then H is a homotopy of Q to the 2-dimensional real Euclidean sphere in $\mathbb{R}^3$. In particular, Q is simply connected and, therefore, D is a simply connected domain.

([17]) In fact, the following general result is true (cf. [13]).

Let $\varOmega \subset \mathbb{C}^n$ be a domain of holomorphy and let M be an analytic subset of $\varOmega$; cf. Exercise 2.7. Then $\mathcal{O}(M) = \mathcal{O}(\varOmega)|_M$, where $\mathcal{O}(M)$ denotes the space of all functions $f : M \longrightarrow \mathbb{C}$ such that for any point $a \in M$ there exist $P = \mathbb{P}_n(a,r)$ and $\widetilde{f}_a \in \mathcal{O}(P)$ with $\widetilde{f}_a = f$ in $P \cap M$.

List of symbols

General symbols

$\mathbb{N} :=$ the set of natural numbers, $0 \notin \mathbb{N}$;
$\mathbb{Z} :=$ the ring of integers;
$\mathbb{Q} :=$ the field of rational numbers;
$\mathbb{R} :=$ the field of real numbers;
$\mathbb{C} :=$ the field of complex numbers;
$\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\} =$ the Riemann sphere;
$\operatorname{Re} z :=$ the real part of $z \in \mathbb{C}$;
$\operatorname{Im} z :=$ the imaginary part of $z \in \mathbb{C}$;
$\overline{z} := \operatorname{Re} z - i \operatorname{Im} z :=$ the conjugate of z;
$\overline{w} := (\overline{w}_1, \dots, \overline{w}_n)$;
$A_* := A \setminus \{0\}$, e.g. $\mathbb{C}_*$;
$A_+ := \{a \in A : a \geq 0\}$, e.g. $\mathbb{Z}_+, \mathbb{R}_+$;
$A_- := \{a \in A : a \leq 0\}$, e.g. $\mathbb{R}_-$;
$A_{>0} := \{a \in A : a > 0\}$, e.g. $\mathbb{R}_{>0}$;
$A^n_* := (A_*)^n$, $A^n_+ := (A_+)^n$, $A^n_{>0} := (A_{>0})^n$ (to simplify notation);
$A + B := \{a + b : a \in A, b \in B\}$, $A, B \subset \mathbb{C}^n$;
$A \cdot B := \{a \cdot b : a \in A, b \in B\}$, $A \subset \mathbb{C}, B \subset \mathbb{C}^n$;
$(e_1, \dots, e_n) :=$ the canonical basis in $\mathbb{C}^n$, $e_j := (e_{j,1}, \dots, e_{j,n})$, $e_{j,k} = 0$ for $j \neq k$ and $e_{j,j} := 1$, $j = 1, \dots, n$;

$\langle z, w \rangle := \sum_{j=1}^n z_j \overline{w}_j =$ the Hermitian scalar product in $\mathbb{C}^n$;
$\|z\| =: \sqrt{\langle z, z \rangle} = \sqrt{|z_1|^2 + \dots + |z_n|^2} =$ the Euclidean norm in $\mathbb{C}^n$;
a $\mathbb{C}$-linear operator $L : \mathbb{C}^n \longrightarrow \mathbb{C}^m$ is unitary if $\langle L(z'), L(z'') \rangle = \langle z', z'' \rangle$, $z', z'' \in \mathbb{C}^n$ (or, equivalently, $\|L(z)\| = \|z\|$, $z \in \mathbb{C}^n$);
$\mathfrak{U}(\mathbb{C}^n) :=$ the group of all unitary isomorphisms of $\mathbb{C}^n$;

$\mathbb{B}(a, r) = \mathbb{B}_n(a, r) := \{z \in \mathbb{C}^n : \|z - a\| < r\} =$ the Euclidean ball with center at $a \in \mathbb{C}^n$ and radius $r > 0$ ($\mathbb{B}(a, +\infty) := \mathbb{C}^n$);
$\mathbb{B}(r) = \mathbb{B}_n(r) := \mathbb{B}_n(0, r)$, $r > 0$;
$\mathbb{B}_n := \mathbb{B}_n(1) =$ the unit Euclidean ball in $\mathbb{C}^n$;
$K(a, r) := \mathbb{B}_1(a, r)$, $a \in \mathbb{C}$, $r > 0$; $K(a, +\infty) := \mathbb{C}$;
$C(a, r) := \partial K(a, r)$, $a \in \mathbb{C}$, $r > 0$; sometimes, we identify $C(a, r)$ with the curve $[0, 2\pi] \ni \theta \longmapsto a + re^{i\theta}$;
$K(r) := \mathbb{B}_1(r) = K(0, r)$, $r > 0$;

$C(r) := \partial K(r)$, $r > 0$;

$\mathbb{D} := \mathbb{B}_1 = \{z \in \mathbb{C} : |z| < 1\}$ = the unit disc;

$\mathbb{T} := \partial\mathbb{D}$;

$\boldsymbol{V}_j := \mathbb{C}^{j-1} \times \{0\} \times \mathbb{C}^{n-j} \subset \mathbb{C}^n$, $j = 1, \dots, n$;

$|z| := \max\{|z_1|, \dots, |z_n|\}$ = the maximum norm in $\mathbb{C}^n$;

$\mathbb{P}(a, r) = \mathbb{P}_n(a, r) := \{z \in \mathbb{C}^n : |z - a| < r\}$ = the polydisc with center at $a \in \mathbb{C}^n$ and radius $r > 0$;

$\mathbb{P}(r) = \mathbb{P}_n(r) := \mathbb{P}_n(0, r)$;

$\mathbb{P}(a, \boldsymbol{r}) = \mathbb{P}_n(a, \boldsymbol{r}) := K(a_1, r_1) \times \cdots \times K(a_n, r_n)$ = the polydisc with center at $a \in \mathbb{C}^n$ and multiradius (polyradius) $\boldsymbol{r} = (r_1, \dots, r_n) \in \mathbb{R}^n_{>0}$;

$\mathbb{P}(\boldsymbol{r}) = \mathbb{P}_n(\boldsymbol{r}) := \mathbb{P}(0, \boldsymbol{r})$;

$\operatorname{conv}(A)$:= the convex hull of A;

$A \subset\subset X \overset{\text{def}}{\Longleftrightarrow} A$ is relatively compact in X;

$\mathit{\Omega}_{a,X} := \{\lambda \in \mathbb{C} : a + \lambda X \in \mathit{\Omega}\}$, $\mathit{\Omega} \subset \mathbb{C}^n$, $a \in \mathbb{C}^n$, $X \in \mathbb{C}^n$;

$f_{a,X}(\lambda) := f(a + \lambda X)$, $\lambda \in \mathit{\Omega}_{a,X}$, $f : \mathit{\Omega} \longrightarrow \mathbb{C}^m$;

$d_{\mathit{\Omega}}(a) := \sup\{r > 0 : \mathbb{P}(a, r) \subset \mathit{\Omega}\}$, $\mathit{\Omega} \in \operatorname{top} \mathbb{C}^n$, $a \in \mathit{\Omega}$;

$\operatorname{top} \mathit{\Omega}$:= the Euclidean topology of $\mathit{\Omega}$, $\mathit{\Omega} \subset \mathbb{C}^n$;

$\operatorname{int}_X A$:= the interior of A in the topology of X, $A \subset X$;

$\operatorname{cl}_X A$:= the closure of A in the topology of X, $A \subset X$;

$\partial_0(A_1 \times \cdots \times A_n) := (\partial A_1) \times \cdots \times (\partial A_n)$:= the distinguished boundary of A;

$D \subset \mathbb{C}$ is regular $\overset{\text{def}}{\Longleftrightarrow}$ ∂D consists of a finite union of pairwise disjoint Jordan piecewise $\mathcal{C}^1$ curves with positive orientation with respect to D;

$R(A) := \{(|z_1|, \dots, |z_n|) : (z_1, \dots, z_n) \in A\}$:= the modular image of $A \subset \mathbb{C}^n$;

$\log A := \{(x_1, \dots, x_n) \in \mathbb{R}^n : (e^{x_1}, \dots, e^{x_n}) \in A\}$ = the logarithmic image of an n-circled set $A \subset \mathbb{C}^n$;

$p(\nu)$ holds for $\nu \gg 1 \overset{\text{def}}{\Longleftrightarrow}$ there exists ν_0 such that $p(\nu)$ holds for $\nu \geq \nu_0$;

$p(\varepsilon)$ holds for $0 < \varepsilon \ll 1 \overset{\text{def}}{\Longleftrightarrow}$ there exists $\varepsilon_0 > 0$ such that $p(\varepsilon)$ holds for $0 < \varepsilon \leq \varepsilon_0$;

$z^\alpha := z_1^{\alpha_1} \cdot \dots \cdot z_n^{\alpha_n}, \quad z = (z_1, \dots, z_n) \in \mathbb{C}^n,\ \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$;

$\alpha! := \alpha_1! \cdot \dots \cdot \alpha_n!,\ \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_+$;

$|\alpha| := |\alpha_1| + \dots + |\alpha_n|,\ \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_+$;

$\alpha \leq \beta \overset{\text{def}}{\Longleftrightarrow} \alpha_j \leq \beta_j,\ j = 1, \dots, n\ \ (\alpha = (\alpha_1, \dots, \alpha_n),\ \beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}^n)$;

$\binom{\alpha}{\beta} := \binom{\alpha_1}{\beta_1} \cdot \dots \cdot \binom{\alpha_n}{\beta_n},\ \alpha = (\alpha_1, \dots, \alpha_n),\ \beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}^n_+,\ \beta \leq \alpha$;

$\mathbf{1} := (1, \dots, 1) \in \mathbb{N}^n$;

$\|f\|_A := \sup\{|f(a)| : a \in A\},\ f : A \longrightarrow \mathbb{C}$;

$\mathcal{F}|_A := \{f|_A : f \in \mathcal{F}\}$, $\mathcal{F}$ is a family of mappings $X \longrightarrow Y$ and $A \subset X$;

$\operatorname{supp} f := \overline{\{x : f(x) \neq 0\}}$ = the support of f;

$\mathcal{C}^{\uparrow}(\mathit{\Omega})$:= the set of all upper semicontinuous functions $u : \mathit{\Omega} \longrightarrow [-\infty, +\infty]$ $(\mathit{\Omega} \in \operatorname{top} \mathbb{C}^n)$;

$\mathcal{L}^N$:= Lebesgue measure in $\mathbb{R}^N$;

$L^p(A) :=$ the space of all Lebesgue measurable functions $u : A \longrightarrow \mathbb{C}$ such that $\int_A |u|^p d\mathcal{L}^N < +\infty$, $A \subset \mathbb{R}^N$;

$L^p(\Omega, \mathrm{loc}) :=$ the space of all Lebesgue measurable functions $u : \Omega \longrightarrow \mathbb{C}$ such that $f|_K \in L^p(K)$ for any compact $K \subset \Omega \in \operatorname{top} \mathbb{R}^N$;

$L^p(\Omega, \mathrm{loc}) \ni f_\nu \overset{L^p(\Omega,\mathrm{loc})}{\longrightarrow} f_0 \in L^p(\Omega, \mathrm{loc}) \overset{\mathrm{def}}{\Longleftrightarrow} \forall_{K \subset\subset \Omega} : f_\nu|_K \overset{L^p(K)}{\longrightarrow} f_0|_K$.

Chapter 1

$f'_{\mathbb{R}}(a) :=$ the real Fréchet differential of f at a;

$f'_{\mathbb{C}}(a) = f'(a) :=$ the complex Fréchet differential of f at a;

$\mathcal{C}^k(\Omega_1, \Omega_2) :=$ the space of all C^k-mappings $F : \Omega_1 \longrightarrow \Omega_2$; $k \in \mathbb{Z}_+ \cup \{\infty\}$;

$\mathcal{C}^k(\Omega) := \mathcal{C}^k(\Omega, \mathbb{C})$;

$\mathcal{C}^k_0(\Omega) := \{f \in \mathcal{C}^k(\Omega) : \operatorname{supp} f \subset\subset \Omega\}$;

$\frac{\partial f}{\partial z_j}(a) := \frac{1}{2}\left(\frac{\partial f}{\partial x_j}(a) - i\frac{\partial f}{\partial y_j}(a)\right), \quad \frac{\partial f}{\partial \bar z_j}(a) := \frac{1}{2}\left(\frac{\partial f}{\partial x_j}(a) + i\frac{\partial f}{\partial y_j}(a)\right)$;

$$J_{\mathbb{R}} f(a) := \det \begin{bmatrix} \frac{\partial f_1}{\partial z_1}(a), \dots, \frac{\partial f_1}{\partial z_n}(a), \frac{\partial f_1}{\partial \bar z_1}(a), \dots, \frac{\partial f_1}{\partial \bar z_n}(a) \\ \dots \\ \frac{\partial f_n}{\partial z_1}(a), \dots, \frac{\partial f_n}{\partial z_n}(a), \frac{\partial f_n}{\partial \bar z_1}(a), \dots, \frac{\partial f_n}{\partial \bar z_n}(a) \\ \frac{\partial \bar f_1}{\partial z_1}(a), \dots, \frac{\partial \bar f_1}{\partial z_n}(a), \frac{\partial \bar f_1}{\partial \bar z_1}(a), \dots, \frac{\partial \bar f_1}{\partial \bar z_n}(a) \\ \dots \\ \frac{\partial \bar f_n}{\partial z_1}(a), \dots, \frac{\partial \bar f_n}{\partial z_n}(a), \frac{\partial \bar f_n}{\partial \bar z_1}(a), \dots, \frac{\partial \bar f_n}{\partial \bar z_n}(a) \end{bmatrix} = \text{the real Jacobian of } f \text{ at } a;$$

$D^{\alpha,\beta} := (\frac{\partial}{\partial z_1})^{\alpha_1} \circ \dots \circ (\frac{\partial}{\partial z_n})^{\alpha_n} \circ (\frac{\partial}{\partial \bar z_1})^{\beta_1} \circ \dots \circ (\frac{\partial}{\partial \bar z_n})^{\beta_n}$, $\alpha, \beta \in \mathbb{Z}^n_+$;

$D^\alpha := D^{\alpha,0} = (\frac{\partial}{\partial z_1})^{\alpha_1} \circ \dots \circ (\frac{\partial}{\partial z_n})^{\alpha_n}$, $\alpha \in \mathbb{Z}^n_+$;

$\frac{\partial f}{\partial z_j}(a) := \lim_{\mathbb{C} \ni \lambda \to 0} \frac{1}{\lambda}(f(a + \lambda e_j) - f(a)) =$ the j-th complex partial derivative of f at a;

$J_{\mathbb{C}} f(a) := \det\left(\left[\frac{\partial f_j}{\partial z_k}(a)\right]_{j,k=1,\dots,n}\right) =$ the complex Jacobian of f at a;

$\mathcal{O}_s(\Omega) :=$ the space of all separately holomorphic functions on Ω;

$\mathfrak{D}(\Sigma) :=$ the domain of convergence of a power series Σ;

$T_a f(z) := \sum_{\alpha \in \mathbb{Z}^n_+} \frac{1}{\alpha!} D^\alpha f(a)(z - a)^\alpha =$ the Taylor series of f at a;

$d(T_a f) := \sup\{r > 0 : \mathbb{P}(a, r) \subset \mathfrak{D}(T_a f)\} =$ the radius of convergence of the series $T_a f$;

$\mathcal{O}(\Omega_1, \Omega_2)$ – the set of all holomorphic mappings $F : \Omega_1 \longrightarrow \Omega_2$;

$\mathcal{O}(\Omega) := \mathcal{O}(\Omega, \mathbb{C})$;

$\widehat{f}_a :=$ the germ of f at a;

$\mathcal{P}(\mathbb{C}^n) :=$ the space of all complex polynomials of n-complex variables;

$\mathcal{H}^\infty(\Omega) :=$ the space of all bounded holomorphic functions on Ω;

$\mathcal{A}^k(\Omega) := \{f \in \mathcal{O}(\Omega) : \forall_{\alpha \in \mathbb{Z}^n_+,\, |\alpha| \le k} : \exists_{\varphi_\alpha \in \mathcal{C}(\overline{\Omega})} : \varphi_\alpha = D^\alpha f \text{ in } \Omega\}, k \in \mathbb{Z}_+ \cup \{\infty\}$;

$L^p_h(\Omega) := \mathcal{O}(\Omega) \cap L^p(\Omega)$.

Chapter 2

$\operatorname{Aut}(\Omega) =$ the group of all automorphisms of $\Omega \subset \mathbb{C}^n$;

$\operatorname{Aut}_a(\Omega) := \{h \in \operatorname{Aut}(\Omega) : h(a) = a\}$, $a \in \Omega$.

$\widehat{K}_{\mathcal{F}} := \{z \in \Omega : \forall\, f \in \mathcal{F} : |f(z)| \le \|f\|_K\}$, $\mathcal{F} \subset \mathcal{O}(\Omega)$;
$\widehat{K} := \widehat{K}_{\mathcal{P}(\mathbb{C}^n)} = \widehat{K}_{\mathcal{O}(\mathbb{C}^n)}$.

Chapter 3

$\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}}$ = the Laplacian operator in $\mathbb{R}^2$;
$\mathcal{H}(\Omega)$:= the space of all harmonic functions on $\Omega \subset \mathbb{C}$;
$\mathbf{P}(u; a, r; z) := \frac{1}{2\pi}\int_0^{2\pi} \frac{r^2 - |z-a|^2}{|re^{i\theta} - (z-a)|^2} u(a + re^{i\theta})\, d\theta$;
$\mathbf{J}(u; a, r) := \mathbf{P}(u; a, r; a) = \frac{1}{2\pi}\int_0^{2\pi} u(a + re^{i\theta})\, d\theta$;
$\mathbf{S}(u; a, r; z) := \frac{1}{2\pi}\int_0^{2\pi} \frac{re^{i\theta} + (z-a)}{re^{i\theta} - (z-a)} u(a + re^{i\theta}) d\theta$;
$\mathcal{SH}(\Omega)$:= the set of all subharmonic functions on $\Omega \subset \mathbb{C}$;
$\mathbf{A}(u; a, r) := \frac{1}{\pi r^2}\int_{K(a,r)} u\, d\mathcal{L}^2$;
$\mathcal{PH}(\Omega)$:= the space of all pluriharmonic functions on $\Omega \subset \mathbb{C}^n$;
$\mathbf{P}(u; a, \boldsymbol{r}; z) := \frac{1}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi} \frac{r_1^2 - |z_1 - a_1|^2}{|r_1 e^{i\theta_1} - (z_1 - a_1)|^2} \cdots \frac{r_n^2 - |z_n - a_n|^2}{|r_n e^{i\theta_n} - (z_n - a_n)|^2} \times$
$u(a_1 + r_1 e^{i\theta_1}, \dots, a_n + r_n e^{i\theta_n})\, d\theta_1 \dots d\theta_n$;
psh: =plurisubharmonic;
$\mathcal{PSH}(\Omega)$:= the set of all plurisubharmonic functions on $\Omega \subset \mathbb{C}^n$;
$\mathbf{J}(u; a, \boldsymbol{r}) := \mathbf{P}(u; a, \boldsymbol{r}; a)$;
$\mathbf{A}(u; a, \boldsymbol{r}) := \frac{1}{(\pi r_1^2)\dots(\pi r_n^2)}\int_{\mathbb{P}(a,\boldsymbol{r})} u\, d\mathcal{L}^{2n}$;
$\mathcal{L}u(a; X) := \sum_{j,k=1}^n \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(a) X_j \overline{X}_k$ = the Levi form;
$\Omega_\varepsilon := \{z \in \Omega : d_\Omega(z) > \varepsilon\}$;
u_ε := the ε-regularization of $u \in \mathcal{PSH}(\Omega)$.

Chapter 4

$\widetilde{K}_{\mathcal{S}} := \{z \in \Omega : \forall_{u \in \mathcal{S}} : u(z) \le \max_K u\}$, $\mathcal{S} \subset \mathcal{PSH}(\Omega)$;
$B_q(a, r) := \{z \in \mathbb{C}^n : q(z - a) < r\}$, $a \in \mathbb{C}^n$, $r > 0$, $q : \mathbb{C}^n \longrightarrow \mathbb{R}_+$ is a $\mathbb{C}$-norm;
$d_{\Omega,q}(a) := \sup\{r > 0 : B_q(a, r) \subset \Omega\}$, $a \in \Omega \subset \mathbb{C}^n$;
$\delta_{\Omega,X}(a) := \sup\{r > 0 : a + K(r) \cdot X \subset \Omega\}$, $a \in \Omega \subset \mathbb{C}^n \ni X$;
$\Xi_p^n := \{I = (i_1, \dots, i_p) \in \mathbb{N}^p : 1 \le i_1 < \dots < i_p \le n\}$;
${\sum}'_{|I|=p} \dots = \sum_{I \in \Xi_p^n} \dots$;
$\mathcal{F}_{(p,q)}(\Omega)$:= the space of all forms of type (p, q);
$\mathcal{C}^k_{(p,q)}(\Omega)$:= the space of all (p, q)-forms with coefficients in $\mathcal{C}^k(\Omega)$;
$\mathcal{D}_{(p,q)}(\Omega)$:= the space of all (p, q)-forms with coefficients in $\mathcal{C}_0^\infty(\Omega)$;

$$\partial u := {\sum_{|I|=p, |J|=q}}' \sum_{j=1}^n \frac{\partial u_{I,J}}{\partial z_j} dz_j \wedge dz_I \wedge d\bar{z}_J; \quad \overline{\partial} u := {\sum_{|I|=p, |J|=q}}' \sum_{j=1}^n \frac{\partial u_{I,J}}{\partial \bar{z}_j} d\bar{z}_j \wedge dz_I \wedge d\bar{z}_J = \text{the } \overline{\partial}\text{-operator};$$

$$\Phi^*(u) := \sum_{|I|=p,|J|=q}' (u_{I,J} \circ \Phi)\, d\Phi_{i_1} \wedge \dots \wedge d\Phi_{i_p} \wedge d\overline{\Phi}_{j_1} \wedge \dots \wedge d\overline{\Phi}_{j_q}$$
$$= \sum_{|I|=p,|J|=q}' (u_{I,J} \circ \Phi)\, \partial\Phi_{i_1} \wedge \dots \wedge \partial\Phi_{i_p} \wedge \overline{\partial\Phi}_{j_1} \wedge \dots \wedge \overline{\partial\Phi}_{j_q}.$$

$S_{p,q} = S_{p,q}(\mathbb{C}^n) :=$ the family of all sets $A \subset \mathbb{C}^n$ such that for every open neighborhood G of A and for every $\overline{\partial}$-closed form $v \in \mathcal{C}^\infty_{(p,q+1)}(G)$ there exist an open neighborhood $\widetilde{G}$ of A (with $\widetilde{G} \subset G$) and a form $u \in \mathcal{C}^\infty_{(p,q)}(\widetilde{G})$ such that $\overline{\partial} u = v$ in $\widetilde{G}$.

Chapter 5

$q_{K,k}(f) := \sum_{\alpha\in\mathbb{Z}_+^N:\, |\alpha|\le k} \sup_K |D^\alpha f|$, $k \in \mathbb{Z}_+$;
$\mathcal{D}(K) := \{f \in \mathcal{C}_0^\infty(\Omega) : \operatorname{supp} f \subset K\}$;
$\mathcal{D}(K) \ni f_\nu \overset{\mathcal{D}(K)}{\longrightarrow} f_0 \in \mathcal{D}(K) \overset{\text{def}}{\iff} \forall_{\alpha\in\mathbb{Z}_+^N} : D^\alpha f_\nu \longrightarrow D^\alpha f_0$ uniformly on K;
$\mathcal{D}(\Omega) := \mathcal{C}_0^\infty(\Omega)$;
$\mathcal{D}(\Omega) \ni f_\nu \overset{\mathcal{D}(\Omega)}{\longrightarrow} f_0 \in \mathcal{D}(\Omega) \overset{\text{def}}{\iff} \exists_{K\subset\subset\Omega} : (f_\nu)_{\nu=1}^\infty \subset \mathcal{D}(K),\ f_\nu \overset{\mathcal{D}(K)}{\longrightarrow} f_0$;
$\mathcal{E}(\Omega) := \mathcal{C}^\infty(\Omega)$;
$\mathcal{E}(\Omega) \ni f_\nu \overset{\mathcal{E}(\Omega)}{\longrightarrow} f_0 \in \mathcal{E}(\Omega) \overset{\text{def}}{\iff} \forall_{\alpha\in\mathbb{Z}_+^N} : D^\alpha f_\nu \longrightarrow D^\alpha f_0$ locally uniformly in Ω;
$\mathcal{D}'(\Omega) :=$ the space of all distributions on Ω;
$\mathfrak{T}_\nu \overset{\mathcal{D}'(\Omega)}{\longrightarrow} \mathfrak{T}_0 \iff \forall_{f\in\mathcal{D}(\Omega)} : \mathfrak{T}_\nu(f) \longrightarrow \mathfrak{T}_0(f)$;
$[u](f) := \int_\Omega uf\, d\mathcal{L}^n$, $f \in \mathcal{D}(\Omega)$, $u \in L^1(\Omega, \mathrm{loc})$;
$\mathfrak{T}_1 \otimes \mathfrak{T}_2 :=$ the tensor product of distributions;
$\mathfrak{T}_1 * \mathfrak{T}_2 :=$ the convolution of distributions;
$\mathfrak{T}_\varepsilon = \mathfrak{T} * \Phi_\varepsilon =$ the ε-regularization of $\mathfrak{T}$;
$L^2(\Omega, \varphi) := \{u \in L^2(\Omega, \mathrm{loc}) : \int_\Omega |u|^2 \exp(-\varphi)\, d\mathcal{L}^{2n} < +\infty\}$, $\Omega \in \operatorname{top} \mathbb{C}^n$;
$\mathcal{V}_{(p,q)}(\Omega)$ – the space of all (p,q)-forms with coefficients in $\mathcal{V}(\Omega)$, e.g. $\mathcal{D}'_{(p,q)}(\Omega)$, $L^2_{(p,q)}(\Omega, \mathrm{loc})$, $L^2_{(p,q)}(\Omega, \varphi)$;

$$\partial\mathfrak{T} := \sum_{|I|=p,|J|=q}' \sum_{j=1}^n \frac{\partial \mathfrak{T}_{I,J}}{\partial z_j} dz_j \wedge dz_I \wedge d\overline{z}_J; \quad \overline{\partial}\mathfrak{T} := \sum_{|I|=p,|J|=q}' \sum_{j=1}^n \frac{\partial \mathfrak{T}_{I,J}}{\partial \overline{z}_j} d\overline{z}_j \wedge dz_I \wedge d\overline{z}_J;$$

$$D^{\alpha,\beta}\mathfrak{T} := \sum_{|I|=p,|J|=q}' (D^{\alpha,\beta}\mathfrak{T}_{I,J}) dz_I \wedge d\overline{z}_J, \quad \alpha, \beta \in \mathbb{Z}_+^n;$$

$$\vartheta\mathfrak{T} := \sum_{|I|=p,|K|=q-1}' \Big(\sum_{j=1}^n \frac{\partial \mathfrak{T}_{I,jK}}{\partial z_j}\Big) dz_I \wedge d\overline{z}_K; \quad \mathfrak{T}_\varepsilon := \sum_{|I|=p,|J|=q}' (\mathfrak{T}_{I,J})_\varepsilon dz_I \wedge d\overline{z}_J;$$

$\varphi_j := \varphi - (3-j)\psi$, $\mathcal{H}_j := L^2_{(p,q+j-1)}(\Omega, \varphi_j)$, $j = 1, 2, 3$;
$\mathrm{Dom}(T) := \{f \in L^2_{(p,q)}(\Omega, \varphi_1) : \overline{\partial} f \in L^2_{(p,q+1)}(\Omega, \varphi_2)\}$;
$T : L^2_{(p,q)}(\Omega, \varphi_1) \supset \mathrm{Dom}(T) \ni f \longrightarrow \overline{\partial} f \in L^2_{(p,q+1)}(\Omega, \varphi_2)$;
$\mathrm{Dom}(S) := \{f \in L^2_{(p,q+1)}(\Omega, \varphi_2) : \overline{\partial} f \in L^2_{(p,q+2)}(\Omega, \varphi_3)\}$;
$S : L^2_{(p,q+1)}(\Omega, \varphi_2) \supset \mathrm{Dom}(S) \ni f \longrightarrow \overline{\partial} f \in L^2_{(p,q+2)}(\Omega, \varphi_3)$;
$\mathcal{R}(L) := L(\mathrm{Dom}(L))$, $L : X \supset \mathrm{Dom}(L) \longrightarrow Y$ is linear;

$\mathcal{W}^k(\Omega) := \{u \in L^2(\Omega) : D^{\alpha,\beta}u \in L^2(\Omega),\ |\alpha| + |\beta| \le k\}$, $k \in \mathbb{Z}_+ \cup \{\infty\}$;
$\mathcal{W}^k(\Omega, \mathrm{loc}) := \{u \in L^2(\Omega, \mathrm{loc}) : D^{\alpha,\beta}u \in L^2(\Omega, \mathrm{loc}),\ |\alpha| + |\beta| \le k\}$, $k \in \mathbb{Z}_+ \cup \{\infty\}$.

Chapter 6

$\mathcal{M}(\Omega) :=$ the set of all meromorphic functions on $\Omega \subset \mathbb{C}^n$;
$\mathfrak{R}(f) :=$ the set of all regular points of $f \in \mathcal{M}(\Omega)$;
$\mathfrak{P}(f) :=$ the set of all poles of $f \in \mathcal{M}(\Omega)$;
$\mathfrak{I}(f) :=$ the set of all points of indeterminacy of $f \in \mathcal{M}(\Omega)$;
$\mathcal{F}^*(\Omega) := \{f \in \mathcal{F}(\Omega) : f(z) \neq 0,\ z \in \Omega\}$, e.g. $\mathcal{O}^*(\Omega)$;
$\Omega \in \mathfrak{CP}^1(\mathcal{F}) \overset{\text{def}}{\iff}$ the first Cousin problem with data in $\mathcal{F}$ has a solution for any open covering of Ω;
$\Omega \in \mathfrak{CP}^1(\mathcal{O}) \overset{\text{def}}{\iff}$ the first holomorphic Cousin problem has a solution for any open covering of Ω;
$\Omega \in \mathfrak{CP}^1[\mathcal{M}] \overset{\text{def}}{\iff}$ the first meromorphic Cousin problem has a solution for any open covering of Ω;
$\Omega \in \mathfrak{CP}^2(\mathcal{F}) \overset{\text{def}}{\iff}$ the second Cousin problem with data in $\mathcal{F}$ has a solution for any open covering of Ω;
$\Omega \in \mathfrak{CP}^2(\mathcal{O}) \overset{\text{def}}{\iff}$ the second holomorphic Cousin problem has a solution for any open covering of Ω;
$\Omega \in \mathfrak{CP}^2[\mathcal{M}] \overset{\text{def}}{\iff}$ the second meromorphic Cousin problem has a solution for any open covering of Ω.

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