Analytic Continuation of Harmonic Functions

1. INTRODUCTION

In this paper by $D$ we denote an arbitrary fixed, open, connected and not empty subset (a region) of $\mathbb{R}^n$, $n \geq 2$. It is known that every function $h$ harmonic on $D$ may be continued to a holomorphic function in an open set $\bar{D}_h \subset C^n$. It may be asked whether there exists an open connected set $\bar{D} \subset C^n$ such that $D \subset \bar{D}$ and every harmonic function on $D$ may be continued to a holomorphic (or only to an analytic multivalued) function on $\bar{D}$. It may also be asked whether there exists a maximal set $\bar{D}$ with these properties. This set will be called a harmonic envelope of holomorphy (or of analyticity) for $D$.

From the paper [2] we can deduce the following

Theorem I. For every region $D \subset \mathbb{R}^n$ there exists a harmonic envelope of analyticity.

In the paper [3] we can find

Theorem II. For every region $D \subset \mathbb{R}^n$ there exists an open connected set $\bar{D} \subset C^n$ such that every harmonic function on $D$ may be continued to a holomorphic function on $\bar{D}$.

Theorem III. If $B = \{x \in \mathbb{R}^n : |x| < r\}$, $n \geq 2$, $r > 0$, then

$$\bar{B} = \{z = x + iy \in C^n : [|x|^2 + |y|^2 + 2(|x|^2|y|^2 - \langle x, y \rangle)^{\frac{1}{2}} < r\}$$

is the harmonic envelope of holomorphy for $B$.

In his paper [2] Lelong presented two methods of construction of a harmonic envelope of analyticity. In Section 2 of this paper these two methods are analysed and used for the effective construction of a harmonic envelope of analyticity for the ball and the spatial ring. By different methods the harmonic envelope
of holomorphy of the ball was obtained in [1] and [3]. The paper is closed by Section 3 in which theorems 7 and 8 are proved. Theorem 7 permits the effective construction of the harmonic envelope of analyticity for \( D \subset C \), if a harmonic envelope of analyticity is known for some region that is biholomorphically equivalent to \( D \). Theorem 8 permits the construction of a harmonic envelope of holomorphy for the set \( D \subset C \) that is biholomorphically equivalent to the unit disc.

Now we present a list of the denotations used in this note.

As usual for \( A \subset C^n \) (or \( A \subset R^n \)) by \( A^0, \bar{A}, \partial A \) we denote, respectively, the interior, the closure and the boundary of \( A \). For \( U = U^0 \subset R^n \), by \( H(U) \) we denote the set of all harmonic functions on \( U \). For \( \Omega = \Omega^0 \subset C^n \), by \( \Theta(\Omega) \) we denote the set of all holomorphic functions on \( \Omega \). For arbitrary \( z, w \in C^n \), by \( \langle z, w \rangle \) we denote the standard scalar product in \( C^n \) (i.e. \( \langle z, w \rangle = \sum_{j=1}^{n} z_j \bar{w}_j \)) and by \( |z| \) the norm induced by the scalar product.

2. LELONG SETS

The whole of this section has been suggested by the ideas contained in Lelong's paper [2]. It has the character of a short report on results relative to the analytic continuation of harmonic functions.

The first part of this section is devoted to generalizations of Lelong's methods of construction of a harmonic envelope of analyticity. The case \( n = 2 \) plays a special role in this theory, therefore we shall devote most attention to this.

Now, we shall define two sets, which play a fundamental role in the following constructions.

Let \( F(z) = \sum_{j=1}^{n} z_j \), \( z = (z_1, ..., z_n) \in C^n \). For \( z_0 \in C^n \), \( t_0 \in R^n \) let \( T(z_0) = \{ t \in R^n : F(z_0 - t) = 0 \} \), \( \Gamma(t_0) = \{ z \in C^n : F(z - t_0) = 0 \} \).

From these definitions we can directly obtain the following

Lemma 1.
(a) For \( z = x + iy \in C^n : T(z) = \{ t \in R^n : |x - t| = |y|, \langle x = t, y \rangle = 0 \} \); (b) for \( z = x + iy \in C^n : T(z) = T(\bar{z}), T(x) = \{ x \} / \bar{z} = x - iy \); (c) in the case \( n \geq 2 \), for \( z = x + iy \in C^n \), \( y \neq 0 \) the set \( T(z) \) is an \((n-2)\) dimensional sphere with the center \( x \), the radius \(|y|\) and \( T(z) \) lies in the hyperplane \( \{ t \in R^n : \langle x - t, y \rangle = 0 \} \); (d) in the case \( n = 2 \), for \( z = (z_1, z_2) \in C^2 : T(z) = \{ z + i\bar{z}_2, \bar{z}_1 + i\bar{z}_2 \} \) (we identify \( C \) and \( R^2 \)); (e) in the case \( n \geq 3 \) the set \( T(z) \) is connected; (f) for \( t \in R^n \) \( \Gamma(t) \cap R^n = \{ t \}, (n = 1 : \Gamma(t) = \{ t \}) \);
(g) for $z \in C^n$, $t \in R^n$ $z \in \Gamma(t) \Leftrightarrow t \in T(z)$;

(h) for $z = x + iy \in C^n$, $a \in R$, $a \neq 0$ : $T(x + iy) = x + aT\left(\frac{iy}{a}\right)$;

(i) if $U$ is an $n$-dimensional real orthogonal matrix, then $T(Uz) = U(T(z))$

(we identify $U$ with the mapping of $C^n$ onto $C^n$).

This lemma (except (i)) will be used many times in the sequel.

Lemma 2. Let $A$ be a subset of $R^n$ such that $\partial A \neq \phi$.

Set $\Gamma(A) = \bigcup_{t \in \partial A} \Gamma(t)$. Then $\Gamma(A) = \overline{\Gamma(A)}$, $\Gamma(A) \cap R^n = \partial A$ $(n = 1: \Gamma(A) = \partial A)$.

Proof. The second part of this lemma follows from lemma 1 if. We shall prove only that $\Gamma(A) = \overline{\Gamma(A)}$.

Let $\{z_k\}_{k \in N} \subset \Gamma(A)$, $\lim z_k = z_0$, $z_k \in \Gamma(t_k)$, $t_k \in \partial A$, $k \geq 1$. We want to prove that $z_0 \in \Gamma(t_0)$, i.e. there exists $t_0 \in \partial A$ such that $z_0 \in \Gamma(t_0)$.

There exists a constant $M > 0$ such that $|z_k| \leq M$, $k \geq 1$. For any $k, l \in N$ we have

$$|t_k - t_l| \leq |t_k - z_k| + |z_k - z_l| + |z_l - t_l| \leq |t_k - z_k| + |t_l - z_l| + 2M,$$

$$|t_k - z_l|^2 = |t_k - (x_k + iy_k)|^2 = |t_k - z_k|^2 + |y_k|^2 = 2|y_k|^2 \leq 2M^2.$$  

Hence $|t_k - t_l| \leq 2(1 + \sqrt{2})M$, $k, l \in N$. Since $\{t_k\}_{k \in N}$ is bounded and $\partial A$ is a closed set then there exists a subsequence $\{t_{k_m}\}_{m \in N}$ and a point $t_0 \in \partial A$ such that $\lim t_{k_m} = t_0 = S_0$, $0 = \lim F(z_0 - t_{k_m}) = F(z_0 - t_0)$, i.e. $z_0 \in \Gamma(t_0)$. The proof is concluded.

Now we give the definition of a Lelong set of the first type.

Let $A$ be a subset of $R^n$ such that $\partial A \neq \phi, A^0 \neq \phi$; by $\overline{A}$ we denote the connected component of the set $C^n \setminus \Gamma(A)$ which contains $A^0$ (in the case $A = R^n$ we set $\overline{A} = C^n$).

Since $A^0 \subset R^n \setminus \partial A \subset C^n \setminus \Gamma(A)$ (see lemma 2) then $\overline{A}$ is well defined. Since $C^n \setminus \Gamma(A)$ is an open set (see lemma 2) then $\overline{A}$, as a connected component of an open set, is a region in $C^n$.

For a region $D \subset R^n$, the region $\overline{D}$ is the same as the region constructed by the method given by Lelong in [2].

The definition of the set $\overline{A}$ is clear with respect to topological properties but it is not useful in concrete constructions.

Below we give the definition of a Lelong set of the second type (denoted by $W(A)$); this definition is useful with respect to its constructive properties. It will be proved that for any region $D \subset R^n$, $n \geq 2 : \overline{D} = W(D)$. This will provide a method for effectively constructing the set $\overline{D}$.

First we give the following auxiliary definitions.

Given $B \subset R^n$, $n \geq 2$, $b \in R^n$, we say that the sphere $T(z)$ is spherically contractible in $B$ to the point $b$ if and only if there exists a continuous mapping $\gamma : I \to C^n$ such that $\gamma(0) = z$, $\gamma(1) = b$ and for every $\tau \in I : T(\gamma(\tau)) \subset B$ ($I = [0, 1] \subset R$).
For $A \subset \mathbb{R}^n$, $n \geq 2$ put 
$W(A) = \{ z \in \mathbb{C}^n : \exists a \in A \text{ such that } T(z) \text{ is spherically contractible in } A \text{ to every point } a \in A \}$. 
The following lemma gives a certain description of the set $W(A)$.

Lemma 3.

(a) If $A \subset \mathbb{R}^n$ is arcwise connected then $W(A) = \{ z \in \mathbb{C}^n : \exists a \in A \text{ such that } T(z) \text{ is spherically contractible in } A \text{ to } a \}, W(A)$ is arcwise connected, $W(A) \cap \mathbb{R}^n = \partial A$ and $\partial A \subset W(A) \setminus W(A)$;

(b) if $n = 2$ and if $A$ is arcwise connected then $W(A) = \{ z \in \mathbb{C}^2 : T(z) \subset A \}$;

(c) if $A$ is starlike with respect to $t_0 \in A$ then $W(A) = \{ z \in \mathbb{C}^n : T(z) \subset A \}$ and $W(A)$ is starlike with respect to $t_0$;

(d) if $\partial A \neq \emptyset$, $A^0 \neq \emptyset$ and $A^0$ is connected then $\overline{A} = W(A^0)$, $\overline{A} \cap \mathbb{R}^n = A^0$ and $\partial(A^0) \subset \partial \overline{A}$.

Proof.

(a) is implied directly by the definition of $W(A)$.

(b) Let $z = (z_1, z_2) \in \mathbb{C}^2$, $T(z) = \{ z_1 + i z_2, \bar{z}_1 + i \bar{z}_2 \} \subset A$, $a \in A$ be an arbitrary but fixed point. There exist two continuous mappings $\sigma_i : I \to A$, $i = 1, 2$ such that $\sigma_i(0) = z_i + i z_2$, $\sigma_2(0) = \bar{z}_1 + i \bar{z}_2$, $\sigma_1(1) = \sigma_2(1) = a$. We define $\gamma : I \to \mathbb{C}^2$ by the formula $\gamma = \left( \frac{\sigma_1 + \sigma_2}{2}, \frac{\sigma_1 - \sigma_2}{2i} \right)$. It is obvious that $\gamma$ is a continuous mapping, $\gamma(0) = z$, $\gamma(1) = a$ and for every $\tau \in I$ $T(\gamma(\tau)) = \{ \sigma_1(\tau), \sigma_2(\tau) \} \subset A$.

(c) Let $z \in \mathbb{C}^n$, $T(z) \subset A$. $A$ is starlike with respect to $t_0$, therefore $A$ is arcwise connected. It suffices to show that $T(z)$ is spherically contractible in $A$ to the point $t_0$. Let $\gamma(\tau) = t_0 + (1 - \tau)z$, $\tau \in I$. It is obvious that $\gamma$ is a continuous mapping of $I$ into $\mathbb{C}^n$, $\gamma(0) = z$, $\gamma(1) = t_0$ and $T(\gamma(\tau)) = t_0 + (1 - \tau)T(z)$, $\tau \in I$ (see lemma 1h). Therefore $T(\gamma(\tau)) \subset A$, $\tau \in I$.

(d) We know that $A^0 \subset W(A^0)$ and $W(A^0)$ is connected. First we shall show that $W(A^0) \cap \Gamma(A) = \emptyset$. Suppose there exists $z \in W(A^0) \cap \Gamma(A)$, so there exists $t \in \partial A$ such that $t \in T(z)$. Therefore $T(z) \cap \partial A \neq \emptyset$ and we have a contradiction to the inclusion $T(z) \subset A^0$. Hence $W(A^0) \subset \overline{A}$.

We shall prove the opposite inclusion. Let $z \in \overline{A}$, $a \in A^0$ be two fixed points and let $\sigma : I \to \overline{A}$ be a continuous mapping such that $\sigma(0) = z$, $\sigma(1) = a$. We want to demonstrate that for every $\tau \in I : T(\sigma(\tau)) \subset A^0$ (in particular from this it follows that $z \in W(A^0)$). Let $K = \bigcup \{ T(\sigma(\tau)) \mid \tau \in I \}$. We shall show that $K$ is connected compact set.

$K$ is a bounded set. Let $t_i \in T(\sigma(\tau)), i = 1, 2$; $|t_1 - t_2| \leq |t_1 - \sigma(\tau_i) + |\sigma(\tau_i) - t_2| \leq 2(1 + \sqrt{2})\max \{|\sigma(\tau)| : \tau \in I\} < +\infty$ (see the proof of lemma 2).

$K$ is a closed set. Let $t_k \in T(\sigma(\tau_k)), k \geq 1, t_0 = \lim t_k$. There exists a subsequence $\{ \tau_{k_m} \}_{m \in \mathbb{N}}$ and a point $\tau_0 \in I$ such that $\tau_0 = \lim \tau_{k_m}$. Since $F$ and $\sigma$ are continuous mappings, we have $0 = \lim F(t_{k_m} - \sigma(\tau_{k_m})) = F(t_0 - \sigma(\tau_0))$. So $t_0 \in T(\sigma(\tau_0)) \subset K$. 

$K$ is connected. We distinguish two cases.

At first — the case $n = 2$. Let $\sigma = (\sigma_1, \sigma_2), \sigma_i : I \to C, i = 1, 2, K_i \overset{\text{df}}{=} (\sigma_1 + i\sigma_2)(I), K_2 \overset{\text{df}}{=} (\sigma_1 + i\sigma_2)(I)$. $K_1, K_2$ are connected, $a \in K_1 \cap K_2$, $K = K_1 \cup K_2$. Therefore $K$ is connected.

Now we consider the case $n \geq 3$. Suppose that $K = K_1 \cup K_2, K_1 \cap K_2 = \varnothing, K_i \neq \varnothing, K_i = K_i, i = 1, 2$. Let $I_i = \{t \in I : T(\sigma(t)) \cap K_i \neq \varnothing\}, i = 1, 2$. It may be checked that $I_i \neq \varnothing, i = 1, 2, I = I_1 \cup I_2$. Since for $\tau_0 \in I \exists I$ the set $T(\sigma(\tau_0))$ is connected, then $T(\sigma(\tau_0)) \subset K_1$ or $T(\sigma(\tau_0)) \subset K_2$. So $I_1 \cap I_2 = \varnothing$. It is easy to prove that $I_i = I_i, i = 1, 2$ and we get a contradiction to connectedness of $I$.

We want to show that $K \subset A_0$. It is obvious that $K \cap \partial A = \varnothing, K \cap A \neq \varnothing$. Suppose that $K \not\subset A_0, \text{i.e. } K = (K \cap A_0) \cup (K \cap (R^n \setminus A))$ where $(K \cap (R^n \setminus A)) \neq \varnothing$. This is contradiction to the connectedness of $K$. This completes the proof.

Lemma 3 implies the following

**Corollary**

I. For a region $D \subset R^n, n \geq 2$:

(a) $W(D) = \{z \in C^n : \exists a \in D : T(z)$ is spherically contractible in $D$ to $a\}$

(b) In the case $n = 2, W(D) = \{z \in C^n : T(z) \subset D\}$

(c) $\tilde{D} = W(D), \tilde{D} \subset R^n = D, \partial D \subset \partial \tilde{D} \subset T(D)$.

II. For $A, B \subset R^n$:

(a) If $A \cap B = \varnothing$ then $W(A) \cap W(B) = \varnothing$;

(b) If $A \subset B$ and $B$ is arcwise connected then $W(A) \subset W(B)$.

The following Lemma 4 gives a characterization of metrical properties in the family $\{T(z)\}_{z \in C^n}$. First we give a known definition of the Hausdorff distance between the sets.

For $A, B \subset C^n, A, B \neq \varnothing$ we define the Hausdorff distance between $A$ and $B$ by the formula

$$\varrho_H(A, B) = \max \{\sup_{x \in A} \varrho(x, B), \sup_{y \in B} \varrho(y, A)\},$$

where $\varrho(z, C)$ denotes the distance between the point $z$ and the set $C$.

**Lemma 4**

(a) Let $z = x + iy, z' = x' + iy' \in C^n, n \geq 2$, then $\varrho_H(T(z), T(z')) = |x - x'| + |y - y'|$.

(b) Let $z \in C^n, t \in R^n$, there exists $z' \in C^n$ such that $t \in T(z'), |z - z'| = \varrho(t, T(z'))$.

**Proof**

(a) Let $t' = x' + t', \xi' \in T(\xi'y')$ be an arbitrary point of $T(z')$. For the distance between $t'$ and $T(z)$ we have

$$\varrho(t', T(z)) = \min \{|t - t'| : t \in T(z)\} = |x - x'| + \min \{|\xi - \xi'| : \xi \in T(\xi'y')\}$$

$$= |x - x'| + \frac{(|y|^2 + |y'|^2 - 2 \max \{\xi, \xi' : \xi \in T(\xi'y')\})^\frac{1}{2}}{2}.$$  

It may easily be proved (by the method of Lagrange factors — see the proof of lemma 8) that

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\[ \max \{ \langle \xi, \xi' \rangle : \xi \in T(iy) \} = \phi(\xi', y) \], where \( \phi \) is a mapping \( R^n \times R^n \to [0, +\infty) \) given by the formula

\[ \phi(x, y) = (|x|^2 + |y|^2 - \langle x, y \rangle^2)^{\frac{1}{2}}, \quad x, y \in R^n. \]

Whence we have

\[ \epsilon(t', t(z)) \leq |x - x'| + |y'|^2 + |y|^2 - 2 \phi(\xi', y)|^{\frac{1}{2}}, \]

therefore

\[ \max \{ \epsilon(t', t(z)) : t' \in T(z') \} \leq |x - x'| + \]

\[ + |y'|^2 + |y|^2 - 2 \left( |y'|^2 + |y|^2 - \max \{ \langle \xi, y \rangle : \xi \in T(iy) \} \right)^{\frac{1}{2}}. \]

Analogously, as previously, we can prove that \( \max \{ \langle \xi, y \rangle : \xi \in T(iy') \} = \phi(y, y') \); after the simple calculation we have \( \max \{ \epsilon(t', t(z)) : t' \in T(z') \} \leq |x - x'| + |y - y'| \leq 2 |x - z'|. \)

Because the assumptions of the lemma are symmetric then

\[ \epsilon_H(T(z), T(z')) \leq |x - x'| + |y - y'| \leq 2 |x - z'|. \]

(b) There exists \( \xi \in T(z) \) such that \( |\xi - t| = \epsilon(t, T(z)) \). We can take \( z' = z + \xi - t \). The proof is completed.

Lemma 5.

(a) Let \( A \subset R^n \), \( n \geq 2 \). If \( z \in \overline{W(\xi)} \) then \( T(z) \subset \overline{A} \);

(b) in the case \( n = 2 \), if \( A \) is arcwise connected and \( T(z) \subset \overline{A} \) then \( z \in \overline{W(\xi)} \);

(c) (see [2]) in the case \( n = 2 \), for a region \( D \subset R^2 \), if \( B \subset \partial D, \quad \overline{B} = \partial D \) then

the set \( B^* = \left\{ z = \zeta_1, \zeta_2 (\frac{\zeta_1 + \overline{\zeta_2}}{2}, \frac{\zeta_1 - \overline{\zeta_2}}{2i}) \in C^2 : \zeta_1 \in B, \epsilon \in B, \zeta_2 \in \partial D \right\} \) is dense in \( \partial \overline{D} \).

Proof.

(a) Let \( z_0 \in \overline{W(\xi)} \), \( \{ z_k \}_{k \in N} \subset W(\xi) \), \( z_0 = \lim_{k \to \infty} z_k \). For every \( t \in T(z) \), let

\[ t_k = T(z_k), \quad k \geq 1 \] such that \( |t - t_k| = \epsilon(t, T(z_k)) \). By lemma 4 \( |t - t_k| \leq \epsilon_H(T(z), T(z_k)) \leq 2 |z - z_k| \). Therefore \( t \equiv t_k \in \overline{A} \).

(b) Let \( z = (z_1, z_2) \in C^2, T(z) = \{ z_1 + iz_2, \overline{z}_1 + iz_2 \} \subset \overline{A}, \{ \zeta_{1,k} \}_{k \in N}, \{ \zeta_{2,k} \}_{k \in N} \subset A, \]

\[ z_1 + iz_2 = \lim_{k \to \infty} \zeta_{1,k}, \overline{z}_1 + iz_2 = \lim_{k \to \infty} \zeta_{2,k}. \]

It is easy to prove that the points

\[ z_k = z_k(\zeta_{1,k}, \zeta_{2,k}) = \left( \frac{\zeta_{1,k} + \overline{\zeta_{2,k}}}{2}, \frac{\zeta_{1,k} - \overline{\zeta_{2,k}}}{2i} \right), \quad k \geq 1 \]

lie in \( W(\xi) \) and \( \lim_{k \to \infty} z_k = z \).

Remark. In the case \( n = 2 \), for the region \( D \neq R^2 \), \( \epsilon \partial D = \{ z \in C^2 : T(z) \subset D, \}

\( T(z) \cap \partial D = \emptyset \} \).

(c) We already know that for every \( z \in B^* : z \in \partial \overline{D} \). Take \( z = (z_1, z_2) \in \partial \overline{D} \), \( \epsilon > 0 \) and suppose that \( z_1 + iz_2 \in \partial D \). There exist \( \zeta_{1}, \zeta_{2} \in R^2 \) such that \( |\zeta_1|, |\zeta_2| < \quad \frac{\epsilon}{2} \) and \( \zeta_1 = z_1 + iz_2 + \zeta_1 \in B, \zeta_2 = \overline{z}_1 + iz_2 + \zeta_2 \in \partial D \). By a simple calculation we
have $|z_1, z_2| = \frac{1}{2}(|\xi_1 + \xi_2|^2 + |\xi_1 - \xi_2|^2)i < \varepsilon$. Therefore $B^*$ is dense in $\partial D$.
This completes the proof.

The following lemma is useful in the construction of the set $\tilde{D}$ (see proof of
Theorem 4).

Lemma 6. [2] Let \( \{D_\varepsilon\}_{\varepsilon \in M} \) be an upper filtrant family of regions in $\mathbb{R}^n$, $D = \bigcup D_\varepsilon$. Then \( \{W(D_\varepsilon)\}_{\varepsilon \in M} \) is also an upper filtrant family and $W(D) = \bigcup \limits_{\varepsilon \in M} W(D_\varepsilon)$ (\( D = \bigcup \limits_{\varepsilon \in M} \tilde{D}_\varepsilon \)).

Proof. From corollary IIb after lemma 3 it follows that \( \{W(D_\varepsilon)\}_{\varepsilon \in M} \) is an upper filtrant. Now, let $z \in W(D)$, $a \in D$ be two fixed points. There exists $\gamma : I \to \mathbb{C}^n$ which contracts the sphere $T(z)$ in $D$ to the point $a$. As previously, we can prove that the set $K = \bigcup \limits_{\varepsilon \in I} T(\gamma(x))$ is a compact set and that $K \subset D$.

Therefore there exists $\mu \in \mathbb{C}^n$ such that $K \subset D_\mu$, i.e. $z \in W(D_\mu)$. Hence $W(D) \subset C \cup W(D_\mu)$. The opposite inclusion is obvious. The proof is concluded.

Below we give two examples of the effective construction of Lelong sets.

Let us fix $r, r_1, r_2$ such that $0 < r < +\infty$, $0 < r_1 < r_2 < +\infty$. By $B$ we
denote the ball with the center zero and the radius $r$, i.e. $B = \{x \in \mathbb{R}^n : |x| < r\}$. By $P$ we denote the spatial ring in $\mathbb{R}^n$ with the center zero and the radii $r_1$ and $r_2$, i.e. $P = \{x \in \mathbb{R}^n : r_1 < |x| < r_2\}$.

We define two mappings $t_-, t_+: \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty)$ by the formulas

$$t_-(x, y) = \frac{1}{2}(|x|^2 + |y|^2 - 2(|x|^2|y|^2 - \langle x, y \rangle^2)i),$$

$$t_+(x, y) = \frac{1}{2}(|x|^2 + |y|^2 + 2(|x|^2|y|^2 - \langle x, y \rangle^2)i),$$ $x, y \in \mathbb{R}^n$. For $z = x + iy$ we write $\phi(z), t_-(z), t_+(z)$ instead of $\phi(x, y), t_-(x, y), t_+(x, y)$.

Directly from the definition we get

Lemma 7.
(a) For $z = x + iy \in \mathbb{C}^n$ the vectors $x$ and $y$ are linearly dependent if and only if $t_+(z) = t_-(z) = |z|$;
(b) for $z = x + iy \in \mathbb{C}^n$ : $\langle x, y \rangle = 0 \iff t(z) = |x| - |y| \iff t_+(z) = |x| + |y|$;
(c) in the case $n = 2$, for $z = (x_1, x_2) \in \mathbb{C}^2$, $x_k = x_k + iy_k$, $k = 1, 2$ : $\phi(z) = |x_2y_1 - y_2x_1|$;
(d) in the case $n = 3$, for $z = x + iy \in \mathbb{C}^3$ : $\phi(z) = |x \times y|$, where $\times$ denotes the vector product in $\mathbb{R}^3$;
(e) in the case $n = 2$, for $z = (z_1, z_2) \in \mathbb{C}^2$:

$$t_-(z) = \min\{|z_1 + iz_2|, |z_1 - iz_2|\}, \quad t_+(z) = \max\{|z_1 + iz_2|, |z_1 - iz_2|\}.$$

Lemma 8. For $z \in \mathbb{C}^n$, $n \geq 2$

(a) $T(z) \subset B \iff t_+(z) < r$;
(b) $T(z) \subset P \iff r_1 < t_+(z) < r_2$.

Proof. We shall prove only part (b) of this lemma (the proof of part (a) is
analogue).
It is obvious that \( T(x) \subset P \iff \forall t \in T(x) : r_1 < |t| < r_2 \iff \forall \xi \in T(iy) : r_1 < |x + \xi| < r_2 \iff \)

\[
r_1 < (|x|^2 + |y|^2 + 2 \min \{ \langle x, \xi \rangle : \xi \in T(iy) \})^{1/2} \leq \leq (|x|^2 + |y|^2 + 2 \max \{ \langle x, \xi \rangle : \xi \in T(iy) \})^{1/2} < r_2 .
\]

For \( \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \) we define the functions

\[
L_0(\xi) = \sum_{j=1}^n \xi_j x_j , \quad L_1(\xi) = \sum_{j=1}^n (\xi_j^2 - \xi_j y_j) , \quad L_2(\xi) = \sum_{j=1}^n \xi_j y_j ,
\]

\((x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n))\). We want to find the maximal points and the minimal points of the function \( L_0 \) on the set \( T(iy) = \{ \xi \in \mathbb{R}^n : L_0(\xi) \leq L_0(\xi) = 0 \} \). We make use of the method of Lagrange factors. Let \( L = L_0 + \lambda_1 L_1 + \lambda_2 L_2 \); \( \frac{\partial L}{\partial \xi_j}(\xi) = x_j + 2 \lambda_1 \xi_j + \lambda_2 y_j, \quad j = 1, \ldots, n \). Additionally, suppose that \( \phi(x, y) > 0 \) (i.e. the vectors \( x \) and \( y \) are linearly independent). In this case a simple reasoning shows that the extremal points must have the form \( \xi = ax + \beta y, \quad a, \beta \in \mathbb{R} \).

Between these points only two lie in \( T(iy) \), namely for \( a = \frac{\varepsilon}{\phi(x, y)} \),

\[
\beta = -\varepsilon \frac{\langle x, y \rangle}{\phi(x, y)} , \quad \varepsilon \in \{-1, 1\} .
\]

For these \( a \) and \( \beta \) we have \( \langle x, \xi \rangle = \langle x, ax + \beta y \rangle = \varepsilon \phi(x, y), \) i.e.

\[
\min \{ \langle x, \xi \rangle : \xi \in T(iy) \} = -\phi(x, y) , \quad \max \{ \langle x, \xi \rangle : \xi \in T(iy) \} = \phi(x, y) .
\]

In the case \( \phi(x, y) = 0 \) it is easy to prove that for every \( \xi \in T(iy) \langle x, x \rangle = 0 = \phi(x, y) \). Therefore \( T(x) \subset P \iff \tilde{r}_2 < (|x|^2 + |y|^2 - 2 \phi(x, y))^{1/2} \leq (|x|^2 + |y|^2 + 2 \phi(x, y))^{1/2} < r_2 \). The proof is completed.

The ball is starlike with respect to zero, whence we get the following (see lemma 3c)

**Corollary.** \( \tilde{B} = \{ x \in \mathbb{C}^n : \tau(x) < r \} \).

**Lemma 9.** For every \( z \in \mathbb{C}^n \), if \( T(z) \subset P \) then there exists \( a \in P \) such that \( T(z) \) is spherically contractible in \( P \) to \( a \).

**Proof.** We distinguish three cases

(a) The vectors \( x \) and \( y \) are linearly dependent;

(a1) \( y = 0 \) — we take \( a = x \) and \( \tau(\tau) = x, \tau \in I \) as a contractible mapping;

(a2) \( y \neq 0, x = 0 \) — we take \( a = y \) and \( \tau(\tau) = \tau y + iy \sqrt{1 - \tau^2} , \tau \in I \) as a mapping which contracts the sphere \( T(x) \) in \( P \) to the point \( a \);

(a3) \( x, y \neq 0, x = ay \), it suffices to contract spherically the sphere \( T(x) \) in \( P \) to the sphere as in \( (a _ 2) \); we can do this by the mapping

\[
\beta(\tau) = (1 - \tau)x + i \frac{y}{|y|} \left( \left[ 1 - (1 - \tau)^2 \right] |x|^2 + |y|^2 \right)^{1/2} , \quad \tau \in I .
\]
(b) \( U(z) = \bigcup_{\tau \in I} T(z + i\gamma_0) \subset P \) — we take \( a = x \) and \( \gamma(\tau) = x + i\tau y, \tau \in I \).

(c) The vectors \(\hat{x}, \hat{y}\) are linearly independent and \( U(z) \neq P \). We define the function \( \psi(\lambda) = \ell^2(x + i\lambda y) - \tau_1^2 = \lambda^2|y|^2 - 2\lambda\phi(x, y) + |x|^2 - \tau_1^2, \lambda \in \mathbb{R} \). Because \( T(z) \subset P \) we have \( \psi(1) > 0 \); because \( U(z) \neq P \) there exists \( \lambda_0 \in [0, 1) \) such that \( \psi(\lambda_0) \leq 0 \). The function \( \psi(\lambda) = \lambda^2|y|^2 - 2\lambda\phi(x, y) + |x|^2 - \tau_1^2, \lambda \in \mathbb{R} \) is the quadratic trinomial with \( A = A(\psi) = 4(|y|^2r_1^2 - \langle x, y \rangle^2) = 4\Delta_1 \geq 0 \). Because \( \psi(1) > 0 \) and \( \psi(\lambda_0) \leq 0 \) for \( \lambda_0 \in [0, 1) \) then the greatest root of the equation \( \psi(\lambda) = 0 \) is less than 1, i.e. \( \lambda_+ = \frac{\phi(x, y) + \sqrt{\Delta_1}}{|y|^2} < 1 \).

For \( \mu = \left\langle x, y \right\rangle \) we have \( \left\langle \mu y - x, y \right\rangle = 0 \). We shall demonstrate that it is possible to contract the sphere \( T(z) \) in \( P \) to a sphere as in (a), We can do this, for example, by the mapping \( \beta(\tau) = x + \tau(\mu y - x) + iy, \tau \in I \). It is easy to obtain that \( t_{\tau_1}^{\beta}(\tau) = \frac{(1 - \tau)^2}{|y|^2} \phi^2(x, y) \pm 2(1 - \tau)\phi(x, y) + \frac{\langle x, y \rangle^2}{|y|^2} + |y|^2 \). Whence \( t_{\tau_1}^{\beta}(\tau) < r_1, \tau \in I \), therefore \( T(\beta(\tau)) \subset P \) for every \( \tau \in I \) if and only if when \( t_{\tau_1}^{\beta}(\tau) > r_1 \) for every \( \tau \in I \). Let \( \varphi(\tau) = t_{\tau_1}^{\beta}(\beta(1 - \tau)) - r_1^2, \tau \in I \). \( \varphi \) as a function of \( \tau \in \mathbb{R} \) is the quadratic trinomial with \( A = A(\varphi) = \frac{\Delta(\varphi)}{|y|^2} \phi^2(x, y) \), therefore

\[ A(\varphi) \geq 0. \]

The smaller root of the equation \( \varphi(\tau) = 0 \) is equal to \( \frac{|y|^2 - \sqrt{\Delta_1}}{\phi(x, y)} \) and is greater than 1. This completes the proof.

Corollary. \( \tilde{P} = \{ z \in \mathbb{C}^n : T(z) \subset P \} = \{ z \in \mathbb{C}^nr_1 < t_{\tau_1}(z) < r_2 \} \).

3. ANALYTIC CONTINUATION OF HARMONIC FUNCTIONS

All denotations in this section are the same as in Section 2.

From the paper [2] we can deduce the following theorem (see Introduction).

Theorem 1. For the region \( D \subset \mathbb{R}^n \), \( n \geq 2 \) and for every \( h \in H(D) \) there exists an analytic function \( \tilde{h} \) on \( \tilde{D} \) which continues \( h \). Moreover, \( \tilde{D} \) is the maximal region with these properties.

We may consider particular simple cases of this theorem.

Theorem 1. For the spatial ring \( P \) the set \( \tilde{P} \) is the harmonic envelope of analyticity.

Theorem 2. In the case \( n = 2 \), for \( h \in H(D) \) \( h \) has a holomorphic continuation on \( \tilde{D} \) if and only if there exists \( f \in \mathcal{O}(D) \) such that \( h = \text{Re}f \).

Proof. Suppose that \( \tilde{h} \in \mathcal{O}(\tilde{D}), \tilde{h}|D = h \). Let us fix \( z_0 \in D \) and set

\[ g(z) = 2\text{Im} \tilde{h} \left( \frac{z + z_0}{2}, \frac{z - z_0}{2i} \right), z \in D. \]
Since \( \tilde{D} = \{(x_1, x_2) \in C^2 : z_1 + iz_2, \bar{z}_1 + i\bar{z}_2 \in D\} \) (see lemma 3 b, d) then the function \( g \) is well defined on \( D \). It is obvious that \( g \) a function of class \( C^\infty(D) \) as a function of two real variables and it may be checked that \( h(x) = g_y(x), \quad h_y(x) = -g_x(x), \quad z \in D \). Therefore \( f = h + ig \in \mathcal{O}(D) \).

Now, suppose that \( h = \text{Ref}, f \in \mathcal{O}(D) \). For \( (x_1, x_2) \in \tilde{D} \) we define \( \tilde{h}(x_1, x_2) = \frac{1}{2} \{f(x_1 + iz_2) + f(\bar{x}_1 + i\bar{z}_2)\} \). The function \( \tilde{h} \) is well defined on \( \tilde{D} \), \( \tilde{h} \) is a function of class \( C^\infty(\tilde{D}) \) as a function of four real variables and \( \tilde{h} \mid D = h \). It may be checked that \( \tilde{h} \) satisfies the Cauchy-Riemann equations in \( \tilde{D} \). This completes the proof of Theorem 2.

The following three theorems are given by Lelong in [2].

**Theorem 3.** In the case \( n = 2 \), for \( h \in H(D) \) the analytic continuation \( \tilde{h} \) of \( h \) on \( \tilde{D} \) has singlevalued real part and \( \text{Re} \tilde{h}(x_1, x_2) = \frac{1}{2} \{h(x_1 + iz_2) + h(\bar{x}_1 + i\bar{z}_2)\} \), \((x_1, x_2) \in \tilde{D} \).

**Theorem 4.** In the case \( n = 2 \) there exists \( h_0 \in H(D) \) such that the holomorphic continuation \( \tilde{h}_0 \) of \( h_0 \) on \( \tilde{D} \) cannot be continued beyond \( \tilde{D} \).

**Proof.** It is known that there exists \( B \subset \partial D \) and \( f_0 \in \mathcal{O}(D) \) such that \( B = \partial D \) and \( \lim_{\xi \to \zeta \Rightarrow B} f_0(\xi) = \infty \) for every \( \zeta \in B \). Let \( h_0 = \text{Ref}_0 \). Hence (see Theorem 2)

\[
\tilde{h}_0(x_1, x_2) = \frac{1}{2} \{f_0(x_1 + iz_2) + f_0(\bar{x}_1 + i\bar{z}_2)\}.
\]

Let \( B^* = \{z = z(\xi_1, \xi_2) = (\xi_1 + \xi_2, -\xi_1 - \xi_2) \in C^2 : \xi_1 \in B, \xi_2 \in D\} \). We know that \( B^* \) is dense in \( \partial \tilde{D} \) (see lemma 5 c). For every \( \xi_1 \in B, \xi_2 \in D \):

\[
\lim_{(x_1, x_2) \to \xi_1, x_2 \to \xi_2} \tilde{h}_0(x_1, x_2) = \frac{1}{2} \{\lim_{\xi \to \zeta} f_0(\xi) + f_0(\xi)\} = \infty.
\]

Whence \( \tilde{h}_0 \) cannot be continued beyond \( \tilde{D} \) and the proof is completed.

**Theorem 5.** In the case \( n = 2p \geq 4 \), every function \( h \in H(D) \) may be continued to a holomorphic function \( \tilde{h} \) on \( \tilde{D} \).

**Proof.** It is known that there exists a sequence \( \{D_k\}_{k \in \mathbb{N}} \) of regions in \( \mathbb{R}^n \) such that \( D_k \subset D_{k+1}, \tilde{D}_k \subset D, k \geq 1, D = \bigcup_{k=1}^{\infty} D_k, D_k \) is bounded and \( \partial D_k \) is the sum of a finite number of surfaces of class \( C^1 \), \( k \geq 1 \).

Let for \( z \in C^n, z \neq 0 \),

\[
E(z) = \frac{1}{(2-n)\theta_n} (F(z))^{2-n},
\]

where \( \theta_n \) denotes the area of the unit \( (n-1) \) — dimensional sphere in \( \mathbb{R}^n \). It is known that for \( x \in D_k, k \geq 1 \),

\[
h(x) = \int_{\partial D_k} \left( h(t) \frac{\partial E(x-t)}{\partial n} - E(x-t) \frac{\partial h(t)}{\partial n} \right) \sigma_k(dt),
\]

where \( \sigma_k \) denotes the \( (n-1) \) — dimensional Lebesgue measure on \( \partial D_k \), \( \{\hat{\sigma}_k\}_{t \in \tilde{D}_k} \) denotes the field of exterior normal vectors to \( \partial D_k \). For \( z \in \tilde{D}_k \) we define

\[
\hat{h}_k(z) = \int_{iD_k} \left( h(t) \frac{\partial E(z-t)}{\partial n} - E(z-t) \frac{\partial h(t)}{\partial n} \right) \sigma_k(dt).
\]
It is easy to prove that \( \tilde{\kappa}_k \in \mathcal{O}(\tilde{D}_k), \tilde{\kappa}_k|_{\tilde{D}_k} = h|_{\tilde{D}_k}, \ k \geq 1. \) Therefore for \( k \leq l \), \( \tilde{\kappa}_k = 1 \). As a holomorphic continuation \( \tilde{\kappa} \) of \( h \) on \( \tilde{D} \) we can take \( \tilde{\kappa} = \bigcup_{k=1}^{\infty} \tilde{\kappa}_k. \) This completes the proof.

**Theorem 6.** In the case \( n = 2 \) (\( n = 2p+1, \ p \in \mathbb{N} \)) the function \( h_0 \in \mathcal{H}(P) \) given by formula \( h_0(x) = \ln |x| (h_0(x) = |x|^{2-n}), x \in P \) cannot be holomorphically continued on \( \tilde{\mathcal{P}} \), i.e. \( \tilde{\mathcal{P}} \) is a harmonic envelope of analyticity but not of holomorphy for \( P \).

**Proof.** In the case \( n = 2 \) follows from Theorem 2. In the case \( n = 2p+1 \) the function \( h_0 \) may be holomorphically continued on \( \tilde{\mathcal{P}} \) if and only if the square root \( \sqrt{F} \) has holomorphic singlevalued branch on \( \tilde{P} \). We shall prove that this impossible. Let \( r \in (v_1^2, r_2^2), \theta \in \mathbb{R}, x = \left( \sqrt{r} \cos \frac{\theta}{2}, 0, ..., 0 \right), y = \left( \sqrt{r} \sin \frac{\theta}{2}, 0, ..., 0 \right), \ z = x + iy. \) It is easy to check that \( t_-(z) = t_+(z) = \sqrt{r} \), therefore \( z \in \tilde{P} \). By a simple calculation we get \( F(z) = r e^{i \theta}. \) Hence \( \{ z \in \mathbb{C} : r^2 < |z| < r_2^2 \} \subset \tilde{\mathcal{F}}(\tilde{\mathcal{P}}). \) So \( \tilde{\mathcal{F}} \) has not a singlevalued branch of the square root in \( \tilde{\mathcal{P}} \). The proof is completed.

**Theorem 7.** Let \( D_1, D_2 \) be two regions in \( \mathbb{C} \), \( D_1, D_2 \not\subset \mathbb{C} \), \( f = u + iv : D_1 \rightarrow D_2 \) be a biholomorphic mapping between these regions. By \( \tilde{u}, \tilde{v} \) we denote the holomorphic continuations of \( u \) and \( v \) (see Theorem 2). Let \( \tilde{f} = (\tilde{u}, \tilde{v}) : \tilde{D}_1 \rightarrow \mathbb{C}^2. \) Then \( \tilde{f}(\tilde{D}_1) = \tilde{D}_2 \) and \( \tilde{f} \) is biholomorphic.

**Proof.** We know that for \( z = (z_1, z_2) \in \tilde{D}_1 \)

\[
(\tilde{u}(z) = \frac{1}{2} \left( f(z_1 + iz_2) + f(\bar{z}_1 + i\bar{z}_2) \right)), \quad (\tilde{v}(z) = \frac{1}{2i} \left( f(z_1 + iz_2) - f(\bar{z}_1 + i\bar{z}_2) \right)).
\]

Since \( T(\tilde{f}(z)) \subset f(T(z)), z \in \tilde{D}_1 \) then \( \tilde{f}(\tilde{D}_1) \subset \tilde{D}_2. \)

Now, let \( w = (w_1, w_2) \in \mathbb{C}^2, T(w) \subset D_2 \) (i.e. \( w \in \tilde{D}_2 \)), \( w_1 + iw_2 = f(\xi), \ w_1 + \bar{w}_2 = f(\bar{\xi}), \ z_1, z_2 \in \tilde{D}_1 \). We define \( z = (z_1, z_2) \in \mathbb{C}^2 \) by formulas \( z_1 = \frac{1}{2} (\xi_1 + \xi_2), \ z_2 = \frac{1}{2i} (\xi_1 - \bar{\xi}_2). \) Hence \( T(z) = (\xi_1, \xi_2) \subset D_1. \) So \( z \in \tilde{D}_1, \ w = \tilde{f}(z) \subset f(\tilde{D}_1), \) i.e. \( \tilde{D}_2 \subset f(\tilde{D}_1). \)

Let \( g : D_2 \rightarrow D_2, \ g = f^{-1} \) and \( \tilde{g} : \tilde{D}_2 \rightarrow \tilde{D}_1 \) is defined analogically as the function \( \tilde{f}. \) The mappings \( \tilde{f}, \tilde{g} \) are holomorphic and \( \tilde{f} \circ \tilde{g} = id_{D_1}, \ g \circ \tilde{f} \circ \tilde{g} = id_{D_2}. \) By the principle of identity for holomorphic functions we have \( \tilde{f} \circ \tilde{g} = id_{D_1}, \ g \circ \tilde{f} \circ \tilde{g} = id_{D_2}. \) This completes the proof.

**Corollary.** If \( D_1, D_2 \) are as in Theorem 7 then if \( \tilde{D}_1 \) is the harmonic envelope of holomorphy for \( D_1 \) then \( \tilde{D}_2 \) is the harmonic envelope of holomorphy for \( D_2. \)

In particular, we get the following.

**Theorem 8.** Let \( D \subset \mathbb{C} \) be a simple connected domain such that \( \partial D \) has at least two distinct points. Let \( f = u + iv : B_1 \rightarrow D \) be the biholomorphic mapping (\( B_1 \) denotes the unit disc in \( \mathbb{C} \)). By \( \tilde{u}, \tilde{v} \) we denote the holomorphic continuations for \( u \) and \( v, f = (u, v). \) Then \( \tilde{D} = \tilde{f}(\tilde{B}_1) \) is the harmonic envelope of holomorphy for \( D \) and the mapping \( \tilde{f} : \tilde{B}_1 \rightarrow \tilde{D} \) is biholomorphic.
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