ZESZYTY NAUKOWE UNIWERSYTETU JAGIELLOŃSKIEGO
CCCCIII PRACE MATEMATYCZNE, ZESZYT 17 1975

Marek Jarnicki

Analytic Continuation of Harmonic Functions

1. INTRODUCTION

In this paper by D we denote an arbitrary fixed, open, connected and not empty subset (a region) of $\boldsymbol{R}^n$, $n \geqslant 2$. It is known that every function h harmonic on D may be continued to a holomorphic function in an open set $\tilde{D}_h \subset \boldsymbol{C}^n$. It may be asked whether there exists an open connected set $\tilde{D} \subset \boldsymbol{C}^n$ such that $D \subset \tilde{D}$ and every harmonic function on D may be continued to a holomorphic (or only to an analytic multivalued) function on $\tilde{D}$. It may also be asked whether there exists a maximal set $\tilde{D}$ with these properties. This set will be called *a harmonic envelope of holomorphy* (or *of analyticity*) for D.

From the paper [2] we can deduce the following

Theorem I. For every region $D \subset \boldsymbol{R}^n$ there exists a harmonic envelope of analyticity.

In the paper [3] we can find

Theorem II. For every region $D \subset \boldsymbol{R}^n$ there exists an open connected set $\tilde{\tilde{D}} \subset \boldsymbol{C}^n$ such that every harmonic function on D may be continued to a holomorphic function on $\tilde{\tilde{D}}$.

Theorem III. If $B = \{x \in \boldsymbol{R}^n: |x| < r\}$, $n \geqslant 2$, $r > 0$, then

$$\tilde{B} = \{z = x+iy \in \boldsymbol{C}^n: [|x|^2+|y|^2+2(|x|^2|y|^2-\langle x, y\rangle^2)^{\frac{1}{2}}]^{\frac{1}{2}} < r\}$$

is the harmonic envelope of holomorphy for B.

In his paper [2] Lelong presented two methods of construction of a harmonic envelope of analyticity. In Section 2 of this paper these two methods are analysed and used for the effective construction of a harmonic envelope of analyticity for the ball and the spatial ring. By different methods the harmonic envelope of holomorphy of the ball was obtained in [1] and [3]. The paper is closed by Section 3 in which theorems 7 and 8 are proved. Theorem 7 permits the effective construction of the harmonic envelope of analyticity for $D\subset \boldsymbol{C}$, if a harmonic envelope of analyticity is known for some region that is biholomorphically equivalent to D. Theorem 8 permits the construction of a harmonic envelope of holomorphy for the set $D\subset \boldsymbol{C}$ that is biholomorphically equivalent to the unit disc.

Now we present a list of the denotations used in this note.

As usual for $A\subset \boldsymbol{C}^n$ (or $A\subset \boldsymbol{R}^n$) by $A^0, \bar{A}, \partial A$ we denote, respectively, the interior, the closure and the boundary of A. For $U=U^0\subset \boldsymbol{R}^n$, by $H(U)$ we denote the set of all harmonic functions on U. For $\Omega=\Omega^0\subset \boldsymbol{C}^n$, by $O(\Omega)$ we denote the set of all holomorphic functions on Ω. For arbitrary $z, w\in \boldsymbol{C}^n$, by $\langle z, w\rangle$ we denote the standard scalar product in $\boldsymbol{C}^n$ (i.e. $\langle z, w\rangle = \sum_{j=1}^{n} z_j\overline{w}_j$) and by $|z|$ the norm induced by the scalar product.

2. LELONG SETS

The whole of this section has been suggested by the ideas contained in Lelong's paper [2]. It has the character of a short report on results relative to the analytic continuation of harmonic functions.

The first part of this section is devoted to generalizations of Lelong's methods of construction of a harmonic envelope of analyticity. The case $n=2$ plays a special role in this theory, therefore we shall devote most attention to this.

Now, we shall define two sets, which play a fundamental role in the following constructions.

Let $F(z)=\sum_{j=1}^{n} z_j^2$, $z=(z_1, \ldots, z_n)\in \boldsymbol{C}^n$. For $z_0\in \boldsymbol{C}^n$, $t_0\in \boldsymbol{R}^n$ let $T(z_0)= \{t\in \boldsymbol{R}^n : F(z_0-t)=0\}$, $\Gamma(t_0)=\{z\in \boldsymbol{C}^n : F(z-t_0)=0\}$.

From these definitions we can directly obtain the following

Lemma 1.

(a) For $z=x+iy\in \boldsymbol{C}^n : T(z)=\{t\in \boldsymbol{R}^n : |x-t|=|y|, \langle x=t, y\rangle=0\}$;

(b) for $z=x+iy\in \boldsymbol{C}^n : T(z)=T(\bar{z}), T(x)=\{x\}/\bar{z}=x-iy/$;

(c) in the case $n\geqslant 2$, for $z=x+iy\in \boldsymbol{C}^n$, $y\neq 0$ the set $T(z)$ is an $(n-2)$ dimensional sphere with the center x, the radius $|y|$ and $T(z)$ lies in the hyperplane $\{t\in \boldsymbol{R}^n : \langle x-t, y\rangle=0\}$;

(d) in the case $n=2$, for $z=(z_1, z_2)\in \boldsymbol{C}^2$: $T(z)=\{z+iz_2, \bar{z}_1+i\bar{z}_2\}$ (we identify $\boldsymbol{C}$ and $\boldsymbol{R}^2$);

(e) in the case $n\geqslant 3$ the set $T(z)$ is connected;

(f) for $t\in \boldsymbol{R}^n$ $\Gamma(t)\cap \boldsymbol{R}^n=\{t\}$, $(n=1 : \Gamma(t)=\{t\})$;

(g) for $z \in \boldsymbol{C}^n, t \in \boldsymbol{R}^n \quad z \in \Gamma(t) \Leftrightarrow t \in T(z)$;

(h) for $z = x+iy \in \boldsymbol{C}^n, a \in \boldsymbol{R}, a \neq 0 : T(x+iy) = x + aT\left(i\dfrac{y}{a}\right)$;

(i) if U is an n-dimensional real orthogonal matrix, then $T(Uz) = U\big(T(z)\big)$ (we identify U with the mapping of $\boldsymbol{C}^n$ onto $\boldsymbol{C}^n$).

This lemma (except (i)) will be used many times in the sequel.

Lemma 2. Let A be a subset of $\boldsymbol{R}^n$ such that $\partial A \neq \phi$.

Set $\Gamma(A) = \bigcup_{t \in \partial A} \Gamma(t)$. Then $\Gamma(A) = \overline{\Gamma(A)}$, $\Gamma(A) \cap \boldsymbol{R}^n = \partial A$ $\big(n = 1 : \Gamma(A) = \partial A\big)$.

Proof. The second part of this lemma follows from lemma 1f. We shall prove only that $\Gamma(A) = \overline{\Gamma(A)}$.

Let $\{z_k\}_{k \in N} \subset \Gamma(A)$, $\lim\limits_{k \to \infty} z_k = z_0$, $z_k \in \Gamma(t_k)$, $t_k \in \partial A$, $k \geqslant 1$. We want to prove that $z_0 \in \Gamma(A)$, i.e. there exists $t_0 \in \partial A$ such that $z_0 \in \Gamma(t_0)$.

There exists a constant $M > 0$ such that $|z_k| \leqslant M$, $k \geqslant 1$. For arbitrary $k, l \in \boldsymbol{N}$ we have

$$|t_k - t_l| \leqslant |t_k - z_k| + |z_k - z_l| + |z_l - t_l| \leqslant |t_k - z_k| + |t_l - z_l| + 2M\,,$$
$$|t_k - z_k|^2 = |t_k - (x_k + iy_k)|^2 = |t_k - x_k|^2 + |y_k|^2 = 2\,|y_k|^2 \leqslant 2M^2\,.$$

Hence $|t_k - t_l| \leqslant 2(1+\sqrt{2})M$, $k, l \in \boldsymbol{N}$. Since $\{t_k\}_{k \in N}$ is bounded and ∂A is a closed set then there exists a subsequence $\{t_{k_m}\}_{m \in N}$ and a point $t_0 \in \partial A$ such that $\lim\limits_{m \to \infty} t_{k_m} = t_0 \cdot S_0$, $0 = \lim\limits_{m \to \infty} F(z_{k_m} - t_{k_m}) = F(z_0 - t_0)$, i.e. $z_0 \in \Gamma(t_0)$. The proof is concluded.

Now we give the definition of a Lelong set of the first type.

Let A be a subset of $\boldsymbol{R}^n$ such that $\partial A \neq \phi$, $A^0 \neq \phi$; by $\widetilde{A}$ we denote the connected component of the set $\boldsymbol{C}^n \setminus \Gamma(A)$ which contains A^0 (in the case $A = \boldsymbol{R}^n$ we set $\widetilde{A} = \boldsymbol{C}^n$).

Since $A^0 \subset R^n \setminus \partial A \subset \boldsymbol{C}^n \setminus \Gamma(A)$ (see lemma 2) then $\widetilde{A}$ is well defined. Since $\boldsymbol{C}^n \setminus \Gamma(A)$ is an open set (see lemma 2) then $\widetilde{A}$, as a connected component of an open set, is a region in C^n.

For a region $D \subset \boldsymbol{R}^n$, the region $\widetilde{D}$ is the same as the region constructed by the method given by Lelong in [2].

The definition of the set $\widetilde{A}$ is clear with respect to topological properties but it is not useful in concrete constructions.

Below we give the definition of a Lelong set of the second type (denoted by $W(A)$); this definition is useful with respect to its constructive properties. It will be proved that for any region $D \subset \boldsymbol{R}^n$, $n \geqslant 2 : \widetilde{D} = W(D)$. This will provide a method for effectively constructing the set $\widetilde{D}$.

First we give the following auxiliary definitions.

Given $B \subset \boldsymbol{R}^n$, $n \geqslant 2$, $b \in \boldsymbol{R}^n$, we say that the sphere $T(z)$ is *spherically contractible in* B *to the point* b if and only if there exists a continuous mapping $\gamma : I \to \boldsymbol{C}^n$ such that $\gamma(0) = z$, $\gamma(1) = b$ and for every $\tau \in I : T\big(\gamma(\tau)\big) \subset B$ $(I \overset{\mathrm{df}}{=} [0, 1] \subset \boldsymbol{R})$.

For $A \subset R^n$, $n \geqslant 2$ put
$W(A) = \{z \in C^n : T(z)$ is spherically contractible in A to every point $a \in A\}$.
The following lemma gives a certain description of the set $W(A)$.

Lemma 3.

(a) If $A \subset R^n$ is arcwise connected then $W(A) = \{z \in C^n : \exists a \in A$ such that $T(z)$ is spherically contractible in A to a$\}$, $W(A)$ is arcwise connected, $W(A) \cap R^n = \partial A$ and $\partial A \subset \overline{W(A)} \setminus W(A)$;

(b) if $n = 2$ and if A is arcwise connected then $W(A) = \{z \in C^2 : T(z) \subset A\}$;

(c) if A is starlike with respect to $t_0 \in A$ then $W(A) = \{z \in C^n : T(z) \subset A\}$ and $W(A)$ is starlike with respect to t_0;

(d) if $\partial A \neq \phi$, $A^0 \neq \phi$ and A^0 is connected then $\widetilde{A} = W(A^0)$, $\widetilde{A} \cap R^n = A^0$ and $\partial(A^\circ) \subset \partial\widetilde{A}$.

Proof.

(a) is implied directly by the definition of $W(A)$.

(b) Let $z = (z_1, z_2) \in C^2$, $T(z) = \{z_1 + iz_2, \bar{z}_1 + i\bar{z}_2\} \subset A$, $a \in A$ be an arbitrary but fixed point. There exist two continuous mappings $\sigma_i : I \to A$, $i = 1, 2$ such that $\sigma_1(0) = z_1 + iz_2$, $\sigma_2(0) = \bar{z}_1 + i\bar{z}_2$, $\sigma_1(1) = \sigma_2(1) = a$. We define $\gamma : I \to C^2$ by the formula $\gamma = \left(\frac{\sigma_1 + \bar{\sigma}_2}{2}, \frac{\sigma_1 - \bar{\sigma}_2}{2i}\right)$. It is obvious that γ is a continuous mapping, $\gamma(0) = z$, $\gamma(1) = a$ and for every $\tau \in I$ $T(\gamma(\tau)) = \{\sigma_1(\tau), \sigma_2(\tau)\} \subset A$.

(c) Let $z \in C^n$, $T(z) \subset A$. A is starlike with respect to t_0, therefore A is arcwise connected. It suffices to show that $T(z)$ is spherically contractible in A to the point t_0. Let $\gamma(\tau) = \tau t_0 + (1-\tau)z$, $\tau \in I$. It is obvious that γ is a continuous mapping of I into C^n, $\gamma(0) = z$, $\gamma(1) = t_0$ and $T(\gamma(\tau)) = \tau t_0 + (1-\tau)T(z)$, $\tau \in I$ (see lemma 1h). Therefore $T(\gamma(\tau)) \subset A$, $\tau \in I$.

(d) We know that $A^0 \subset W(A^0)$ and $W(A^0)$ is connected. First we shall show that $W(A^0) \cap \Gamma(A) = \phi$. Suppose there exists $z \in W(A^0) \cap \Gamma(A)$, so there exists $t \in \partial A$ such that $t \in T(z)$. Therefore $T(z) \cap \partial A \neq \phi$ and we have a contradiction to the inclusion $T(z) \subset A^0$. Hence $W(A^0) \subset \widetilde{A}$.

We shall prove the opposite inclusion. Let $z \in \widetilde{A}$, $a \in A^0$ be two fixed points and let $\sigma : I \to \widetilde{A}$ be a continuous mapping such that $\sigma(0) = z$, $\sigma(1) = a$. We want to demonstrate that for every $\tau \in I : T(\sigma(\tau)) \subset A^0$ (in particular from this it follows that $z \in W(A^0)$). Let $K = \bigcup_{\tau \in I} T(\sigma(\tau))$. We shall show that K is connected compact set.

K is a bounded set. Let $t_i \in T(\sigma(\tau))$, $i = 1, 2$; $|t_1 - t_2| \leqslant |t_1 - \sigma(\tau_1)| + |\sigma(\tau_1) - \sigma(\tau_2)| + |\sigma(\tau_2) - t_2| \leqslant 2(1+\sqrt{2})\max\{|\sigma(\tau)| : \tau \in I\} < +\infty$ (see the proof of lemma 2).

K is a closed set. Let $t_k \in T(\sigma(\tau_k))$, $k \geqslant 1$, $t_0 = \lim t_k$. There exists a subsequence $\{\tau_{k_m}\}_{m \in N}$ and a point $\tau_0 \in I$ such that $\tau_0 = \lim_{m\to\infty} \tau_{k_m}$. Since F and σ are continuous mappings, we have $0 = \lim_{m\to\infty} F(t_{k_m} - \sigma(\tau_{k_m})) = F(t_0 - \sigma(\tau_0))$. So $t_0 \in T(\sigma(\tau_0)) \subset K$.

K is connected. We distinguish two cases.

At first — the case $n = 2$. Let $\sigma = (\sigma_1, \sigma_2)$, $\sigma_i : I \to C$, $i = 1, 2$, $K_1 \overset{\text{df}}{=} (\sigma_1 + i\sigma_2)(I)$, $K_2 \overset{\text{df}}{=} (\bar{\sigma}_1 + i\bar{\sigma}_2)(I)$. K_1, K_2 are connected, $a \in K_1 \cap K_2$, $K = K_1 \cup K_2$. Therefore K is connected.

Now we consider the case $n \geqslant 3$. Suppose that $K = K_1 \cup K_2$, $K_1 \cap K_2 = \phi$, $K_i \neq \phi$, $K_i = \bar{K}_i$, $i = 1, 2$. Let $I_i = \{\tau \in I : T(\sigma(\tau)) \cap K_i \neq \phi\}$, $i = 1, 2$. It may be checked that $I_i \neq \phi$, $i = 1, 2$, $I = I_1 \cup I_2$. Since for $\tau_0 \in I$ the set $T(\sigma(\tau_0))$ is connected, then $T(\sigma(\tau_0)) \subset K_1$ or $T(\sigma(\tau_0)) \subset K_2$. So $I_1 \cap I_2 = \phi$. It is easy to prove that $I_i = \bar{I}_i$, $i = 1, 2$ and we get a contradiction to connectedness of I.

We want to show that $K \subset A^0$. It is obvious that $K \cap \partial A = \phi$, $K \cap A \neq \phi$. Suppose that $K \not\subset A^0$, i.e. $K = (K \cap A^0) \cup (K \cap (R^n \setminus \bar{A}))$ where $(K \cap (R^n \setminus \bar{A})) \neq \phi$. This is contradiction to the connectedness of K. This completes the proof.

Lemma 3 implies the following

Corollary

I. For a region $D \subset \boldsymbol{R}^n$, $n \geqslant 2$:

(a) $W(D) = \{z \in \boldsymbol{C}^n : \exists a \in D : T(z)$ is spherically contractible in D to a$\}$;

(b) in the case $n = 2$ $W(D) = \{z \in \boldsymbol{C}^2 : T(z) \subset D\}$;

(c) $\tilde{D} = W(D)$, $\tilde{D} \cap \boldsymbol{R}^n = D$, $\partial D \subset \partial \tilde{D} \subset \Gamma(D)$.

II. For $A, B \subset \boldsymbol{R}^n$:

(a) if $A \cap B = \phi$ then $W(A) \cap W(B) = \phi$;

(b) if $A \subset B$ and B is arcwise connected then $W(A) \subset W(B)$.

The following Lemma 4 gives a characterization of metrical dependences in the family $\{T(z)\}_{z \in C^n}$. First we give a known definition of the Hausdorff distance between the sets.

For $A, B \subset \boldsymbol{C}^n$, $A, B \neq \phi$ we define the Hausdorff distance between A and B by the formula

$$\varrho_H(A, B) = \max\{\sup_{x \in A} \varrho(x, B), \sup_{y \in B} \varrho(y, A)\},$$

where $\varrho(z, C)$ denotes the distance between the point z and the set C.

Lemma 4

(a) Let $z = x + iy$, $z' = x' + iy' \in \boldsymbol{C}^n$, $n \geqslant 2$, then $\varrho_H(T(z), T(z')) \leqslant |x - x'| + |y - y'|$;

(b) let $z \in \boldsymbol{C}^n$, $t \in \boldsymbol{R}^n$, there exists $z' \in \boldsymbol{C}^n$ such that $t \in T(z')$, $|z - z'| = \varrho(t, T(z))$.

Proof

(a) Let $t' = x' + \xi'$, $\xi' \in T(iy')$ be an arbitrary point of $T(z')$. For the distance between t' and $T(z)$ we have

$$\varrho(t', T(z)) = \min\{|t - t'| : t \in T(z)\} \leqslant |x - x'| + \min\{|\xi - \xi'| : \xi \in T(iy)\}$$
$$= |x - x'| + (|y|^2 + |y'|^2 - 2\max\{\langle \xi, \xi' \rangle : \xi \in T(iy)\})^{\frac{1}{2}}.$$

It may easily be proved (by the method of Lagrange factors — see the proof of lemma 8) that

$\max\{\langle\xi, \xi'\rangle : \xi \in T(iy)\} = \phi(\xi', y)$, where ϕ is a mapping $R^n \times R^n \to [0, +\infty)$ given by the formula

$$\phi(x, y) = (|x|^2 + |y|^2 - \langle x, y\rangle^2)^{\frac{1}{2}}, \quad x, y \in R^n .$$

Whence we have

$$\varrho\big(t', T(z)\big) \leqslant |x - x'| + \big(|y|^2 + |y'|^2 - 2\phi(\xi', y)\big)^{\frac{1}{2}},$$

therefore

$$\max\{\varrho\big(t', T(z)\big) : t' \in T(z')\} \leqslant |x - x'| + \\ + \{|y'|^2 + |y|^2 - 2\big[|y'|^2|y|^2 - (\max\{\langle\xi', y\rangle : \xi' \in T(iy)\})^2\big]^{\frac{1}{2}}\}^{\frac{1}{2}} .$$

Analogically, as previously, we can prove that $\max\{\langle\xi', y\rangle : \xi' \in T(iy')\} = \phi(y, y')$; after the simple calculation we have $\max\{\varrho\big(t' T(z)\big) : t' \in T(z')\} \leqslant |x - x'| + |y - y'|$.

Because the assumptions of the lemma are symmetric then

$$\varrho_H\big(T(z), T(z')\big) \leqslant |x - x'| + |y - y'| \leqslant 2|z - z'| .$$

(b) There exists $\xi \in T(z)$ such that $|\xi - t| = \varrho\big(t, T(z)\big)$. We can take $z' = z + \xi - t$. The proof is completed.

Lemma 5.

(a) Let $A \subset R^n$, $n \geqslant 2$. If $z \in \overline{W(A)}$ then $T(z) \subset \bar{A}$;

(b) in the case $n = 2$, if A is arcwise connected and $T(z) \subset \bar{A}$ then $z \in \overline{W(A)}$;

(c) (see [2]) in the case $n = 2$, for a region $D \subset R^2$, if $B \subset \partial D$, $\bar{B} = \partial D$ then the set $B^* = \left\{z = z(\zeta_1, \zeta_2) = \left(\frac{\zeta_1 + \bar{\zeta}_2}{2}, \frac{\zeta_1 - \bar{\zeta}_2}{2i}\right) \in C^2 : \zeta_1 \in B, \in B, \zeta_2 \in D\right\}$ is dence in $\partial\tilde{D}$.

Proof.

(a) Let $z_0 \in \overline{W(A)}$, $\{z_k\}_{k \in N} \subset W(A)$, $z_0 = \lim_{k\to\infty} z_k$. For every $t \in T(z)$, let $t_k \in T(z_k)$, $k \geqslant 1$ such that $|t - t_k| = \varrho\big(t, T(z_k)\big)$. By lemma 4a $|t - t_k| \leqslant \varrho_H\big(T(z), T(z_k)\big) \leqslant 2|z - z_k|$. Therefore $t = \lim_{k\to\infty} t_k \in \bar{A}$.

(b) Let $z = (z_1, z_2) \in C^2$, $T(z) = \{z_1 + iz_2, \bar{z}_1 + i\bar{z}_2\} \subset \bar{A}$, $\{\zeta_{1,k}\}_{k\in N}$, $\{\zeta_{2,k}\}_{k\in N} \subset A$, $z_1 + iz_2 = \lim_{k\to\infty} \zeta_{1,k}$, $\bar{z}_1 + i\bar{z}_2 = \lim_{k\to\infty} \zeta_{2,k}$. It is easy to prove that the points $z_k = z_k(\zeta_{1,k}, \zeta_{2,k}) = \left(\frac{\zeta_{1,k} + \overline{\zeta_{2,k}}}{2}, \frac{\zeta_{1,k} - \overline{\zeta_{2,k}}}{2i}\right)$, $k \geqslant 1$ lie in $W(A)$ and $\lim_{k\to\infty} z_k = z$.

Remark. In the case $n = 2$, for the region $D \not\subseteq R^2$ $\partial\tilde{D} = \{z \in C^2 : T(z) \subset D, T(z) \cap \partial D \neq \phi\}$.

(c) We already know that for every $z \in B^* : z \in \partial\tilde{D}$. Take $z = (z_1, z_2) \in \partial\tilde{D}$, $\varepsilon > 0$ and suppose that $z_1 + iz_2 \in \partial D$. There exist $\xi_1, \xi_2 \in R^2$ such that $|\xi_1|, |\xi_2| < \frac{\varepsilon}{2}$ and $\zeta_1 = z_1 + iz_2 + \xi_1 \in B$, $\zeta_2 = \bar{z}_1 + i\bar{z}_2 + \xi_2 \in D$. By a simple calculation we have $|z(\xi_1, \xi_2)-z| = \frac{1}{2}(|\xi_1+\bar{\xi}_2|^2+|\xi_1-\bar{\xi}_2|^2)^{\frac{1}{2}} < \varepsilon$. Therefore B^* is dence in $\partial\tilde{D}$. This completes the proof.

The following lemma is useful in the construction of the set $\tilde{D}$ (see proof of Theorem 4).

Lemma 6. [2] Let $\{D_v\}_{v\in M}$ be an upper filtrant family of regions in $\boldsymbol{R}^n$, $D = \bigcup_{v\in M} D_v$. Then $\{W(D_v)\}_{v\in M}$ is also an upper filtrant family and $W(D) = \bigcup_{v\in M} W(D_v)$ $(D = \bigcup_{v\in M} \tilde{D}_v)$.

Proof. From corollary IIb after lemma 3 it follows that $\{W(D_v\}_{v\in M}$ is an upper filtrant. Now, let $z \in W(D)$, $a \in D$ be two fixed points. There exists $\gamma : I \to \boldsymbol{C}^n$ which contracts the sphere $T(z)$ in D to the point a. As previously, we can prove that the set $K = \bigcup_{\tau\in I} T(\gamma(\tau))$ is a compact set and that $K \subset D$. Therefore there exists $\mu \in M$ such that $K \subset D_\mu$ i.e. $z \in W(D_\mu)$. Hence $W(D) \subset \subset \bigcup_{v\in M} W(D_v)$. The opposite inclusion is obvious. The proof is concluded.

Below we give two examples of the effective construction of Lelong sets.

Let us fix r, r_1, r_2 such that $0 < r < +\infty, 0 \leqslant r_1 < r_2 \leqslant +\infty$. By B we denote the ball with the center zero and the radius r, i.e. $B = \{x \in \boldsymbol{R}^n : |x| < r\}$. By P we denote the spatial ring in $\boldsymbol{R}^n$ with the center zero and the radii r_1 and r_2, i.e. $P = \{x \in \boldsymbol{R}^n : r_1 < |x| < r_2\}$.

We define two mappings $t_-, t_+ \boldsymbol{R}^n \times \boldsymbol{R}^n \to [0, +\infty)$ by the formulas

$$t_-(x, y) = [|x|^2+|y|^2-2(|x|^2y^2-\langle x, y\rangle^2)^{\frac{1}{2}}]^{\frac{1}{2}},$$

$$t_+(x, y) = [|x|^2+|y|^2+2(|x|^2|y|^2-\langle x, y\rangle^2)^{\frac{1}{2}}]^{\frac{1}{2}}, \quad x, y \in \boldsymbol{R}^n.$$ For $z = x+iy$ we write $\phi(z), t_-(z), t_+(z)$ instead of $\phi(x, y), t_-(x, y), t_+(x, y)$.

Directly from the definition we get

Lemma 7.

(a) For $z = z+iy \in \boldsymbol{C}^n$ the vectors x and y are linearly dependent if and only if $t_-(z) = t_+(z) = |z|$;

(b) for $z = x+iy \in \boldsymbol{C}^n : \langle x, y\rangle = 0 \Leftrightarrow t(z) = ||x|-|y|| \Leftrightarrow t_+(z) = |x|+|y|$;

(c) in the case $n = 2$, for $z = (z_1, z_2) \in \boldsymbol{C}^2$, $z_k = x_k+iy_k$, $k = 1, 2 : \phi(z) = |x_1y_2-y_1x_2|$;

(d) in the case $n = 3$, for $z = x+iy \in \boldsymbol{C}^3 : \phi(z) = |x\times y|$, where $\times$ denotes the vector product in $\boldsymbol{R}^3$;

(e) in the case $n = 2$, for $z = (z_1, z_2) \in \boldsymbol{C}^2$:

$$t_-(z) = \min\{|z_1+iz_2|, |z_1-iz_2|\}, \qquad t_+(z) = \max\{|z_1+iz_2|, |z_1-iz_2|\}.$$

Lemma 8. For $z \in C^n$, $n \geqslant 2$

(a) $T(z) \subset B \Leftrightarrow t_+(z) < r$;

(b) $T(z) \subset P \Leftrightarrow r_1 < t_-(z) \leqslant t_+(z) < r_2$.

Proof. We shall prove only part (b) of this lemma (the proof of part (a) is analogical).

It is obvious that $T(z)\subset P \Leftrightarrow \forall t\in T(z): r_1<|t|<r_2 \Leftrightarrow \forall \xi\in T(iy): r_1< <|x+\xi|<r_2 \Leftrightarrow$

$$r_1<\left(|x|^2+|y|^2+2\min\{\langle x,\xi\rangle:\xi\in T(iy)\}\right)^{\frac{1}{2}}\leqslant$$
$$\leqslant\left(|x|^2+|y|^2+2\max\{\langle x,\xi\rangle:\xi\in T(iy)\}\right)^{\frac{1}{2}}<r_2\,.$$

For $\xi=(\xi_1,\ldots,\xi_n)\in \boldsymbol{R}^n$ we define the functions

$$L_0(\xi)=\sum_{j=1}^{n}\xi_j x_j\,,\qquad L_1(\xi)=\sum_{j=1}^{n}(\xi_j^2-y_j^2)\,,\qquad L_2(\xi)=\sum_{j=1}^{n}\xi_j y_j$$

$\big(x=(x_1,\ldots,x_n),\ y=(y_1,\ldots,y_n)\big)$. We want to find the maximal points and the minimal points of the function L_0 on the set $T(iy)=\{\xi\in \boldsymbol{R}^n L_1(\xi)=L_2(\xi)=0\}$. We make use of the method of Lagrange factors. Let $L=L_0+\lambda_1L_1+\lambda_2L_2$; $\frac{\partial L}{\partial\xi_j}(\xi)=x_j+2\lambda_1\xi_j+\lambda_2y_j,\ j=1,\ldots,n$. Additionally, suppose that $\phi(x,y)>0$ (i.e. the vectors x and y are linearly independent). In this case a simple reasoning shows that the extremal points must have the form $\xi=ax+\beta y,\ a,\beta\in\boldsymbol{R}$. Between these points only two lie in $T(iy)$, namely for $a=\varepsilon\frac{|y|^2}{\phi(x,y)}$, $\beta=-\varepsilon\frac{\langle x,y\rangle}{\phi(x,y)}$, $\varepsilon\in\{-1,1\}$. For these a and β we have $\langle x,\xi\rangle=\langle x,ax+\beta y\rangle$ $=\varepsilon\phi(x,y)$, i.e.
$\min\{\langle x,\xi\rangle:\xi\in T(iy)\}=-\phi(x,y)\,,\ \max\{\langle x,\xi\rangle:\xi\in T(iy)\}=\phi(x,y)\,.$

In the case $\phi(x,y)=0$ it is easy to prove that for every $\xi\in T(iy)\langle\xi,x\rangle= =0=\phi(x,y)$. Therefore $T(z)\subset P\Leftrightarrow r_1<\left(|x|^2+|y|^2-2\phi(x,y)\right)^{\frac{1}{2}}\leqslant\left(|x|^2+|y|^2+ +2\phi(x,y)\right)^{\frac{1}{2}}<r_2$. The proof is completed.

The ball is starlike with respect to zero, whence we get the following (see lemma 3c)

Corollary. $\widetilde{B}=\{z\in \boldsymbol{C}^n: t_+(z)<r\}$.

Lemma 9. For every $z\in\boldsymbol{C}^n$, if $T(z)\subset P$ then there exists $a\in P$ such that $T(z)$ is spherically contractible in P to a.

Proof. We distinguish three cases

(a) The vectors x and y are linearly dependent;

(a_1) $y=0$ — we take $a=x$ and $\gamma(\tau)\equiv x,\ \tau\in I$ as a conractible mapping;

(a_2) $y\neq0,\ x=0$ — we take $a=y$ and $\gamma(\tau)=\tau y+iy\sqrt{1-\tau^2},\ \tau\in I$ as a mapping which contracts the sphere $T(z)$ in P to the point $\bar{a}$;

(a_3) $x,y\neq0,\ x=ay$, it suffices to contract spherically the sphere $T(z)$ in P to the sphere as in (a_2); we can do this by the mapping

$$\beta(\tau)=(1-\tau)x+i\frac{y}{|y|}\{[1-(1-\tau)^2]|x|^2+|y|^2\}^{\frac{1}{2}}\,,\qquad \tau\in I\,.$$

(b) $U(z) \overset{\mathrm{df}}{=} \bigcup_{\lambda \in I} T(x+i\lambda y) \subset P$ — we take $a = x$ and $\gamma(\tau) = x+i\tau y, \tau \in I$.

(c) The vectors x, y are linearly independent and $U(z) \not\subset P$. We define the function $\psi(\lambda) = t_-^2(x+i\lambda y) - r_1^2 = \lambda^2|y|^2 - 2\lambda\phi(x, y) + |x|^2 - r_2^2$, $\lambda \in I$. Because $T(z) \subset P$ we have $\psi(1) > 0$; because $U(z) \not\subset P$ there exists $\lambda_0 \in [0, 1)$ such that $\psi(\lambda_0) \leqslant 0$. The function $\psi(\lambda) = \lambda^2|y|^2 - 2\lambda\phi(x, y) + |x|^2 - r_1^2$, $\lambda \in \boldsymbol{R}$ is the quadratic trinomial with $\Delta = \Delta(\psi) = 4(|y|^2 r_1^2 - \langle x, y\rangle^2) \overset{\mathrm{df}}{=} 4\Delta_1 \geqslant 0$. Because $\psi(1) > 0$ and $\psi(\lambda_0) \leqslant 0$ for $\lambda_0 \in [0, 1)$ then the greatest root of the equation $\psi(\lambda) = 0$ is less than 1, i.e. $\lambda_+ = \dfrac{\phi(x, y) + \sqrt{\Delta_1}}{|y|^2} < 1$.

For $\mu = \dfrac{\langle x, y\rangle}{|y|^2}$ we have $\langle \mu y - x, y\rangle = 0$. We shall demonstrate that it is possible to contract the sphere $T(z)$ in P to a sphere as in (a_3). We can do this, for example, by the mapping $\beta(\tau) = x + \tau(\mu y - x) + iy$, $\tau \in I$. It is easy to obtain that $t_{\substack{+\\1}}^2(\beta(\tau)) = \dfrac{(1-\tau)^2}{|y|^2}\phi^2(x, y) \pm 2(1-\tau)\phi(x, y) + \dfrac{\langle x, y\rangle^2}{|y|^2} + |y|^2$. Whence $t_+(\beta(\tau)) < r_2$, $\tau \in I$, therefore $T(\beta(\tau)) \subset P$ for every $\tau \in I$ if and only if when $t_-(\beta(\tau)) > r_1$ for every $\tau \in I$. Let $\varphi(\tau) = t_-^2(\beta(1-\tau)) - r_1^2$, $\tau \in I$. φ as a function of $\tau \in \boldsymbol{R}$ is the quadratic trinomial with $\Delta = \Delta(\varphi) = \dfrac{\Delta(\psi)}{|y|^4}\phi^2(x, y)$, therefore $\Delta(\varphi) \geqslant 0$. The smaller root of the equation $\varphi(\tau) = 0$ is equal to $\dfrac{|y|^2 - \sqrt{\Delta_1}}{\phi(x, y)}$ and is greater than 1. This completes the proof.

Corollary. $\widetilde{P} = \{z \in \boldsymbol{C}^n : T(z) \subset P\} = \{z \in \boldsymbol{C}^n r_1 < t_-(z) \leqslant t_+(z) < r_2\}$.

3. ANALYTIC CONTINUATION OF HARMONIC FUNCTIONS

All denotations in this section are the same as in Section 2.

From the paper [2] we can deduce the following theorem (see Introduction).

Theorem I. For the region $D \subset \boldsymbol{R}^n$, $n \geqslant 2$ and for every $h \in H(D)$ there exists an analytic function $\widetilde{h}$ on $\widetilde{D}$ which continues h. Moreover, $\widetilde{D}$ is the maximal region with these properties.

We may consider particular simple cases of this theorem.

Theorem 1. For the spatial ring P the set $\widetilde{P}$ is the harmonic envelope of analyticity.

Theorem 2. In the case $n = 2$, for $h \in H(D) h$ has a holomorphic continuation on $\widetilde{D}$ if and only if there exists $f \in \mathcal{O}(D)$ such that $h = Re f$.

Proof. Suppose that $\widetilde{h} \in \mathcal{O}(\widetilde{D})$, $\widetilde{h}|_D = h$. Let us fix $z_0 \in D$ and set $g(z) = 2 Im\, \widetilde{h}\left(\dfrac{z+\bar{z}_0}{2}, \dfrac{z-\bar{z}_0}{2i}\right)$, $z \in D$.

Since $\widetilde{D} = \{(z_1, z_2) \in C^2: z_1 + iz_2, \bar{z}_1 + i\bar{z}_2 \in D\}$ (see lemma 3b, d) then the function g is well defined on D. It is obvious that g a function of class $C^\infty(D)$ as a function of two real variables and it may be checked that $h_x(z) = g_y(z)$, $h_y(z) = -g_x(z)$, $z \in D$. Therefore $f = h + ig \in \mathcal{O}(D)$.

Now, suppose that $h = Re f$, $f \in \mathcal{O}(D)$. For $(z_1, z_2) \in \widetilde{D}$ we define $\widetilde{h}(z_1, z_2) = \frac{1}{2}\big(f(z_1 + iz_2) + \overline{f(\bar{z}_1 + i\bar{z}_2)}\big)$. The function $\widetilde{h}$ is well defined on $\widetilde{D}$, $\widetilde{h}$ is a function of class $C^\infty(\widetilde{D})$ as a function of four real variables and $\widetilde{h}\,|_D = h$. It may be checked that $\widetilde{h}$ satisfies the Cauchy-Riemann equations in $\widetilde{D}$. This completes the proof of Theorem 2.

The following three theorems are given by Lelong in [2].

Theorem 3. In the case $n = 2$, for $h \in H(D)$ the analytic continuation $\widetilde{h}$ of h on $\widetilde{D}$ has singlevalued real part and $Re\widetilde{h}(z_1, z_2) = \frac{1}{2}\big(h(z_1 + iz_2) + h(\bar{z}_1 + i\bar{z}_2)\big)$, $(z_1, z_2) \in \widetilde{D}$.

Theorem 4. In the case $n = 2$ there exists $h_0 \in H(D)$ such that the holomorphic continuation $\widetilde{h}_0$ of h_0 on $\widetilde{D}$ cannot be continued beyond $\widetilde{D}$.

Proof. It is known that there exists $B \subset \partial D$ and $f_0 \in \mathcal{O}(D)$ such that $\bar{B} = \partial D$ and $\lim\limits_{\zeta \to \zeta_0} f_0(\zeta) = \infty$ for every $\zeta_0 \in B$. Let $h_0 = Re f_0$. Hence (see Theorem 2)

$$\widetilde{h}_0(z_1, z_2) = \tfrac{1}{2}\big(f_0(z_1 + iz_2) + \overline{f_0(\bar{z}_1 + i\bar{z}_2)}\big).$$

Let $B^* = \left\{z = z(\zeta_1, \zeta_2) = \left(\dfrac{\zeta_1 + \bar{\zeta}_2}{2}, \dfrac{\zeta_1 - \bar{\zeta}_2}{2i}\right) \in C^2 : \zeta_1 \in B,\ \zeta_2 \in D\right\}$. We know that B^* is dense in $\partial\widetilde{D}$ (see lemma 5c). For every $\zeta_1 \in B$, $\zeta_2 \in D$: $\lim\limits_{(z_1, z_2) \to z(\zeta_1, \zeta_2)} \widetilde{h}_0(z_1, z_2) = \frac{1}{2}\big(\lim\limits_{\zeta \to \zeta_1} f_0(\zeta) + \overline{f_0(\zeta_2)}\big) = \infty$. Whence $\widetilde{h}_0$ cannot be continued beyond $\widetilde{D}$ and the proof is completed.

Theorem 5. In the case $n = 2p \geqslant 4$, every function $h \in H(D)$ may be continued to a holomorphic function $\widetilde{h}$ on $\widetilde{D}$.

Proof. It is known that there exists a sequence $\{D_k\}_{k \in N}$ of regions in R^n such that $D_k \subset D_{k+1}$, $\widetilde{D}_k \subset D$, $k \geqslant 1$, $D = \bigcup\limits_{k=1}^{\infty} D_k$, D_k is bounded and ∂D_k is the sum of a finite number of surfaces of class C^1, $k \geqslant 1$.

Let for $z \in C^n$, $z \neq 0$, $E(z) = \dfrac{1}{(2-n)\theta_n}\big(F(z)\big)^{1-\frac{n}{2}}$, where θ_n denotes the area of the unit $(n-1)$ — dimensional sphere in R^n. It is known that for $x \in D_k$, $k \geqslant 1$, $h(x) = \int\limits_{\partial D_k} \left(h(t)\dfrac{\partial E(x-t)}{\partial \vec{n}} - E(x-t)\dfrac{\partial h(t)}{\partial \vec{n}}\right)\sigma_k(dt)$, where σ_k denotes the $(n-1)$ — dimensional Lebesgue measure on ∂D_k, $\{\vec{n}_t\}_{t \in \partial D_k}$ denotes the field of exterior normal vectors to ∂D_k. For $z \in \widetilde{D}_k$ we define

$$\widetilde{h}_k(z) = \int\limits_{\partial D_k} \left(h(t)\frac{\partial E(z-t)}{\partial \vec{n}} - E(z-t)\frac{\partial h(t)}{\partial \vec{n}}\right)\sigma_k(dt).$$

It is easy to prove that $\widetilde{h}_k \in \mathcal{O}(\widetilde{D}_k)$, $\widetilde{h}_k|_{D_k} = h|_{D_k}$, $k \geqslant 1$. Therefore for $k \leqslant l$ $\widetilde{h}_k = \widetilde{h}_l|_{\widetilde{D}_k}$. As a holomorphic continuation $\widetilde{h}$ of h on $\widetilde{D}$ we can take $\widetilde{h} = \bigcup_{k=1}^{\infty} \widetilde{h}_k$. This completes the proof.

Theorem 6. In the case $n = 2$ $(n = 2p+1,\ p \in N)$ the function $h_0 \in H(P)$ given by formula $h_0(x) = \ln|x|$ $\left(h_0(x) = |x|^{2-n}\right)$, $x \in P$ cannot be holomorphically continued on $\widetilde{P}$, i.e. $\widetilde{P}$ is a harmonic envelope of analyticity but not of holomorphy for P.

Proof. Proof in the case $n = 2$ follows from Theorem 2. In the case $n = 2p+1$ the function h_0 may be holomorphically continued on $\widetilde{P}$ if and only if the square root $\sqrt{F}$ has holomorphic singlevalued branch on $\widetilde{P}$. We shall prove that this impossible. Let $r \in (r_1^2, r_2^2)$, $\theta \in R$, $x = \left(\sqrt{r}\cos\frac{\theta}{2}, 0, \ldots, 0\right)$, $y = \left(\sqrt{r}\sin\frac{\theta}{2}, 0, \ldots, 0\right)$, $z = x + iy$. It is easy to check that $t_-(z) = t_+(z) = \sqrt{r}$, therefore $z \in \widetilde{P}$. By a simple calculation we get $F(z) = re^{i\theta}$. Hence $\{z \in C : r_1^2 < |z| < r_2^2\} \subset F(\widetilde{P})$. So F has not a simplevalued branch of the square root in $\widetilde{P}$. The proof is completed.

Theorem 7. Let D_1, D_2 be two regions in C, $D_1, D_2 \notin C$, $f = u + iv : D_1 \to D_2$ be a biholomorphic mapping between these regions. By $\widetilde{u}, \widetilde{v}$ we denote the holomorphic continuations of u and v (see Theorem 2). Let $\widetilde{f} = (\widetilde{u}, \widetilde{v}) : \widetilde{D}_1 \to C^2$. Then $\widetilde{f}(\widetilde{D}_1) = \widetilde{D}_2$ and $\widetilde{f}$ is biholomorphic.

Proof. We know that for $z = (z_1, z_2) \in \widetilde{D}_1$

$$\widetilde{u}(z) = \frac{1}{2}\left(f(z_1 + iz_2) + \overline{f(\bar{z}_1 + i\bar{z}_2)}\right), \qquad \widetilde{v}(z) = \frac{1}{2i}\left(f(z_1 + iz_2) - \overline{f(\bar{z}_1 + i\bar{z}_2)}\right).$$

Since $T\left(\widetilde{f}(z)\right) \subset f\left(T(z)\right)$, $z \in \widetilde{D}_1$ then $\widetilde{f}(\widetilde{D}_1) \subset \widetilde{D}_2$.

Now, let $w = (w_1, w_2) \in C^2$, $T(w) \subset D_2$ (i.e. $w \in \widetilde{D}_2$), $w_1 + iw_2 = f(\xi_1)$, $w_1 + iw_2 = f(\xi_2)$, $\xi_1, \xi_2 \in D_1$. We define $z = (z_1, z_2) \in C^2$ by formulas $z_1 = \frac{1}{2}(\xi_1 + \bar{\xi}_2)$, $z_2 = \frac{1}{2i}(\xi_1 - \bar{\xi}_2)$. Hence $T(z) = \{\xi_1, \xi_2\} \subset D_1$. So $z \in \widetilde{D}_1$, $w = \widetilde{f}(z) \subset \widetilde{f}(\widetilde{D}_1)$, i.e, $\widetilde{D}_2 \subset \widetilde{f}(\widetilde{D}_1)$.

Let $g : D_2 \to D_1$, $g = f^{-1}$ and $\widetilde{g} : \widetilde{D}_2 \to \widetilde{D}_1$ is defined analogically as the function $\widetilde{f}$. The mappings $\widetilde{f}, \widetilde{g}$ are holomorphic and $\widetilde{f} \circ \widetilde{g}|_{D_2} = id_{D_2}$, $\widetilde{g} \circ \widetilde{f}|_{D_1} = id_{D_1}$. By the principle of identity for holomorphic functions we have $\widetilde{f} \circ \widetilde{g} = id_{\widetilde{D}_2}$, $\widetilde{g} \circ \widetilde{f} = id_{\widetilde{D}_1}$, i.e. $(\widetilde{f})^{-1} = \widetilde{g}$. This completes the proof.

Corollary. If D_1, D_2 are as in Theorem 7 then if $\widetilde{D}_1$ is the harmonic envelope of holomorphy for D_1 then $\widetilde{D}_2$ is the harmonic envelope of holomorphy for D_2.

In particular, we get the following.

Theorem 8. Let $D \subset C$ be a simple connected domain such that ∂D has at least two distinct points. Let $f = u + iv : B_1 \to D$ be the biholomorphic mapping (B_1 denotes the unit disc in C). By $\widetilde{u}, \widetilde{v}$ we denote the holomorphic continuations for u and v, $f = (u, v)$. Then $\widetilde{D} = \widetilde{f}(\widetilde{B}_1)$ is the harmonic envelope of holomorphy for D and the mapping $\widetilde{f} : \widetilde{B}_1 \to \widetilde{D}$ is biholomorphic.

REFERENCES

[1] C. O. Kiselman, *Prolongement des solutions d'une équation aux dérivées partielles à coefficients constants*, Bull. Soc. Math. France 97 (4) (1969), 329—356.

[2] P. Lelong, *Prolongement analytique et singularités complexes des fonctions harmoniques*, Bull. Soc. Math. Belgique 7 (1) (1954), 10—23.

[3] J. Siciak, *Holomorphic continuation of harmonic functions*, Ann. Pol. Math. XXIX (1974), 67—73.