Marek Jarnicki

Analytic Continuation of Pluriharmonic Functions

ABSTRACT

In this paper we shall present a construction of the pluriharmonic envelope of analyticity and of holomorphy for a region in \( \mathbb{R}^{2n} \). We shall prove that this envelope is "invariant" with respect to biholomorphic transformations of the region. We shall also construct the polyharmonic envelope of analyticity and of holomorphy for a polycylindrical region.

INTRODUCTION

Let \( U \) be an open set in \( \mathbb{R}^n \); by \( A(U) \) we denote the space of all real analytic functions of \( n \) real variables on \( U \).

Let \( \Omega \) be open in \( \mathbb{C}^n \); by \( \mathcal{O}(\Omega) \) we denote the space of all holomorphic functions on \( \Omega \).

By \( \mathcal{O}_n \) we denote the sheaf of germs of holomorphic functions on \( \mathbb{C}^n \), and by \( \pi_n \) the natural projection \( \mathcal{O}_n \rightarrow \mathbb{C}^n \).

Every region (i.e. a non-empty, open and connected set) in \( \mathcal{O}_n \) will be called an analytic function.

An analytic function \( F \subset \mathcal{O}_n \) will be called arbitrarily continuable if:

\[(\ast) \ \forall z \in \pi_n(F), \ \forall F_z \in \pi^{-1}_n(z) \cap F \ \text{and} \ \text{for every continuous mapping} \ \gamma: I = [0, 1] \rightarrow \pi_n(F) \ \text{such that} \ \gamma(0) = z, \ \text{there exists a continuous mapping} \ \hat{\gamma}: I \rightarrow F \ \text{such that} \ \hat{\gamma}(0) = F_z \ \text{and} \ \pi_n \circ \hat{\gamma} = \gamma.\]

It is known that \((\ast)\) is equivalent to

\[(\ast\ast) \ \forall z \in \pi_n(F), \ \forall F_z \in \pi^{-1}_n(z) \cap F, \ \forall \varphi \in F_z: \ \varphi \ \text{may be holomorphically extended on every polydisc} \ P(z; r) \subset \pi_n(F). \]

Fix a region \( D \) in \( \mathbb{R}^n, n \geq 2 \), and a vector subspace \( S \) in \( A(D) \). We consider the two following problems:

\[(A) \ \text{Whether there exists a set} \ \Omega \ \text{in} \ \mathbb{C}^n \ \text{such that:}

(\text{A1}) \ \Omega \ \text{is a connected domain of holomorphy containing} \ D; \]

\[(A2) \ \text{for every} \ f \in S \ \text{there exists an arbitrarily continuable analytic function} \ F \ \text{over} \ \Omega \ \text{(i.e.} \ \pi_n(F) = \Omega) \ \text{such that} \ \forall x \in D \ \text{the germ} \ f_x \ \text{belongs to} \ F;\]
there exists a function \( f_0 \in S \) such that its continuation \( F_0 \) (in the sense of (A2)) has the following property:

\[
\forall z \in \Omega, \; \forall (F_0)_z \in \pi^{-1}(z) \cap F_0, \; \forall \varphi \in (F_0) : \varphi \text{ cannot be holomorphically extended on any polydisc } P(z; R) \text{ if } P(z; R) \cap \Omega \neq \emptyset.
\]

(H) Whether there exists a set \( \Omega \) in \( C^n \) such that:

(H1) \( \Omega \) satisfies (A1);

(H2) \( \forall f \in S \exists \hat{f} \in \Theta(\Omega) : \hat{f}|_\partial = f \);

(H3) there exists \( f_0 \in S \) such that its holomorphic continuation \( \hat{f}_0 \) on \( \Omega \) cannot be holomorphically continued beyond \( \Omega \).

Remarks. The function \( F \) in (A2) is uniquely determined by \( f \).

The solution of (A) is uniquely determined by \( D \) and \( S \), so if \( \Omega \) satisfies (A), we write \( \Omega = D^A_S \) and we call \( D^A_S \) the \( S \) — envelope of analyticity for \( D \).

Similarly, the solution of (H) is uniquely determined by \( D \) and \( S \), we write \( \Omega = D^H_S \) and we call \( D^H_S \) the \( S \) — envelope of holomorphy for \( D \).

If (H) has the solution, then (A) has the solution and \( D^A_S = D^H_S \).

If (A) has a solution which satisfies (H2), then (H) has the solution and \( D^H_S = D^A_S \).

If (A) has a solution which is homotopically simply connected, then \( D^A_S \) satisfies (H2).

If \( \Omega \) satisfies (A1), (A2) and (H3) then \( \Omega = D^A_S \).

It is possible to show that for \( S = A(D) \) the problem (A) has not any solution. On the other hand, if \( S \) is too small (for example, if \( S = R[x_1, ..., x_n]|_\partial \)), then (H) has the only solution \( D^H_S = C^n \); this case is not interesting.

We shall present some solutions of (A) and (H) for particular cases of \( S \).

1° \( S = H(D) \) — the space of all real harmonic functions on \( D \). P. Lelong proved that here the answer to the problem (A) is always positive and that in the case \( n = 2p \geq 4 \), \( p \in N \), the answer to the problem (H) is also positive. More exactly we have

Lelong’s theorem [4]. Given \( z = (z_1, ..., z_n) \in C^n \), put

\[
T(z) = \{ t = (t_1, ..., t_n) \in R^n : \sum_{j=1}^{n} (t_j - z_j)^2 = 0 \}.
\]

Set \( D = \{ z \in C^n : \exists a \in D, \exists \gamma : I \rightarrow C^n \text{ such that } \gamma \text{ is continuous, } \gamma(0) = a, \gamma(1) = z \text{ and } \forall \tau \in I : T(\gamma(\tau)) \subset D \} \).

Then \( D = D^A(D) \) and in the case \( n = 2p \geq 4 \) \( D = D^H_{H(D)} \) (see [4], theorems 2, 4 and 6).

Note that in the cases \( n = 2 \) and \( n = 2p + 1, p \in N \), there exist examples of regions \( D \) for which \( D \) is not any solution of (H) (see [4], p. 15, also [2] theorem 6).

2° \( S = H_L(D) \) — the space of all solutions of a linear elliptic differential operator \( L \) with constant coefficients. C. O. Kiselman in [3] proved that for every convex region \( D \) there exists a maximal convex region \( \Omega \) in \( C^n \) which satisfies (H1) and (H2).

In Section 1 of this paper we answer the problems (A) and (H) for a region \( D \subset R^{2n} \) and for \( S = PH(D) \) — the space of all pluriharmonic functions on \( D \). This will be an extension of Lelong’s theorem and of theorems 2, 3, 4, 7 from [2] (see also [4], p. 17). In Section 2 we consider the problems (A) and (H) for the space \( H_0(D) \) consisting of all polyharmonic functions of the given type on a polycylindrical region \( D \).
1. ANALYTIC CONTINUATION OF PLURIHARMONIC FUNCTIONS

In this section $D$ denotes a region in $\mathbb{R}^{2n}$. For $z \in C^k$ and the positive numbers $r_1, \ldots, r_k$, by $P(z; r_1, \ldots, r_k)$ we denote the polydisc in $C^k$ with the center $z$ and the radii $r_1, \ldots, r_k$. If $r_1 = \ldots = r_k = r$ we write $P(z; r; k)$ instead of $P(z; r, \ldots, r)$.

For $z = (z_1, z_2, \ldots, z_{2n-1}, z_{2n}) \in C^{2n}$ set

$$
\phi(z) \overset{\text{def}}{=} (z_1 + iz_2, \ldots, z_{2n-1} + iz_{2n}) \in C^n.
$$

Let

$$
\hat{D} = \{z \in C^{2n}: \phi(z), \phi(\bar{z}) \in D\};
$$

we identify $R^{2n}$ with $C^n$.

Remarks. $\hat{D}$ is a region in $C^{2n}$ symmetric with respect to the mapping $C^{2n} \ni z \mapsto \bar{z} \in C^{2n}$, $\hat{D} \cap R^{2n} = D$; we identify $R^{2n} \times \{0\} \subset C^{2n}$ with $C^n$.

$D$ is starlike with respect to $\xi \in D$ if and only if $\hat{D}$ is starlike with respect to $\xi$.

$D$ is convex if and only if $\hat{D}$ is convex.

$D$ is homotopically simply connected if and only if $\hat{D}$ is homotopically simply connected.

The following theorem (analogical to Lelong's theorem) plays the fundamental role in our considerations.

Theorem 1. $\hat{D}$ satisfies $(A2)$ for $S = PH(D)$, moreover for every $f \in PH(D)$ the analytic arbitrarily continuable continuation $F$ of $f$ over $\hat{D}$ has the single-valued real part on $\hat{D}$ and

$$
\text{Re} F(z) = \frac{1}{2} \{f(\phi(z)) + f(\phi(\bar{z}))\}, z \in \hat{D}.
$$

Proof. Let $f \in PH(D)$ be fixed. Locally in $D$, $f$ is the real part of a holomorphic function, so there exists an analytic arbitrarily continuable function $G \subset \theta_n$ over $D$ such that $f = \text{Re} G$.

We shall give a construction of the continuation of $f$ over $\hat{D}$. Let $z \in \hat{D}$, take $G_\phi(z) \in \pi_n^{-1}(\phi(z)) \cap G$ and $G_\phi(\bar{z}) \in \pi_n^{-1}(\phi(\bar{z})) \cap G$. Let $\varphi \in G_\phi(z)$, $\varphi \in \partial P(\phi(z); q; n)$, $P(\phi(z); q; n) \subset D$, $\psi \in G_\phi(\bar{z})$, $\psi \in \partial P(\phi(\bar{z}); q; n)$, $P(\phi(\bar{z}); q; n) \subset D$. Set

$$
\lambda(w) = \frac{1}{2} \{\varphi(\phi(w)) + \overline{\psi(\phi(w))}\}, w \in P(z; 1/q; 2n).
$$

$P(z; 1/q; 2n) \subset \hat{D}$, so $\lambda$ is well defined and $\lambda \in \partial P(z; 1/q; 2n))$. We take the germ $\lambda_w$ of $\lambda$ at $w \in P(z; 1/q; 2n)$. Now we change, if possible, $w \in P(z; 1/q; 2n)$, $G_\phi(z) \in \pi_n^{-1}(\phi(z)) \cap G$, $G_\phi(\bar{z}) \in \pi_n^{-1}(\phi(\bar{z})) \cap G$ and $z \in \hat{D}$. The set of all germs of the type $\lambda_w$, obtained in this way, we denote by $F$. It is obvious that $F$ is an arbitrarily continuable analytic function over $\hat{D}$ which extends $f$. Since $f = \text{Re} G$, (4) implies (3). This completes the proof.

The mapping $C^n \ni z \rightarrow (\phi(z), \phi(\bar{z})) \in C^n \times C^n$ is a homeomorphism and its inverse mapping $\Lambda$ is given by the formula

$$
C^n \times C^n \ni (\xi = (\xi_1, \ldots, \xi_n), \eta = (\eta_1, \ldots, \eta_n)) \mapsto 
\left(\frac{\xi_1 + \bar{\eta}_1}{2}, \frac{\xi_1 - \bar{\eta}_1}{2i}, \ldots, \frac{\xi_n + \bar{\eta}_n}{2}, \frac{\xi_n - \bar{\eta}_n}{2i}\right) \in C^{2n}.
$$
Analogously to Theorem 2 in [2], we can prove the following.

**Lemma 1.** A function \( h \in \mathcal{D}(D) \) has a holomorphic continuation \( \hat{h} \) on \( \hat{D} \) if and only if there exists \( f \in \mathcal{D}(D) \) such that \( h = \text{Re} f \), moreover:

\[
(6) \quad f(\xi) = h(\xi) + i(2\text{Im} \hat{h}(A(\xi, \eta)) + \text{const.}), \quad \xi \in D; \quad \eta \in D \text{ fixed};
\]

\[
(7) \quad \hat{h}(z) = \frac{1}{2} \left( f(\phi(z)) + \overline{f(\phi(\bar{z}))} \right), \quad z \in \hat{D}.
\]

**Proposition 1.** If \( D \) is homotopically simply connected then \( \hat{D} \) satisfies (H2) for \( \hat{S} = \mathcal{D}(D) \).

**Proof.** If \( D \) is homotopically simply connected then every function from \( \mathcal{D}(D) \) is the real part of a holomorphic function from \( \mathcal{D}(D) \), so we can use Lemma 1.

The following theorem is analogous to Theorem 7 in [2].

**Theorem 2.** Let \( D, \ G \) be regions in \( \mathbb{C}^n \), \( f = (f_1, \ldots, f_n) : D \to G \) be biholomorphic; \( f_k = u_k + iv_k \), \( u_k, v_k \) denote the corresponding holomorphic continuations of \( u_k \) and \( v_k \) on \( \hat{D} \), \( k = 1, \ldots, n \); \( \hat{f} \) defines \( \hat{u}_k, \hat{v}_k, \hat{u}_k, \hat{v}_k : \hat{D} \to \mathbb{C}^{2n} \). Then \( \hat{f}(\hat{D}) = \hat{G} \) and \( \hat{f} : \hat{D} \to \hat{G} \) is biholomorphic.

**Proof.** The proof is analogous as in the case \( n = 1 \).

By Lemma 1:

\[
\hat{u}_k(z) = \frac{1}{2i} \left( f_k(\phi(z)) + \overline{f_k(\phi(\bar{z}))} \right), \quad z \in \hat{D}, \quad k = 1, \ldots, n.
\]

Set, for \( z \in \mathbb{C}^{2n} \),

\[
(8) \quad \hat{\mathcal{T}}(z) = \{ \phi(z), \phi(\bar{z}) \}.
\]

It is easy to show that for every \( z \in \hat{D} \), \( \hat{\mathcal{T}}(\hat{f}(z)) = f(\hat{\mathcal{T}}(z)) \), so \( \hat{f}(\hat{D}) \subset \hat{G} \).

Now, let \( w \in \hat{G} \) be fixed. There exist \( \xi, \eta \in D \) such that \( \phi(w) = f(\xi), \phi(\bar{w}) = f(\eta) \), so \( w = \hat{f}(A(\xi, \eta)), \) (see (5)), hence \( \hat{G} \subset \hat{f}(\hat{D}) \).

For the mapping \( g = \hat{f}^{-1} : \hat{D} \to D \) we construct \( \hat{g} \) (in the same way as \( \hat{f} \) for \( f \)). Then \( \hat{f}, \hat{g} \) are holomorphic, \( \hat{f} \circ \hat{g} = \text{id}_D, \hat{g} \circ \hat{f} = \text{id}_\hat{D} \), so \( \hat{g} = (\hat{f})^{-1} \). This completes the proof.

**Corollary 1.** Let \( \hat{D}, \ G, \ f \) be as in Theorem 2. Then \( \hat{D} \) satisfies (H2) if and only if \( \hat{G} \) satisfies (H2).

If \( \hat{D} \) satisfies (H2), \( \hat{D} \) need not satisfy (H3). For example, if we take \( D = \Omega \setminus K \) such that:

(a) \( \Omega \) is a region in \( \mathbb{C}^n \),
(b) \( K \subset \Omega, \ K \) is a non-empty compact set,
(c) \( D \) is homotopically simply connected, then \( \hat{D} \) satisfies (H2) for \( \mathcal{D}(D) \) but \( \hat{D} \not\subset \hat{\Omega} \).

Now we shall discuss situations when \( \hat{D} \) is the solution of (A) or (H).

**Theorem 3.** If \( D \) is a domain of holomorphy in \( \mathbb{C}^n \) then \( \hat{D} \) is the solution of (A).

**Proof.** By Theorem 1 it suffices to show that \( \hat{D} \) satisfies (H3).
Let \( f \in \Phi(D) \) be a function which cannot be holomorphically continued beyond \( D \). Let \( \bar{h} \) be given by the formula (7). It suffices to show that \( \bar{h} \) cannot be holomorphically continued beyond \( \bar{D} \).

Suppose that there exist \( z \in \bar{D}, r > 0 \) and \( \varphi \in \Phi(P(z; r; 2n)) \) such that \( P(z; r; 2n) \not\equiv \phi \) and \( \varphi \) is equal to \( \bar{h} \) in a neighbourhood of \( z \). It is easy to prove that \( \phi(P(z; r; 2n)) = P(\phi(z); 2r; n), \psi(P(z; r; 2n)) = P(\psi(z); 2r; n), \) where \( \psi(z) = \phi(z) \), \( z \in C^{2n} \). We have \( P(\phi(z); 2r; n) \not\equiv \phi \) or \( P(\psi(z); 2r; n) \not\equiv \phi \); suppose, for example, that \( P(\phi(z); 2r; n) \not\equiv \phi \).

Set \( g(\xi) = 2\varphi(\Lambda(\xi, \phi(z))) - f(\phi(z)), \xi \in P(\phi(z); 2r; n) \). \( g \) is well defined, \( g \in \Phi(P(\phi(z); 2r; n)) \) and \( g \) is equal to \( f \) in a neighbourhood of \( \phi(z) \). Since \( f \) cannot be continued beyond \( D \), this gives a contradiction. This completes the proof.

Conversely, we have

**Theorem 4.** If \( \bar{D} \) satisfies (A) then \( D \) is a domain of holomorphy.

**Proof.** We shall use the following well known theorem (see [1], Theorem 2.5.14):

Let \( \Omega \) and \( \Omega' \) be holomorphy domains in \( \mathbb{C}^m \) and in \( \mathbb{C}^n \), respectively, and let \( u \) be a holomorphic map of \( \Omega \) into \( \mathbb{C}^n \). Then \( \Omega_u = \{ \xi \in \Omega: u(\xi) \in \Omega' \} \) is a domain of holomorphy.

In our situation we set, for fixed \( \eta \in D, m = 2n, \Omega = \mathbb{C}^n, \Omega' = \bar{D}, u(\xi) = \Lambda(\xi, \eta), \xi \in \mathbb{C}^n \). Then \( \Omega_u = D, \) so \( D \) is a domain of holomorphy. The proof is completed.

Note that if we put \( \Omega = \mathbb{C}^{2n}, \Omega' = \mathbb{C} \times \mathbb{C}, u(z) = (\phi(z), \phi(\xi)), z \in C^{2n} \), where \( D^* = \{ \xi \in \mathbb{C}^n: \xi \in D \} \), then from the assumption that \( D \) is a domain of holomorphy, we may deduce that \( \bar{D} \) is a domain of holomorphy. Hence the essential meaning of Theorem 3 is such that \( \bar{D} \) is a domain of holomorphy with respect to the space of all holomorphic functions in \( D \) which are the continuations of functions from \( PH(D) \).

Theorems 1, 3 and 4 imply

**Corollary 2.** \( \bar{D} \) is the solution of (A) if and only if \( D \) is a domain of holomorphy.

**Corollary 3.** \( \bar{D} \) is the solution of (H) if and only if \( D \) is a domain of holomorphy and

\[
\forall h \in PH(D) \exists \eta \in \Phi(D): h = Re f.
\]

**Corollary 4.** If \( D \) is not any domain of holomorphy, \( D \) satisfies (R) and the envelope of holomorphy \( \Omega \) of \( D \) is univalent, then \( \bar{D} \) is the solution of (H) for \( PH(D) \).

Directly from the definitions of \( T \) (see Lelong's theorem) and — of \( \bar{T} \) (see (8)) we have:

\[
T(z) \sqsubset T(z_1, z_2)^{n-1} \times T(z_{2n-1}, z_{2n}) \subset \bar{T}(z), \ z = (z_1, z_2, ..., z_{2n-1}, z_{2n}) \in \mathbb{C}^{2n}.
\]

Whence \( \bar{D} \subset \bar{D} \) (more exactly — \( \{ \xi \in \mathbb{C}^{2n}: T(z) \subset \bar{D} \} \subset \bar{D} \) and for \( D = D_1 \times \ldots \times D_n, D_i — \)

a region in \( \mathbb{C}, i = 1, ..., n, \bar{D} = \bar{D}_1 \times \ldots \times \bar{D}_n \) (in the case \( n = 1: \bar{D} = \bar{D} \)).

Below we shall give an example of a situation when \( \bar{D} \not\subset \bar{D} \). Let \( D = B = \{ \xi \in \mathbb{R}^{2n}: |\xi| < r \} — \)

the ball in \( \mathbb{R}^{2n}, n \geq 2 \). It is possible to show that \( \bar{B} = \{ z \in \mathbb{C}^{2n}: t(z) < r \} \), where for \( z = x + iy \in \mathbb{C}^{2n} \): \( t(z) = (|x|^2 + |y|^2 + 2 \sqrt{|x|^2|y|^2 - \langle x, y \rangle^2})^{1/2} \), see [3], [5] also [2]. Let \( \theta \in (2, 2 \sqrt{2}), z = \frac{r}{\theta} ((1, 1, 0, ..., 0) + i(0, 0, 1, 1, 0, ..., 0)) \). It is easy to check \( z \in \bar{B} \).
2. ANALYTIC CONTINUATION OF POLYHARMONIC FUNCTIONS

Fix \( k \in \mathbb{N} \), \( \alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k \) and let \( \Omega \) be an open set in \( \mathbb{R}^|\alpha| = \mathbb{R}^{\alpha_1} \times \cdots \times \mathbb{R}^{\alpha_k} \).

The function \( u : \Omega \to \mathbb{R} \) is called \( \alpha \)-polyharmonic if for every \( a = (a_1, \ldots, a_k) \in \Omega \) the function \( x_i \mapsto u(a_1, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots, a_k) \) is harmonic in a neighbourhood of \( a_i \), \( i = 1, \ldots, k \).

By \( H_\alpha(\Omega) \) we denote the space of all \( \alpha \)-polyharmonic functions on \( \Omega \).

We consider the problems \((A)\) and \((H)\) for \( S = H_\alpha(D) \), where \( D = D_1 \times \cdots \times D_k \), \( D_i \) is a region in \( \mathbb{R}^{\alpha_i} \), \( i = 1, \ldots, k \).

First, note that in this case we can reduce the problem to the case \( \alpha_i \geq 2 \), \( i = 1, \ldots, k \).

Further we always make this assumption.

The main result of this section is the following

**Theorem 5.** The set \( \overline{D}_1 \times \cdots \times \overline{D}_k \) is the \( \alpha \)-polyharmonic envelope of analyticity for \( D \).

Moreover, if \( \alpha_i = 2p_i \geq 4 \), \( p_i \in \mathbb{N} \), \( i = 1, \ldots, k \), then \( \overline{D}_1 \times \cdots \times \overline{D}_k \) is the \( \alpha \)-polyharmonic envelope of holomorphy for \( D \).

**Proof.** Obviously \( \overline{D}_1 \times \cdots \times \overline{D}_k \) is a domain of holomorphy, so \((A)I = (H)I\) is satisfied.

By iteration of the classical integral representation with the Newton kernel for harmonic functions we obtain an integral representation for \( \alpha \)-polyharmonic functions; more exactly we get the following

**Lemma 2.** Let \( E \) denote the Newton kernel in \( \mathbb{R}^{\alpha_i} \). Set \( E(x) = E_1(x_1) \cdots E_k(x_k) \), \( x = (x_1, \ldots, x_k) \in \mathbb{R}^{|\alpha|} \), \( x_i \neq 0 \), \( i = 1, \ldots, k \). Let \( G_i \) be a region in \( \mathbb{R}^{\alpha_i} \) such that \( G_i \subseteq D_i \), \( \partial G_i \) is the union of a finite number of surfaces of class \( C^1 \), \( i = 1, \ldots, k \). Let \( f \in H_\alpha(D) \).

Then, for every \( x = (x_1, \ldots, x_k) \in G_1 \times \cdots \times G_k \):

\[
f(x) = \int_{\partial G_1} \cdots \int_{\partial G_k} W_\alpha(f, x, t_1, \ldots, t_k) \sigma_1(dt_1) \cdots \sigma_k(dt_k),
\]

where \( \sigma_i \) denotes the \((\alpha_i - 1)\) — dimensional Lebesgue measure on \( \partial G_i \), \( i = 1, \ldots, k \);

\[
W_\alpha(f, x_1, \ldots, x_k, t_1, \ldots, t_k) = \sum_{I,J} (-1)^p \frac{\partial^p E(x_1 - t_1, \ldots, x_k - t_k)}{\partial n_{t_1} \cdots \partial n_{t_k} / \partial n_{t_p}},
\]

where \( I = (i_1, \ldots, i_p), J = (j_1, \ldots, j_p), I \cap J = \emptyset, p + r = k, \{n_{t_i}\}_{t_i \in \partial G_i} \) denotes the field of exterior normal vectors to \( \partial G_i \), \( i = 1, \ldots, k \).

Having this representation, in the proof that every \( \alpha \)-polyharmonic function on \( D \) may be continued to arbitrarily continuous analytic (or, in the case \( \alpha_i = 2p_i \geq 4 \), \( i = 1, \ldots, k \), to holomorphic) function on \( \overline{D}_1 \times \cdots \times \overline{D}_k \), we can apply (with only formal changes) the method of [4]. Hence \( \overline{D}_1 \times \cdots \times \overline{D}_k \) satisfies \((A)2\) (or \((H)2\)).

Let \( f_i \in H(D_i) \) satisfy \((A)3\) for \( S_i = H(D_i) \), \( i = 1, \ldots, k \). Then the function \( f(x) = f_1(x_1), \ldots, f_k(x_k) \), \( x = (x_1, \ldots, x_k) \in D \), is \( \alpha \)-polyharmonic on \( D \). Let \( F_i \in \mathcal{O}_{\alpha_i} \) be an arbitrarily continuous continuation of \( f_i \) over \( \overline{D}_i \), \( i = 1, \ldots, k \); let \( z = (z_1, \ldots, z_k) \in \overline{D}_1 \times \cdots \times \overline{D}_k \), \( (F_i)_{a_i} \in \pi_{a_i}^{-1}(z_i) \cap F_i \), \( \phi_1 \in (F_i)_{a_i} \), \( \phi_i \in \mathcal{O}(U_i), z_i \in U_i = U_i^\circ \subseteq D_i, i = 1, \ldots, k \).

Set \( \varphi(z) = \varphi_1(w_1) \cdots \varphi_k(w_k), w = (w_1, \ldots, w_k) \in U = U_1 \times \cdots \times U_k \); \( \varphi \in \mathcal{O}(U) \). We take the germ \( \varphi_\omega \) of \( \varphi \) at \( w \). Now we change \( w \in U \), \( (F_i)_{a_i} \in \pi_{a_i}^{-1}(z_i) \cap F_i \) and \( z \in \overline{D}_1 \times \cdots \times \overline{D}_k \).
The set of all the germs, obtained in this way, we denote by $F$. Obviously, $F$ is an arbitrarily
continuabile continuation of $f$ over $\bar{D}_1 \times \ldots \times \bar{D}_k$, which satisfies
(A3) for $S = H_+(D)$. In the case $z_i = 2p_i \geq 4$, the proof of (H3) is analogical. The proof is completed.

**Corollary 5.** If $z_1 = \ldots = z_k = 2$, then
$D_{PH(D)}^A = \bar{D}_1 \times \ldots \times \bar{D}_k = \bar{D} = D_{PH(D)}^A$;
if, moreover, $D_i$ is simply connected, $i = 1, \ldots, k$, then
$D_{PH(D)}^I = \bar{D}_1 \times \ldots \times \bar{D}_k = \bar{D} = D_{PH(D)}^I$.

Note that if $k \geq 2$ then $PH(D) \not\subseteq H_+(D)$.

It is easy to show that for $z = (z_1, \ldots, z_k) \in C^{[a]}$: $T(z_1) \times \ldots \times T(z_k) \subset T(z)$, so
$\bar{D} = \bar{D}_1 \times \ldots \times \bar{D}_k$. Below, we shall give an example of a situation when $\bar{D} \not\subseteq \bar{D}_1 \times \ldots \times \bar{D}_k$.

Let $k = z_1 = z_2 = 2$, let $D_0$ be a region in $C$ such that $1 + i, -1 + i, -1 - i, 1 - i \in D_0$,
but $1 \notin D_0$. Set $D = D_0 \times D_0$. Then the point $z = (1 + \sqrt{5}, -1 + \sqrt{5}, 1, 0) \in T(z)$.

Theorem 5 implies the following

**Proposition 2.** In the general case, if $D$ is only a region in $R^{[a]}$ (not necessarily poly-
cylindrical) and $z_i \geq 2$, $i = 1, \ldots, k$, then the set \( \bigcup \bar{D}_1 \times \ldots \times \bar{D}_k \), where $D_i$ is a
convex region in $R^{[a]}$, is the region in $C^{[a]}$ containing $D$ and satisfying (H2) for $H_+(D)$.

**REFERENCES**


zeszyt 17 (1975), 93–104.

