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Holomorphic functions with bounded growth on Riemann domains over $\mathbf{C}^n$

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Abstract. The aim of this paper is to extend some results of the theory of δ-tempered holomorphic functions in pseudoconvex domains in $\mathbf{C}^n$ (cf. [3]) to the case of Riemann-Stein domains over $\mathbf{C}^n$.

1. Introduction

First we fix the following denotations:

$\mathscr{R}_n$: = the class of all *Riemann domains over* $\mathbf{C}^n$;

$\mathscr{R}_n^c$: = the class of all *connected* Riemann domains over $\mathbf{C}^n$;

$\mathscr{R}_n^\infty$: = the class of all *countable at infinity* Riemann domains over $\mathbf{C}^n$;

if $(X, p) \in \mathscr{R}_n$, $x \in X$, $r>0$ then $\hat{B}(x, r)$ (resp. $\hat{P}(x, r)$) denotes an open neighbourhood of x which is mapped homemorphically by p onto the Euclidean ball $B(p(x), r) \subset \mathbf{C}^n$ (resp.-onto the polydisc $P(p(x), r) \subset \mathbf{C}^n$);

$\varrho_X(x)$: $= \sup\{r>0\colon \hat{B}(x, r) \text{ exists}\}$; $\quad \delta_X$: $= \min\{(1+|p|^2)^{-1/2}, \varrho_X\}$;

$d_X(x)$: $= \sup\{r>0\colon \hat{P}(x, r) \text{ exists}\}$;

$d_X(A)$: $= \inf\{d_X(x)\colon x \in A\}$ $(A \subset X)$;

$\|f\|_A$: $= \sup\{|f(x)|\colon x \in A\}$ $(f\colon X \to \mathbf{C}, A \subset X)$;

$\mathscr{O}(X)$: = the space of all *holomorphic functions* on X $((X, p) \in \mathscr{R}_n)$;

$PSH(X)$: = the class of all *plurisubharmonic* (psh.) *functions* on X $((X, p) \in \mathscr{R}_n)$;

$\hat{K}_S$: $= \{x \in X\colon \forall f \in S\colon |f(x)| \leqslant \|f\|_K\}$ $(K \subset X, S \subset \mathscr{O}(X))$;

∂^α: = the differential operator on $\mathscr{O}(X)$ given by the formula:

$$\partial^\alpha f(x) := \frac{\partial^{|\alpha|}(f \circ p_x^{-1})}{\partial z_1^{\alpha_1} \ldots \partial z_n^{\alpha_n}}(p(x)), \qquad \alpha = (\alpha_1, \ldots, \alpha_n) \in Z_+^n, f \in \mathscr{O}(X), x \in X,$$

where p_x: $= p|\hat{B}(x, \varrho_X(x))$;

S^*: $= \{\partial^\alpha f\colon \alpha \in Z_+^n, f \in S\}$ $(S \subset \mathscr{O}(X))$; *a family* $S \subset \mathscr{O}(X)$ *will be called* ∂*-stable if* $S^* = S$;

if $(X, p) \in \mathscr{R}_\infty^n$ then $d\mu = d\mu_X$ denotes *the element of volume on* X defined by the form $(2i)^{-n} d\bar{p}_1 \wedge \ldots \wedge d\bar{p}_n \wedge dp_1 \wedge \ldots \wedge dp_n$ (cf. [7], § 2.9);

τ_n: $= \dfrac{(2\pi)^n}{2n!}$ = the volume of the unit ball in $\mathbf{C}^n$.

Now we give some basic definitions related to the theory of δ-tempered holomorphic functions on Riemann domains (comp. [3], §§ 1.1, 1.2).

Let $(X, p) \in \mathscr{R}_n$ be fixed.

Definition 1. *Let $\delta\colon X \to [0, +\infty)$ be such that the set $X^\delta := \{x \in X\colon \delta(x) > 0\}$ is open and non-empty. A function $f \in \mathcal{O}(X^\delta)$ is said to be a δ-tempered holomorphic function on X of degree $\leqslant r$ $(f \in \mathcal{O}^{(r)}(\delta))$ if $\delta^r f$ is a bounded function on X^δ $(r \geqslant 0)$; let us put $\mathcal{O}(\delta) := \bigcup_{r \geqslant 0} \mathcal{O}^{(r)}(\delta) =:$ the class of all δ-tempered holomorphic functions on X.*

Definition 2. *A function $\delta\colon X \to [0, +\infty)$ is called a Lipschitz function on X $(\delta \in L(X))$ if*

(L1) $\delta \leqslant \varrho_X$,

(L2) $|\delta(x) - \delta(x')| \leqslant |p(x) - p(x')|$, $x \in X$, $x' \in \hat{B}(x, \varrho_X(x))$.

Definition 3. *A function $\delta\colon X \to [0, +\infty)$ is called a weight function on X $(\delta \in W(X))$ if*

(W1) $\delta \leqslant \delta_X$,

(W2) $\delta \in L(X)$.

Note that if $\delta \in L(X)$ (resp. $W(X)$) then $\delta|_{X^\delta} \in L(X^\delta)$ (resp. $W(X^\delta)$).

Example: $(X, p) \in \mathscr{R}_n$, $\delta = \delta_X$. Functions in $\mathcal{O}(\delta_X)$ are called *holomorphic functions with polynomial growth on X* (note that $X^{\delta_X} = X$). It may easily be verified that $\delta_X \in W(X)$.

2. General properties of δ-tempered holomorphic functions

Directly from Definition 1 we get the following:

Remark 1. (comp. [3], §§ 1.1, 2.1)

(i) *$\mathcal{O}^{(r)}(\delta)$ is a complex vector space;*

(ii) *the function $\mathcal{O}^{(r)}(\delta) \ni f \to \|\delta^r f\|_{X^\delta} \in [0, +\infty)$ is a norm on $\mathcal{O}^{(r)}(\delta)$;*

(iii) *if δ is lower semi-continuous on X then for every compact set $K \subset X^\delta$*

$$\|f\|_K \leqslant (\min_K \delta)^{-r} \|\delta^r f\|_{X^\delta}, \quad f \in \mathcal{O}^{(r)}(\delta),$$

in particular, $\mathcal{O}^{(r)}(\delta)$ with the norm given in (ii) *is a Banach space;*

(iv) $\mathcal{O}^{(r)}(\delta)\mathcal{O}^{(s)}(\delta) \subset \mathcal{O}^{(r+s)}(\delta)$;

(v) *if δ is bounded then $\mathcal{O}^{(r)}(\delta) \subset \mathcal{O}^{(s)}(\delta)$, $0 \leqslant r \leqslant s$, in particular, $\mathcal{O}(\delta)$ is a complex algebra and $\mathcal{O}(\delta) = \bigcup_{N=1}^{\infty} \mathcal{O}^{(N)}(\delta)$.*

Below we shall prove some fundamental properties of δ-tempered holomorphic functions.

PROPOSITION 1. *Let $(X, p) \in \mathscr{R}_n^\infty$ and let $\delta\colon X \to [0, +\infty)$ be lower semi-continuous. Then $\mathcal{O}^{(r)}(\delta)$ is of the first Baire category in $\mathcal{O}(X^\delta)$ ($\mathcal{O}(X^\delta)$ with the topology of almost uniform convergence in X^δ is a Fréchet algebra).*

Note that if δ is moreover bounded then, in view of Remark 1 (v), *$\mathcal{O}(\delta)$ is of the first Baire category in $\mathcal{O}(X^\delta)$.*

Proof. We may assume that $X^\delta = X$. For $m \in \mathbf{N}$, set

$$\mathcal{O}_m^{(r)}(\delta) := \{f \in \mathcal{O}^{(r)}(\delta) : \|\delta^r f\|_X \leqslant m\}.$$

Obviously $\mathcal{O}_m^{(r)}(\delta)$ is a closed subset of $\mathcal{O}(X)$, $m \in \mathbf{N}$, and $\mathcal{O}^{(r)}(\delta) = \bigcup_{m=1}^{\infty} \mathcal{O}_m^{(r)}(\delta)$. Hence it is sufficient to prove that the interior of $\mathcal{O}_m^{(r)}(\delta)$ in $\mathcal{O}(X)$ is empty, $m \in \mathbf{N}$. Suppose by absurd that there exists $m \in \mathbf{N}$ such that $\mathcal{O}_m^{(r)}(\delta)$ has a non-empty interior. It is easily seen that in this case $\mathcal{O}^{(r)}(\delta) = \mathcal{O}(X)$ and the topology of $\mathcal{O}(X)$ admits a bounded neighbourhood of zero. Since $\mathcal{O}(X)$ is an infinite-dimensional Fréchet space we get the contradiction.

PROPOSITION 2. (comp. [3], § 1.3) *Let $(X, p) \in \mathcal{R}_n$, $\delta \in L(X)$. Then*

$$\|\delta^{r+|\alpha|}\partial^\alpha f\|_{X^\delta} \leqslant D(n; r, \alpha)\|\delta^r f\|_{X^\delta}, \quad r \geqslant 0, \alpha \in Z_+^n, f \in \mathcal{O}^{(r)}(\delta),$$

where $D(n; r, \alpha) := \alpha!\sqrt{n}^{|\alpha|}2^{r+|\alpha|}$.

In particular, $\mathcal{O}(\delta)$ is ∂-stable in $\mathcal{O}(X^\delta)$.

Proof. We may assume that $X^\delta = X$. Fix $r \geqslant 0$, $\alpha \in \mathbf{Z}_+^n$, $f \in \mathcal{O}^{(r)}(\delta)$ and $x \in X$. By the Cauchy inequality:

$$(*) \qquad |\partial^\alpha f(x)| \leqslant \alpha!\tau^{-|\alpha|}\|f\|_{\overline{\hat{P}(x,\tau)}}, \quad 0 < \tau < d_X(x).$$

In view of Remark 1 (iii):

$$(**) \qquad \|f\|_{\overline{\hat{P}(x,\tau)}} \leqslant (\min_{\overline{\hat{P}(x,\tau)}} \delta)^{-r}\|\delta^r f\|_X, \quad 0 < \tau < d_X(x).$$

By the condition (L2) of Definition 2:

$$\left(\substack{**\\ *}\right) \qquad \delta(x') \geqslant \delta(x) - \sqrt{n}\tau, \quad x' \in \overline{\hat{P}(x,\tau)}, 0 < \tau < \frac{\varrho_X(x)}{\sqrt{n}}.$$

From $(^*)$, $(^{**})$, $\left(\substack{**\\ *}\right)$ we get:

$$\left(\substack{**\\ **}\right) \qquad |\partial^\alpha f(x)| \leqslant \alpha!\tau^{-|\alpha|}(\delta(x) - \sqrt{n}\tau)^{-r}\|\delta^r f\|_X, \quad 0 < \tau < \frac{\delta(x)}{\sqrt{n}}.$$

Putting in $\left(\substack{**\\ **}\right)$ $\tau := \dfrac{\delta(x)}{2\sqrt{n}}$ we obtain the required formula.

PROPOSITION 3. (comp. [3], § 1.3) *Let $(X, p) \in \mathcal{R}_n^\infty$, $\delta \in L(X)$. Then*

$$\|\delta^{r+2n/q} f\|_{X^\delta} \leqslant I(n; q, r, \theta)\left(\int_X |f|^q \delta^r d\mu\right)^{1/q}, \quad q > 0, r \geqslant 0, \theta \in (0,1), f \in \mathcal{O}(X^\delta),$$

where $I(n; q, r, \theta) := [(1-\theta)^r \theta^{2n} \tau_n]^{-1/q}$.

In particular we have:

$$\text{(I)} \quad \|\delta^{r+n} f\|_{X^\delta} \leqslant [(1-\theta)^r \theta^n \tau_n^{1/2}]^{-1}\left(\int_X |f|^2 \delta^{2r} d\mu\right)^{1/2}, \quad r \geqslant 0, \theta \in (0,1), f \in \mathcal{O}(X^\delta).$$

Proof. As previously we may assume that $X^\delta = X$. Fix $q > 0$, $r \geqslant 0$, $\theta \in (0, 1)$, $f \in \mathcal{O}(X)$ and $x \in X$. The function $|f|^q$ is psh. on X, so

$$|f(x)|^q \leqslant [\mathrm{vol}(B(p(x), \theta\delta(x)))]^{-1} \int_{\hat{B}(x,\theta\delta(x))} |f|^q d\mu.$$

Since $\delta \in L(X)$, so $\delta(x') \geqslant (1-\theta)\delta(x)$, $x' \in \hat{B}(x, \theta\delta(x))$ and therefore

$$\delta^r(x)|f(x)|^q \leqslant \left[(1-\theta)^r \operatorname{vol}\left(B(p(x), \theta\delta(x))\right)\right]^{-1} \int_{\hat{B}(x,\theta\delta(x))} |f|^q \delta^r d\mu$$
$$\leqslant I(n; q, r, \theta)^q \int_X |f|^q \delta^r d\mu .$$

This completes the proof.

3. Approximation theorem for weight functions

In this section we shall prove the following:

THEOREM 1. (comp. [3], § 7.3) *Let (X, p) be a Riemann-Stein domain over $\mathbf{C}^n$, let $\delta \in W(X)$. Then the following conditions are equivalent:*

(C1) $\forall s > 0$ $\exists C_s > 0$ *(depending only on n and s)*, $\exists F_s \subset \mathcal{O}(X^\delta)$:

$$1/\delta^s \leqslant \sup\{|f|: f \in F_s\} \leqslant C_s/\delta^{s+6n}, \ \lim_{s \to +\infty} C_s^{1/s} = 1 ;$$

(C2) $\exists \{n_\beta\}_\beta \subset N$, $\exists \{f_\beta\}_\beta \subset \mathcal{O}(X^\delta)$: $-\log\delta = \sup\limits_\beta \left\{\frac{1}{n_\beta} \log|f_\beta|\right\}$;

(C3) $-\log\delta \in PSH(X^\delta)$.

Proof.

(C1) $\Rightarrow$(C2) From (C1) we get:

$$-\frac{N}{N+6n}(\log\delta + \log C_N^{1/N}) \leqslant \sup\left\{\frac{1}{N+6n}\log\frac{|f|}{C_N}: f \in F_N\right\} \leqslant -\log\delta, \ N \in \mathbf{N} .$$

Since the left hand side of the above inequalities tends to $-\log\delta$ as $N \to +\infty$, so it is sufficient to put:

$$\beta := (N, f), \ N \in \mathbf{N}, f \in F_N, \quad n_\beta := \frac{1}{N+6n}, \ f_\beta := \frac{f}{C_N} .$$

The implication (C2)$\Rightarrow$(C3) is obvious.

Before the proof of the implication (C3)$\Rightarrow$(C1) we need some auxiliary theorems.

Let $(X, p) \in \mathscr{R}_n$. For $\varphi \in PSH(X)$, $q, r \in \mathbf{Z}_+$, we denote by $L^2_{(q,r)}(X, \varphi)$ the space of all forms of type (q, r) with coefficients in $L^2(X, e^{-\varphi}d\mu)$. Repeating almost exactly the methods of Chapter IV in [5], we obtain the following generalization of Theorem 4.4.2 in [5]:

THEOREM 2. *Let (X, p) be a Riemann-Stein domain over $\mathbf{C}^n$, let $\varphi \in PSH(X)$. Then for every $g \in L^2_{(q,r+1)}(X, \varphi)$ $(q, r \in \mathbf{Z}_+)$ with $\bar\partial g = 0$ there exists $v \in L^2_{(q,r)}(X, \varphi + 2\log(1+|p|^2))$ such that $\bar\partial v = g$ and*

$$\int_X |v|^2 e^{-\varphi}(1+|p|^2)^{-2} d\mu \leqslant \int_X |g|^2 e^{-\varphi} d\mu$$

($\bar\partial$ is taken in the sence of distribution theory).

Taking this theorem as basis (with $q = 0, r = 1$), we shall prove the following analogue of Theorem 4.4.4 in [5].

THEOREM 3. *Let (X, p) be a Rieman-Stein domain over $\mathbf{C}^n$. Let $\varphi \in PSH(X)$ be such that $e^{-\varphi}$ is locally integrable on X. Then for every $a \in X$, $0<\tau<d_X(a)$ there exists $u \in \mathcal{O}(X)$ such that $u(a) = 1$ and*

$$\int_X |u|^2 e^{-\varphi}(1+|p|^2)^{-3n} d\mu \leqslant \Big(\int_{\hat{P}(a,\tau)} e^{-\varphi} d\mu\Big)[2+C\tau^{-4}(1+|p(a)|^2]^n,$$

where $C \geqslant 0$ is a constant independent of n, (X, p), φ, a, τ.

Proof. (The method of the proof is taken from [5]) Let $\psi \in C_0^\infty(\mathbf{C}, [0, 1])$ be such that $\psi(z) = 1$ for $|z| \leqslant \frac{1}{3}$, $\psi(z) = 0$ for $|z| \geqslant \frac{1}{2}$. Set $c := \max\left\{1, \left(\left\|\frac{\partial\psi}{\partial\bar{z}}\right\|_{\mathbf{C}}\right)^2\right\}$, $C := 36c$.

Let $\psi_\tau(z) := \psi\left(\frac{z}{\tau}\right)$, $z \in \mathbf{C}$.

Let $X_k := \{x \in X: |p_j(x) - p_j(a)| < \tau, k < j \leqslant n\}$, $k = 0, \ldots, n$. It may easily be verified that $(X_k, p|_{X_k})$ is also a Riemann-Stein domain and that $\hat{P}(x, \tau)$ is a connected component of X_0.

It is sufficient to show that there esists $u_k \in \mathcal{O}(X_k)$ such that $u_k(a) = 1$ and

$$\int_{X_k} |u_k|^2 e^{-\varphi}(1+|p|^2)^{-3k} d\mu \leqslant M_k, \qquad k = 0, \ldots, n,$$

where $M_0 := \int_{\hat{P}(a,\tau)} e^{-\varphi} d\mu$, $M_k := M_{k-1}[2+C\tau^{-4}(1+|p(a)|^2)]$, $k = 1, \ldots, n$.

Put $u_0 := 1$ on $\hat{P}(a, \tau)$, $u_0 := 0$ on $X_0 \setminus \hat{P}(a, \tau)$. It is easily seen that u_0 satisfies all required conditions.

Suppose that $u_0, \ldots, u_{k-1}$ are already constructed, $1 \leqslant k \leqslant n$. Consider the form g of type $(0, 1)$ given by the following formula:

$$g(x) := \frac{\partial\psi_\tau}{\partial\bar{z}}(p_k(x) - p_k(a)) \frac{u_{k-1}(x)}{p_k(x) - p_k(a)} d\bar{p}_k \text{ if } x \in X_{k-1} \text{ and } p_k(x) \neq p_k(a),$$

$$g(x) := 0 \text{ if } x \in X_k \setminus X_{k-1} \text{ or } p_k(x) = p_k(a).$$

It may easily be proved that the coefficient of g is of class C^∞ in X_k, g is equal to zero in a neighbourhood of a, $\bar{\partial} g = 0$ and

$$\int_{X_k} |g|^2 e^{-\varphi}(1+|p|^2)^{-3(k-1)} d\mu \leqslant 9c\tau^{-4} M_{k-1}.$$

By Theorem 2, there exists $v \in L^2(X_k, \varphi + (3k-1)\log(1+|p|^2))$ such that $\bar{\partial} v = g$ and

$$\int_{X_k} |v|^2 e^{-\varphi}(1+|p|^2)^{-3k+1} d\mu \leqslant 9c\tau^{-4} M_{k-1}.$$

Set

$$u_k := \psi_\tau(p_k - p_k(a)) u_{k-1} - (p_k - p_k(a)) \cdot v \text{ on } X_{k-1},$$

$$u_k := \qquad -(p_k - p_k(a)) \cdot v \text{ on } X_k \setminus X_{k-1}.$$

It may easily be verified that $\bar\partial u_k = 0$ in X_k, $u_k(a) = u_{k-1}(a) = 1$ and

$$\int_{X_k} |u_k|^2 e^{-\varphi}(1+|p|^2)^{-3k} d\mu \leqslant$$
$$\leqslant 2 \int_{X_{k-1}} |\psi_\tau(p_k - p_k(a)) u_{k-1}|^2 e^{-\varphi}(1+|p|^2)^{-3(k-1)} d\mu +$$
$$+2 \int_{X_k} [|p_k - p_k(a)|^2 (1+|p|^2)^{-1}] |v|^2 e^{-\varphi}(1+|p|^2)^{-3k+1} d\mu \leqslant$$
$$\leqslant 2M_{k-1} + 4(1+|p(a)|^2) \int_{X_k} |v|^2 e^{-\varphi}(1+|p|^2)^{-3k+1} d\mu \leqslant M_{k-1}[2 + C\tau^{-4}(1+|p(a)|^2)].$$

Induction on k finishes the proof.

Now we pass to the proof of the implication (C3)⇒(C1) in Theorem 1.

Since $-\log\delta \in PSH(X)$, so $(X^\delta, p|_{X^\delta})$ is also a Riemann-Stein domain and therefore we may assume that $X^\delta = X$.

Fix $s>0$. From Theorem 3 with $\varphi := -(4n+2s)\log\delta$, $\tau := \frac{\theta}{\sqrt{n}}\delta(a)$, where $\theta := \frac{1}{2+s}$, we find that there exists a function $u = u_{s,a} \in \mathcal{O}(X)$ such that $u(a) = 1$ and

$$(*) \qquad \int_X |u|^2 \delta^{4n+2s}(1+|p|^2)^{-3n} d\mu \leqslant \Big(\int_{\hat{P}(a,\tau)} e^{-\varphi} d\mu \Big)[2 + C\tau^{-4}(1+|p(a)|^2)]^n.$$

Since $\delta \leqslant (1+|p|^2)^{-1/2}$, $\hat{P}(a,\tau) \subset \hat{B}(a, \theta\delta(a))$ and $\delta(x') \leqslant (1+\theta)\delta(a)$, $x' \in \hat{B}(a, \theta\delta(a))$, so from $(*)$ we get:

$$(**) \qquad \int_X |u|^2 \delta^{10n+2s} d\mu \leqslant (1+\theta)^{4n+2s} \theta^{2n} \tau_n \delta(a)^{6n+2s} [2 + Cn^2\theta^{-4}\delta(a)^{-6}]^n.$$

Let us put $f = f_{s,a} := (\delta(a))^{-s} u_{s,a}$. In view of $(**)$ we have:

$$(\overset{**}{*}) \qquad \int_X |f|^2 \delta^{10n+2s} d\mu \leqslant (1+\theta)^{4n+2s} \theta^{2n} \tau_n (2 + Cn^2\theta^{-4})^n.$$

Using the formula (I) of Proposition 2, from $(\overset{**}{*})$ we can easily deduce that:

$\|\delta^{6n+s} f\|_X \leqslant \left(\frac{1+\theta}{1-\theta}\right)^s (Mn\theta^{-2})^n$, where $M>0$ is a constant independent of n, (X,p), δ, s, a.

Now it is sufficient to put $C_s := \left(\frac{1+\theta}{1-\theta}\right)^s (Mn\theta^{-2})^n$ $\left(\theta = \frac{1}{2+s}\right)$, $F_s := \{f_{s,a} \colon a \in X\}$.

The proof of Theorem 1 is completed.

4. $\mathcal{O}(\delta)$-Domains of holomorphy

The following two theorems are the main result of this paper.

THEOREM 4. (comp. [3], § 4.5, Theorem 4) *Let $(X,p) \in \mathcal{R}_n^c$ be a Riemann-Stein domain. Let $\delta \in W(X)$ be such that $X^\delta = X$ and $-\log\delta \in PSH(X)$ (for instance: $\delta = \delta_X$). Then the following conditions are equivalent:*

(i) $\mathcal{O}(\delta)$ *is dense in* $\mathcal{O}(X)$ *in the topology of almost uniform convergence;*

(ii) $\mathcal{O}(\delta)$ *separates points in* X;

(iii) (X, p) *is an* $\mathcal{O}(\delta)$*-domain of holomorphy.*

THEOREM 5. (comp. [3], § 4.5, Theorem 4) *Assume that* (X, p) *and* δ *satisfy the assumptions of Theorem* 4. *Then, for* $r > 6n$, *the following conditions are equivalent:*

(i) *the family* $(\mathcal{O}^{(r)}(\delta))^*$ *separates points in* X;

(ii) (X, p) *is an* $\mathcal{O}^{(r)}(\delta)$*-domain of holomorphy;*

(iii) *there exists a function* $f \in \mathcal{O}^{(r)}(\delta)$ *such that* (X, p) *is an* $\{f\}$*-domain of holomorphy.*

The proofs of these theorems will be based on the following auxiliary results.

THEOREM 6. *Let* $(X, p) \in \mathcal{R}_n^c$ *and let* $S \subset \mathcal{O}(X)$ *be a* ∂*-stable closed subalgebra of* $\mathcal{O}(X)$ *such that* $p = (p_1, \ldots, p_n) \in S^n$. *Then the following conditions are equivalent:*

(S1) (X, p) *is a Riemann-Stein domain and* $S = \mathcal{O}(X)$;

(S2) (X, p) *is an* S*-domain of holomorphy;*

(S3) S *separates points in* X *and* (X, p) *is* S*-pseudoconvex, i.e. for every compact* $K \subset X$: $d_X(K) = d_X(\hat{K}_S)$;

(S4) S *separates points in* X *and* (X, p) *is* S*-convex, i.e. for every compact* $K \subset X$: $\hat{K}_S$ *is also compact.*

Proof. The implication (S1)$\Rightarrow$(S2) is well known — see [4], p. 283, Theorem 4. The implication (S2)$\Rightarrow$(S3) is a consequence of Theorem 1, p. 110 in [8] (*it is true without the assumption that S is closed*). The proof of the implication (S3)$\Rightarrow$(S4) may be based on Bishop's proof that every pseudoconvex Riemann domain is holomorphically convex — see [2], Theorem 5.1; [4], p. 54, Theorem 17; [8], p. 139, Theorem 1. The implication (S4)$\Rightarrow$(S1) is a consequence of Satz 10, p. 145 in [1] (comp. also [4], p. 213, the method of proof of Theorem 6; [6], Theorem 4.2) (*note that this implication is true without the assumption that S is* ∂*-stable if we assume that* $1 \in S$).

COROLLARY 1. *Let* $(X, p) \in \mathcal{R}_n^c$. *Let* $F \subset \mathcal{O}(X)$ *be such that* (X, p) *is an* F*-domain of holomorphy. Put*

$\tilde{F}$: = *the* **C**-*algebra generated in* $\mathcal{O}(X)$ *by the family* $\{1, p_1, \ldots, p_n\} \cup F^*$.

Then $\tilde{F}$ *is dense in* $\mathcal{O}(X)$.

Proof. We only need to put in Theorem 6 $S := cl_{\mathcal{O}(X)} F$ and to use the implication (S2)$\Rightarrow$(S1).

COROLLARY 2. *Let* $(X, p) \in \mathcal{R}_n^c$. *Let* $\delta \in W(X)$ *be such that* $X^\delta = X$ *and* (X, p) *is an* $\mathcal{O}(\delta)$*-domain of holomorphy. Then* $\mathcal{O}(\delta)$ *is dense in* $\mathcal{O}(X)$.

Proof. By Proposition 2, $(\mathcal{O}(\delta))^* = \mathcal{O}(\delta)$. In view of the condition (W1) of Definition 3, $p_1, \ldots, p_n \in \mathcal{O}(\delta)$. Hence $\mathcal{O}(\delta) = \widetilde{\mathcal{O}(\delta)}$ and we can use Corollary 1.

LEMMA 1. *Let* $(X, p) \in \mathcal{R}_n^c$. *Let* $\delta\colon X \to (0, +\infty)$ *be lower semi-continuous and such that* $\delta \leqslant \varrho_X$. *Assume that there exist* $0 < r_0 \leqslant r$, $M > 0$ *and* $F \subset \mathcal{O}(X)$ *such that:*

$$1/\delta^{r_0} \leqslant \sup\{|f| : f \in F\} \leqslant M/\delta^r .$$

Then

(i) *if, for every* $x \in X$, *the family* $(\mathcal{O}^{(r)}(\delta))^*$ *separates points in* $p^{-1}(p(x))$ *then* (X, p) *is an* $\mathcal{O}^{(r)}(\delta)$*-domain of holomorphy;*

(ii) *if, for every $x \in X$, the family $(\mathcal{O}(\delta))^*$ separates points in $p^{-1}(p(x))$ then (X, p) is an $\mathcal{O}(\delta)$-domain of holomorphy.*

Proof. (i) Let $((X', p'), j, S')$ be a maximal analytic extension of $((X, p), \mathcal{O}^{(r)}(\delta))$ (see [9], Definition 3.2). By Lemma 2, p. 96 in [8], j is injective, so we can assume that X is an open subset of X', $p' = p$ on X and $j = id_X$. We wish to prove that $X = X'$. Suppose that $X \subsetneq X'$. This implies that there exists a point x_0 belonging to the boundary of X in X'. It may be proved (see [9]) that S' with the topology generated by seminorms

$$S' \ni f' \to \|\delta^r(f' \circ j)\|_X \in [0, +\infty),$$

$$S' \ni f' \to \|f'\|_K \in [0, +\infty), \; K \subset\subset X',$$

is a Fréchet space and the mapping

$$S' \ni f' \overset{j^*}{\to} (f' \circ j) \in \mathcal{O}^{(r)}(\delta)$$

is continuous in this topology ($\mathcal{O}^{(r)}(\delta)$ is endowed with the standard topology given by the norm $\|\delta^r . \|_X$). Let U denote an open relatively compact neighbourhood of x_0 in X'. By Banach theorem the mapping $(j^*)^{-1}$ is also continuous and therefore there exists a constant $C>0$ such that

$$\|f'\|_U \leqslant C \|\delta^r(f' \circ j)\|_X, \; f' \in S'.$$

In particular

$$\|f\|_{X \cap U} \leqslant CM, \; f \in F.$$

This implies that $\varrho_X(x) \geqslant \varkappa > 0$, $x \in X \cap U$, where $\varkappa$ is a constant. Since $\lim\limits_{\substack{x \to x_0 \\ x \in X}} \varrho_X(x) = 0$, we get a contradiction.

(ii) Let $(X', p'), j, S'')$ be a maximal analytic extension of $((X, p), \mathcal{O}(\delta))$. Analogously as in the proof of (i) we may assume that X is open in X', $p' = p$ on X and $j = id_X$. Put

$$S' := \{f' \in S'' : f' \circ j \in \mathcal{O}^{(r)}(\delta)\}.$$

Then $((X', p'), j, S')$ is an analytic extension of $((X, p), \mathcal{O}^{(r)}(\delta))$ and, for the proof that $X = X'$, we can exactly repeat the proof of (i).

Now we can present the proof of Theorem 4.

The implication (i)⇒(ii) is obvious. The implication (ii)⇒(iii) is a consequence of the implication (C3)⇒(C1) in Theorem 1 and of Lemma 1 (ii). The implication (iii)⇒(i) is a consequence of Corollary 2.

The proof of Theorem 5 is analogous: the implication (i)⇒(ii) is a consequence of Theorem 1 and Lemma 1 (i), the implication (ii)⇒(iii) is a consequence of Theorem 10.1 in [9] and the implication (iii)⇒(i) is a consequence of Lemma 2, p. 96 in [8].

COROLLARY 3. *Let $\delta \in W(\mathbf{C}^n)$ be such that $\Omega := \{z \in \mathbf{C}^n : \delta(z) > 0\}$ is connected and $-\log\delta \in PSH(\Omega)$ (for instance: $\delta = \delta_\Omega$, where Ω is a pseudoconvex connected subset of $\mathbf{C}^n$). Then the following conditions are equivalent:*

(R1) *Ω is a Runge domain;*

(R2) *functions in $\mathcal{O}(\delta)$ may be almost uniformly in Ω approximated by polynomials;*

(R3) *there exists* $\varepsilon>0$ *such that functions in* $\mathcal{O}^{(6n+\varepsilon)}(\delta)$ *may be almost uniformly in* Ω *approximated by polynomials*;

(R4) *there exists* $\varepsilon>0$ *and a function* $f\in\mathcal{O}^{(6n+\varepsilon)}(\delta)$ *which cannot be holomorphically continued beyond* Ω *and which may be approximated by polynomials*;

(R5) *there exists* $f\in\mathcal{O}(\Omega)$ *which cannot be holomorphically continued beyond* Ω *and which may be approximated by polynomials.*

Proof. The implications (R1)$\Rightarrow$(R2)$\Rightarrow$(R3), (R4)$\Rightarrow$(R5) are obvious. The implication (R3)$\Rightarrow$(R4) is a consequence of the implication (i)$\Rightarrow$(iii) in Theorem 5. The implication (R5)$\Rightarrow$(R1) is a consequence of Corollary 1 with $F:=\{f\}$.

5. List of problems

Assume that (X,p) and δ satisfy the assumptions of Theorem 4.

(i) Does $\mathcal{O}(\delta)$ separate points in X?

(ii) Is the following implication true?: $\big(\mathcal{O}(\delta)$ separates points in $X\big)\Rightarrow$ (there exists $r\geqslant 0$ such that $\big(\mathcal{O}^{(r)}(\delta)\big)^*$ separates points in X).

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