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Holomorphic functions with restricted growth on complex manifolds

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Abstract. The aim of this paper is to extend some results of the theory of holomorphic functions with restricted growth in pseudoconvex domains in C^n to the case of Stein manifolds.

0. Introduction. Throughout the paper we denote by M a fixed countable at infinity complex analytic manifold of dimension n. We assume that M is endowed with a Hermitian metric and let μ denote the measure (the volume element) generated by this metric.

We denote by $\mathcal{O}(M)$ the space of all holomorphic functions on M. We shall always assume that $\mathcal{O}(M)$ is endowed with the topology of almost uniform convergence (i.e. uniform convergence on every compact subset of M).

For a continuous function $\delta\colon M \to (0, 1]$ we set

$$H^{(k)}(\delta) := \{f \in \mathcal{O}(M): \int_M |f|^2 \delta^k d\mu < +\infty\}, \quad k \in IN,$$

$$H(\delta) := \bigcup_{k=1}^{\infty} H^{(k)}(\delta).$$

It is easily seen that $H^{(k)}(\delta)$ is a vector subspace of $\mathcal{O}(M)$. Since $H^{(k)}(\delta) \subset H^{(k+1)}(\delta)$, so $H(\delta)$ is also a vector subspace of $\mathcal{O}(M)$. In the case of open subsets of C^n or, more generally, in the case of Riemann domains spread over C^n spaces of type $H(\delta)$ are strictly connected to the theory of δ-tempered holomorphic functions (see [3], § 1.3, Proposition 2; [5]).

The aim of this paper is to study some relations between the spaces $H(\delta)$ and $\mathcal{O}(M)$. The main result of Section 1 of the paper is the following:

THEOREM I. *If $\mathcal{O}(M)$ contains non-constant functions then $H^{(k)}(\delta)$ is of the first Baire category in the Fréchet space $\mathcal{O}(M)$. In particular $H(\delta)$ is of the first Baire category in $\mathcal{O}(M)$.*

The analogous theorem in the case of Riemann domains over C^n and δ-tempered holomorphic functions was proved by the author in [5].

The greater part of the results of this paper heavily depends on Hörmander's theory of non-homogeneous Cauchy–Riemann equations on complex manifolds ([4], § 5.2).

Let us denote by $P^k(M)$ (resp. $SP^k(M)$) the set of all plurisubharmonic (resp. strictly plurisubharmonic) functions on M of class C^k, $k \in Z_+ \cup \{+\infty\}$. To simplify notations let us introduce the following:

Definition 1. A function δ: $M \to (0, 1]$ is said to be *a weight function for $\bar{\partial}$-problem* if $-\log\delta \in P^2(M)$ and for every $\varphi \in P^2(M)$, for every $p, q \in Z_+$, for every $g \in L^2_{(p,q+1)}(M, \varphi)$ with $\bar{\partial}g = 0$ there exists $u \in L^2_{(p,q)}(M, \varphi - 4\log\delta)$ such that $\bar{\partial}u = g$ and

$$\int_M |u|^2 e^{-\varphi}\delta^4 d\mu \leqslant \int_M |g|^2 e^{-\varphi} d\mu$$

(all notations used above have the same meaning as in [4]).

Obviously the class of weight functions for $\bar{\partial}$-problem on M depends on Hermitian metric on M.

It is well known that the function $\delta(z) := (1+|z|^2)^{-1/2}$, $z \in C^n$, is a universal weight function for $\bar{\partial}$-problem in all pseudoconvex domains in C^n with respect to the standard Hermitian metric (Cf. [4], Theorem 4.4.2). More generally, the author proved in [5] that if (X, p) is a Riemann domain over C^n then the function $\delta := (1+|p|^2)^{-1/2}$ is a weight function for $\bar{\partial}$-problem on X with respect to the Hermitian metric generated by the projection p.

In Section 2 we present a characterization of weight functions for $\bar{\partial}$-problem on Stein manifolds (Theorem 1) and a construction of such functions (Proposition 1).

In Section 3 we prove the following approximation theorem:

THEOREM II. *Let δ be a weight function for $\bar{\partial}$-problem on M. Assume that*

(*) *for every $\tau > 0$, the set $\{x \in M\colon \delta(x) \geqslant \tau\}$ is compact in M.*

Then for every function $\varphi \in P^2(M)$ the set

$$F(\delta, \varphi) := \{f \in O(M)\colon \exists k \in N\colon \int_M |f|^2 e^{-\varphi}\delta^k d\mu < +\infty\}$$

is dense in $\mathcal{O}(M)$.

The above theorem is a generalization of Proposition 3, § 7.4 in [3]. In the case of Riemann domains the analogous theorem was proved by the author in [6].

In Section 4 we present a generalization of the fundamental spectral theorem for δ-tempered holomorphic functions in pseudoconvex domains in C^n (Cf. [2])

Definition 2. Let $\Phi = (\Phi_1, \ldots, \Phi_N)$: $M \to C^N$ be a holomorphic mapping. A continuous function δ: $M \to (0, 1]$ is said to be *a spectral function for Φ* if there exist mappings $u_0, \ldots, u_N$: $M \times M \to C$ and constants $k \in N$, $c > 0$ such that

(i) $u_j(s, .) \in \mathcal{O}(M)$, $\quad s \in M$, $j = 0, \ldots, N$,

(ii) $\delta(s)u_0(s, x) + \sum_{j=1}^{N} [\Phi_j(x) - \Phi_j(s)]u_j(s, x) = 1$, $\quad s, x \in M$,

(iii) $\int_M |u_j(s, .)|^2 \delta^k d\mu \leqslant c$, $s \in M$, $j = 0, \ldots, N$.

The main result of Section 4 is the following:

THEOREM III. *Let $\Phi = (\Phi_1, \ldots, \Phi_N)$: $M \to C^N$ be a holomorphic mapping such that $|d\Phi_j| \leqslant 1$, $j = 1, \ldots, N$. Let δ be a function satisfying the following conditions*:

δ is a weight function for $\bar{\partial}$-problem;

$\delta \leqslant \delta_0 := (1+|\Phi|^2)^{-1/2}$, *where* $|\Phi|^2 := \sum_{j=1}^{N} |\Phi_j|^2$;

$|\delta(x)-\delta(x')| \leqslant |\Phi(x)-\Phi(x')|$, $x, x' \in M$, *for some* $m \in N$, $\int_M \delta^m d\mu < +\infty$.

Then δ is spectral for Φ.

In Section 4 we also present a construction of functions δ satisfying the assumptions of the above theorem in the case when Φ is regular and proper.

In Section 5 we present an application of the results of Section 4 to the theory of analytic extensions of complex manifolds.

1. Basic properties of spaces $H(\delta)$. By standard reasoning (based, for example, on the local application of Theorem 2.2.3 from [4]) one can easily get:

LEMMA 1. *Let $\delta\colon M \to (0, 1]$ be a continuous function. Then for all compact sets $K, L \subset M$ with $K \subset \operatorname{int} L$ there exists a constant $c>0$ such that*

$$\|f\|_K^2 \leqslant c(\min_L \delta)^{-k} \int_L |f|^2 \delta^k d\mu, \quad f \in \mathcal{O}(M),\ k \in N,$$

where $\|f\|_K := \sup\{|f(x)|\colon x \in K\}$.

In particular the space $H^{(k)}(\delta)$ with the scalar product $(f, g) \to \int_M f\bar{g}\delta^k d\mu$ is a complex Hilbert space with a topology stronger than the topology of almost uniform convergence in M.

Now we can prove Theorem I (see Introduction).

Let us set $E_{kl} := \{f \in H^{(k)}(\delta)\colon \int_M |f|^2 \delta^k d\mu \leqslant l\}$, $l \in N$. Obviously $H^{(k)}(\delta) = \bigcup_{l=1}^{\infty} E_{kl}$. It may be easily verified that the set E_{kl} is closed in $\mathcal{O}(M)$. Hence it is sufficient to prove that the interior of E_{kl} in $\mathcal{O}(M)$ is empty.

Suppose by absurd that for some $f \in E_{kl}$ and for some open neighbourhood U of zero in $\mathcal{O}(M)$: $f+U \subset E_{kl}$. By dint of Lemma 1 U has to be bounded in $\mathcal{O}(M)$. Under our assumptions, $\mathcal{O}(M)$ is an infinite dimensional space so we get the contradiction.

2. Weight functions for $\bar{\partial}$-problem. Let $\varphi \in SP^2(M)$. For any local orthonormal basis $\omega^1, \ldots, \omega^n$ of forms of type $(1,0)$ if $\partial\bar{\partial}\varphi = \sum_{j,k=1}^{n} \varphi_{jk}\omega^j \bar{\omega}^k$, then the form $\sum_{j,k=1}^{n} \varphi_{jk}\xi_j\bar{\xi}_k$, $\xi = (\xi_1, \ldots, \xi_n) \in C^n$, is positive definite (comp. [4], § 5.2). A continuous function $\lambda\colon M \to (0, +\infty)$ will be called *a lower bound for the plurisubharmonicity of φ* if

$$\sum_{j,k=1}^{n} \varphi_{jk}\xi_j\bar{\xi}_k \geqslant \lambda|\xi|^2, \quad \xi \in C^n.$$

Note that for $\varphi \in SP^2(M)$ such a function always exists (obviously it depends on Hermitian metric on M).

The main result of this section is the following:

THEOREM 1. *Let M be a Stein manifold. Let $\delta\colon M \to (0, 1]$ be a function such that $-\log\delta$ is of class $SP^2(M)$. Assume that δ^4 is a lower bound for the plurisubharmonicity of $-\log\delta$. Then δ is a weight function for $\bar\partial$-problem.*

The method of the proof will be completely based on results of [4], so we only present a sketch of the proof.

Let $(\eta_\nu)_{\nu=1}^\infty \subset C_0^\infty(M, [0, 1])$ be such that for any compact set $K \subset M$ there exists $\nu_0 = \nu_0(K)$ with $\eta_\nu|K \equiv 1$, $\nu \geqslant \nu_0$. Let $\psi \in C^\infty(M, \boldsymbol{R})$ satisfy: $|\bar\partial\eta_\nu|^2 \leqslant e^\psi$, $\nu \in \boldsymbol{N}$. Let $\varphi \in SP^2(M)$ have a *lower bound for the plurisubharmonicity* λ. Put $\varphi_j := \varphi + (3-j)\psi$, $j = 1, 2, 3$. Let

$$L^2_{(p,q)}(M, \varphi_1) \supset DT \overset{T}{\longrightarrow} L^2_{(p,q+1)}(M, \varphi_2)\,,$$

$$L^2_{(p,q+1)}(M, \varphi_2) \supset DS \overset{S}{\longrightarrow} L^2_{(p,q+2)}(M, \varphi_3)$$

denote the operators generated by the operator $\bar\partial$ (taken in the sense of distribution theory) (comp. [4], § 5.2). Let

$$L^2_{(p,q+1)}(M, \varphi_2) \supset DT^* \overset{T^*}{\longrightarrow} L^2_{(p,q)}(M, \varphi_1)$$

denote the operator conjugate to T.

By a reasoning analogous as in the proofs of Lemmas 4.1.3, 5.2.1 in [4] we get:

LEMMA 2. *The space $D_{(p,q+1)}(M)$ is dense in $DT^* \cap DS$ in sense of the norm*

$$DT^* \cap DS \ni f \to \|T^*f\|_{\varphi_1} + \|f\|_{\varphi_2} + \|Sf\|_{\varphi_3}\,.$$

Repeating almost exactly the proof of Theorem 5.2.3 in [4] we get:

LEMMA 3. *There exists a continuous function $c\colon M \to (0, +\infty)$ (which is independent of the functions ψ, φ) such that*

$$\int_M (\lambda - c - 4|\partial\psi|^2)\,|f|^2 e^{-\varphi}\,d\mu \leqslant 4(\|T^*f\|^2_{\varphi_1} + \|Sf\|^2_{\varphi_3}), \quad f \in D_{(p,q+1)}(M)\,.$$

Substituting in the proof of Lemma 4.4.1 in [4], Lemma 4.1.3 by Lemma 2 and the inequality (4.2.9) by Lemma 3 we obtain:

LEMMA 4. *Let M be a Stein manifold. Let $\varphi \in SP^2(M)$ have a lower bound for the plurisubharmonicity λ. Then for every form $g \in L^2_{(p,q+1)}(M, \varphi) \cap L^2_{(p,q+1)}(M, \varphi + \log\lambda)$ with $\bar\partial g = 0$ there exists a form $u \in L^2_{(p,q)}(M, \varphi)$ such that $\bar\partial u = g$ and*

$$\int_M |u|^2 e^{-\varphi}\,d\mu \leqslant 4 \int_M |g|^2 e^{-\varphi} \frac{1}{\lambda}\,d\mu\,.$$

Now for the proof of Theorem 1 we can repeat the first part of the proof of Theorem 4.4.2 in [4] with the function $(1 + |z|^2)^{-1/2}$ substituted by δ and with Lemma 4.4.1 substituted by Lemma 4.

It is natural to ask whether for a given Stein manifold with a Hermitian metric there exist functions δ satisfying the assumptions of Theorem 1.

PROPOSITION 1. *Let M be a Stein manifold endowed with a Hermitian metric. Then exists a function $\delta\colon M \to (0, 1]$ such that:*

$-\log\delta \in SP^{\infty}(M)$, *the function δ^4 is a lower bound for the plurisubharmonicity of $-\log\delta$.*

δ satisfies the condition $(*)$ (see Introduction, Theorem II).

Proof. It is well known that there exists a function $s \in SP^{\infty}(M)$, $s \geqslant 0$, such that for any $t \geqslant 0$ the set $K_t := \{x \in M\colon s(x) \leqslant t\}$ is compact (comp. [4], Theorem 5.1.6).

Observe that such a function may be obtained by the embedding theorem for Stein manifolds (comp. [4], Theorem 5.3.9), namely, let $\Phi = (\Phi_1, \ldots, \Phi_N)\colon M \to C^N$ be a regular and proper holomorphic mapping, let $\delta_0 := (1+|\Phi|^2)^{-1/2}$, then $s := -\log\delta_0$ meets all the required conditions.

Let λ be a *lower bound for the plurisubharmonicity* of s. Take a function $\psi\colon [0, +\infty) \to [0, +\infty)$ of class C^{∞} with $\psi' \geqslant 0$, $\psi'' \geqslant 0$ growing so rapidly that $e^{4\psi(t)} \geqslant \max_{K_t} \frac{1}{\lambda}$, $t \geqslant 0$. Put $\chi(t) := 1+t+\psi(t)$, $t \geqslant 0$, and let $\delta := e^{-\chi(s)}$. It is easily seen that $\delta\colon M \to (0, 1]$, $-\log\delta \in SP^{\infty}(M)$ and a *lower bound for the plurisubharmonicity* of $-\log\delta$ is equal to $\lambda\cdot\chi'(s)$. By dint of the choice of the function ψ $\lambda\cdot\chi'(s) \geqslant \delta^4$. Since $\delta \leqslant e^{-s}$ so the condition $(*)$ is also fulfilled. The proof is completed.

3. An approximation theorem. In this section we present a proof of Theorem II.

The method of the proof will be based on the proof of Theorem 4.4.7 in [4].

Let us fix $f \in \mathcal{O}(M)$ and a compact set $K \subset M$. We want to find a sequence $(f_\nu)_{\nu=1}^{\infty} \subset F(\delta, \varphi)$ which tends to f uniformly on K. We may assume that for some $\tau > 0$, $K = \{x \in M\colon \delta(x) \geqslant \tau\}$.

Set $M_\varepsilon := \{x \in M\colon \delta(x) > \varepsilon\}$, $\varepsilon > 0$. Let us fix α, β such that $0 < \alpha < \beta < \tau$. Let

$$\varphi_\nu = \varphi + \chi_\nu\left(\log\frac{\beta}{\delta}\right),$$

where $\chi_\nu\colon R \to R$ is an arbitrarily fixed function of class C^{∞} such that $\chi_\nu' \geqslant 0$, $\chi_\nu'' \geqslant 0$, $\chi_\nu(t) = 0$ for $t \leqslant 0$, $\chi_\nu(t) \leqslant \nu t$ for $t \geqslant 0$ and $\chi_\nu(t) \nearrow +\infty$ as $\nu \nearrow +\infty$ for $t \geqslant \log\frac{\beta}{\alpha}$.

It is easily seen that $\varphi_\nu \in P^2(M)$, $\varphi_\nu = \varphi$ in M_β, $\varphi_\nu(x) \nearrow +\infty$ as $\nu \nearrow +\infty$ for $x \in M \setminus M_\alpha$.

Let us fix a function $\Phi \in C_0^{\infty}(M, [0, 1])$ such that $\Phi = 1$ in M_α and let $\psi := f\psi$, $g := \bar{\partial}\psi$.

It is seen that g is a form of class $D_{(0,1)}(M)$, $\bar{\partial}g = 0$ and $g = 0$ in $M_\alpha \cup (M \setminus \operatorname{supp}\Phi)$. Set

$$I_\nu := \int_M |g|^2 e^{-\varphi_\nu} d\mu, \qquad \nu \in N.$$

Observe that $I_\nu \searrow 0$ as $\nu \nearrow +\infty$. Since δ is a weight function for $\bar{\partial}$-problem, so there exists a form $u_\nu \in L^2(M, \varphi_\nu - 4\log\delta)$ such that $\bar{\partial}u_\nu = g$ and

$$\int_M |u_\nu|^2 e^{-\varphi_\nu} \delta^4 d\mu \leqslant I_\nu, \qquad \nu \in N.$$

Observe that $u_\nu \in \mathcal{O}(M_\alpha)$, so by Lemma 1, $u_\nu \to 0$ uniformly on K. Let us put $f_\nu := \psi - u_\nu$, $\nu \in N$. Obviously $f_\nu \in \mathcal{O}(M)$ and $f_\nu \to f$ uniformly on K. By dint of the choice of the functions χ_ν,

$$\int_M |f_\nu|^2 e^{-\varphi} \delta^{\nu+4} d\mu < +\infty, \qquad \nu \in N$$

The proof is completed.

Putting in Theorem II $\varphi \equiv 0$ we get:

COROLLARY 1. *If δ satisfies the assumptions of Theorem* II *then $H(\delta)$ is dense in $\mathcal{O}(M)$* (comp. Theorem I).

4. Spectral functions. The assumptions of Theorem III (see Introduction) are in such a way chosen that by repeating (with some evident modifications) the Cnop's proof of the fundamental spectral theorem in C^n ([2]) we can easily get:

LEMMA 5. *Under the assumptions of Theorem* III *there exist mappings $u_0, \ldots, u_N: M \times M \to C$ and constants $k, l \in N$, $c > 0$ for which the conditions* (i), (ii) *of Definition* 2 *are satisfied and*

(iii') $\delta_0^{2l}(s) \int_M |u_j(s, .)|^2 \delta^k d\mu \leqslant c, \quad s \in M,\ j = 0, \ldots, N.$

Now the assertion of Theorem III follows from the following generalization of a part of the proof of Lemma 3, § 3.4 in [3]:

LEMMA 6. *Let $\Phi = (\Phi_1, \ldots, \Phi_N)$: $M \to C^N$ be a holomorphic mapping. Let δ: $M \to (0, 1]$ be a continuous function such that $\delta \leqslant \delta_0$ and for some $m \in N$, $\int_M \delta^m d\mu < +\infty$. Assume that mappings $u_0, \ldots, u_N$ and constants k, l, c satisfy the conditions* (i), (ii) *of Definition* 2 *and the condition* (iii'). *Then δ is spectral for Φ.*

Proof. Let

$$U(s, x) := 1 + \sum_{k=1}^{N} \overline{\Phi_k(s)} \Phi_k(x), \qquad s, x \in M,$$

$$w_0(s, x) := \delta_0^2(s) U(s, x) u_0(s, x),$$

$$w_j(s, x) := -\delta_0^2(s) \overline{\Phi_j(s)} + \delta_0^2(s) U(s, x) u_j(s, x), \qquad s, x \in M,\ j = 1, \ldots, N.$$

It may be easily verified that

$$w_j(s, .) \in \mathcal{O}(M), \qquad s \in M,\ j = 0, \ldots, N,$$

$$\delta(s) w_0(s, x) + \sum_{j=1}^{N} [\Phi_j(x) - \Phi_j(s)] w_j(s, x) = 1, \qquad s, x \in M,$$

for some constants $k_1 \in N$, $c_1 > 0$: $\delta_0^{2(l-1)}(s) \int_M |w_j(s, .)|^2 \delta^{k_1} d\mu \leqslant c_1$, $s \in M$, $j = 0, \ldots, N$.

The decreasing induction over l finishes the proof.

Observe that for a given holomorphic mapping $\Phi: M \to C^N$ we can always find a Hermitian metric in which $|d\Phi_j| \leqslant 1, j = 1, \ldots, N$; it is sufficient to take first an arbitrarily chosen Hermitian metric and next to multiply it by a sufficiently rapidly growing function of class $C^\infty(M, (0, +\infty))$.

The problem is to construct a function δ satisfying the assumptions of Theorem III. Below we shall present a construction of a function of this type under some additional assumptions on Φ.

PROPOSITION 2. *Assume that a holomorphic mapping $\Phi: M \to C^N$ is regular and* and proper. Then there exists a function $\delta: M \to (0, 1]$ such that:

(a) $-\log\delta \in SP^\infty(M)$, *the function δ^4 is a lower bound for the plurisubharmonicity of* $-\log\delta$,

(b) *for every $\tau > 0$, the set $\{x \in M: \delta(x) \geqslant \tau\}$ is compact,*

(c) $\delta \leqslant \delta_0$,

(d) $|\delta(x) - \delta(x')| \leqslant |\Phi(x) - \Phi(x')|, \quad x, x' \in M,$

(e) $\int\limits_M \delta\, d\mu \leqslant 1$.

Proof. We already know (comp. Proposition 1) that if $\delta = e^{-\chi(-\log\delta_0)}$, where $\chi(t) = 1 + t + \psi(t)$, $\psi: [0, +\infty) \to [0, +\infty)$ is a sufficiently rapidly growing function of class C^∞ with $\psi' \geqslant 0$, $\psi'' \geqslant 0$, then δ satisfies (a), (b), (c). It may be easily proved that the condition (e) will be also satisfied if ψ grows sufficiently rapidly. Note that if, moreover, $\psi' \leqslant e^\psi$ then the function

$$(0, 1] \ni t \to e^{-\chi(-\log t)}$$

is a Lipschitz function with the Lipschitz constant 1 (comp. [3], § 1.5, Lemma 2). Since

$$|\delta_0(x) - \delta_0(x')| \leqslant |\Phi(x) - \Phi(x')|\,,$$

so in the case when $\psi' \leqslant e^\psi$ the condition (d) will be also satisfied.

In this way the proof of Proposition 2 is reduced to the following:

LEMMA 7. *Let $F: [0, +\infty) \to [0, +\infty)$ be a function of class C^∞ with $F' > 0$, $F'' \geqslant 0$, $F(+\infty) = +\infty$. Then there exists a function $\psi: [0, +\infty) \to [0, +\infty)$ such that $\psi \geqslant F$, $0 \leqslant \psi' \leqslant e^\psi$, $\psi'' \geqslant 0$.*

Proof. The method of the proof is due to A. Kleiner.

The function

$$F: [0, +\infty) \to A := [F(0), +\infty)$$

is bijective. Set $G := F^{-1}$ and let $\varphi(u) := G(u) - e^{-u}$, $u \in A$. It is easily seen that φ is of class C^∞, $\varphi \leqslant G$, $\varphi' > 0$, $\varphi'' \leqslant 0$. The function

$$\varphi: B := \varphi^{-1}([0, +\infty)) \to [0, +\infty)$$

is bijective. Set $\psi := \varphi^{-1}$. It may be easily verified that ψ meets all the required conditions.

6. Analytic extensions.

Throughout this section we additionally assume that M is connected.

At first we recall some definitions (Cf. [1], [7]).

Definition 3. Let $F\subset\mathcal{O}(M)$. A pair (M',α) is said to be *an analytic extension* (a.e.) *of* (M,F) if

(i) M' is a connected countable at infinity complex analytic manifold of dimension n,

(ii) $\alpha\colon M\to M'$ is a holomorphic mapping,

(iii) for every $f\in F$ there exists exactly one $f'\in\mathcal{O}(M')$ such that $f'\circ\alpha=f$.

Definition 4. A manifold M is said to be *an F-domain of holomorphy* if for every a.e. (M',α) of (M,F) the mapping α is an analytic isomorphism of M onto M'.

Observe that if F separates points in M then for every a.e. (M',α) of (M,F) the mapping α has to be injective, hence the set $\alpha(M)$ is open in M'.

It is well known that any Stein manifold M is an $\mathcal{O}(M)$-domain of holomorphy ([1], [4]).

We shall show that if M is a Stein manifold then M is an $H^p(\delta)$-domain of holomorphy for some δ and p (we recall that $H^p(\delta)$ is of the first Baire category in $\mathcal{O}(M)$).

Lemma 8. *Let E be a vector subspace of $\mathcal{O}(M)$. Assume that the space E is endowed with a norm $\|\ \|$ such that $(E,\|\ \|)$ is a Banach space with a topology stronger than the topology of almost uniform convergence in M. Let $\delta\colon M\to(0,1]$ be a continuous function satisfying the condition* (*). *Let $B\subset E$ and $c>0$ be such that*

$$\|f\|\leqslant c,\ f\in B,\qquad \sup\{|f|: f\in B\}\geqslant\frac{1}{\delta}.$$

Assume that $E\subset F\subset\mathcal{O}(M)$ and F separates points in M. Then M is an F-domain of holomorphy.

Proof. Let (M',α) be an a.e. of (M,F). We already know that α has to be an analytic isomorphism of M onto an open set $\alpha(M)$. Suppose by absurd that $\alpha(M)\nsubseteq M'$.

Set $E':=\{f'\in\mathcal{O}(M')\colon f'\circ\alpha\in E\}$. Obviously E' is a vector subspace of $\mathcal{O}(M')$. Let us consider the following family of seminorms:

$$E'\ni f'\to\|f'\circ\alpha\|$$
$$E'\ni f'\to\|f'\|_K,\qquad K\subset\subset M'.$$

It may be easily verified that the space E' endowed with the topology generated by this family of seminorms is a Fréchet space. The mapping

$$E'\ni f'\overset{\alpha^*}{\to}f'\circ\alpha\in E$$

is continuous and it is an algebraic isomorphism. By Banach theorem $(\alpha^*)^{-1}$ is continuous.

Let a be a boundary point of $\alpha(M)$ in M'. Let K be a compact neighbourhood of a in M'. Since $(\alpha^*)^{-1}$ is continuous so there exists a constant $c_1>0$ such that

$$\|f'\|_K\leqslant c_1\|f'\circ\alpha\|,\qquad f'\in E'.$$

In particular

$$\|f\|_{\alpha^{-1}(K)}\leqslant c_1 c,\qquad f\in B.$$

Hence $\alpha^{-1}(K) \subset \left\{x \in M\colon \delta(x) \geqslant \frac{1}{c_1 c}\right\}$ and therefore $\alpha^{-1}(K)$ is compact. This implies that $\alpha(M) \cap K$ is compact in $\alpha(M)$. We get the contradiction with the choice of K.

PROPOSITION 3. *Let $\Phi\colon M \to C^N$ be a holomorphic mapping. Let $\delta\colon M \to (0, 1]$ be a continuous function satisfying (*). Assume that δ is spectral for Φ. Then*

(i) *if $H(\delta)$ separates points in M then M is an $H(\delta)$-domain of holomorphy,*

(ii) *if for some $l \in N$ the space $H^{(l)}(\delta)$ separates points in M then there exists $p \in N$* such that *M is an $H^{(p)}(\delta)$-domain of holomorphy.*

Observe that

— $H(\delta)$ separates points in M if moreover δ is a weight function for $\bar{\partial}$-problem and $\mathcal{O}(M)$ separates points (Theorem II),

— $H^{(m+2)}(\delta)$ separates points in M if moreover Φ is injective, $\delta \leqslant \delta_0$ and $\int_M \delta^m d\mu < +\infty$.

Proof. Let $u_0, \ldots, u_N$, k, c be associated to Φ and δ accordingly to Definition 2. Set $E := H^{(k)}(\delta)$, $\|f\|^2 := \int_M |f|^2 \delta^k d\mu$, $f \in E$, $B := \{u_0(s, .)\colon s \in M\}$. Since

$$u_0(s, s) = \frac{1}{\delta(s)}, \qquad \int_M |u_0(s, .)|^2 \delta^k d\mu \leqslant c, \qquad s \in M,$$

so the above defined E and B satisfy the assumptions of Lemma 7.

Now for the proof of (i) we only need to put in Lemma 1 $F := H(\delta)$ and for the proof of (ii) $-F := H^{(\max\{k,l\})}(\delta)$.

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