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Holomorphic continuation of functions with restricted growth

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Abstract. Let (X, p) be a connected Riemann–Stein domain over C^n. Let δ be a weight function on X such that $-\log\delta$ is plurisubharmonic. Let $G_1, \ldots, G_m$ be δ-tempered holomorphic functions on X and let $\Gamma = \bigcap_{j=1}^{m} G_j^{-1}(0)$.

In the paper we present examples of normed spaces $\mathscr{E}$ of holomorphic functions on Γ for which there exists a linear continuous operator T of $\mathscr{E}$ into $\mathcal{O}(X) \cap L^2(X, \delta^{2l} d\mu_X)$ such that $Tf = f$ on Γ.

1. Introduction. First we shall present some basic notations, definitions and auxiliary theorems which will be used in the paper (the details may be found in [3]).

Let (X, p) be a connected Riemann domain spread over C^n.

We denote by $\hat{B}_X(x, r)$ an open neighbourhood of $x \in X$ which is mapped homeomorphically by the projection p onto the Euclidean ball $B(p(x), r)$ in C^n. We put

$$\varrho_X(x) = \sup\{r>0\colon \hat{B}_X(x, r) \text{ exists}\}, \quad x \in X,$$

$$\delta_X = \min\{\varrho_X, \delta_0 \circ p\},$$

where $\delta_0(z) = (1+|z|^2)^{-1/2}$, $|z|^2 = \sum_{j=1}^{n} |z_j|^2$, $z = (z_1, \ldots, z_n) \in C^n$.

A function $\delta\colon X \to (0, 1]$ will be called *a weight function on* X $(\delta \in W(X))$ if:

(1.1) $$\delta \leqslant \delta_X,$$

(1.2) $$|\delta(x)-\delta(x')| \leqslant |p(x)-p(x')|, \quad x \in X,\ x' \in \hat{B}_X(x, \varrho_X(x)).$$

Notice that $\delta_X \in W(X)$.

Let μ_X denote the measure on X generated by the volume element $(2i)^{-n} d\bar{p}_1 \wedge \ldots \wedge d\bar{p}_n \wedge dp_1 \wedge \ldots \wedge dp_n$, where $(p_1, \ldots, p_n) = p$.

For a continuous function $\varphi\colon X \to (0, 1]$ and for a number $k \in [0, +\infty)$, we put

$$\mathcal{O}^{(k)}(X, \varphi) = \{f \in \mathcal{O}(X)\colon \|\varphi^k f\|_\infty < +\infty\},$$

$$\mathscr{H}^{(k)}(X, \varphi) = \{f \in \mathcal{O}(X)\colon \|\varphi^k f\|_2 = (\int_X |f|^2 \varphi^{2k} d\mu_X)^{1/2} < +\infty\},$$

where $\mathcal{O}(X)$ denotes the space of all complex-valued holomorphic functions on X.

(1.3) Notice that $\mathscr{H}^{(k)}(X, \varphi)$ *is a complex Hilbert space whose topology is stronger than the topology of uniform convergence on compact subsets of* X ($\mathscr{H}^{(k)}(X, \varphi)$ is *a natural Hilbert space in* $\mathcal{O}(X)$).

(1.4) Observe that if $\|\varphi^{k_0}\|_2 < +\infty$, then

$$\mathcal{O}^{(k)}(X, \varphi) \subset \mathscr{H}^{(k+k_0)}(X, \varphi)$$

and

$$\|\varphi^{k+k_0} f\|_2 \leqslant \|\varphi^{k_0}\|_2 \, \|\varphi^k f\|_\infty .$$

(1.5) In the case when X is an open subset of C^n, $p = id_X$, if $\delta \in W(X)$ then in view of (1.1) we have:

$$\|\delta^{n+1/2}\|_2 \leqslant \|\delta_0{}^{n+1/2}\|_2 < +\infty .$$

It may be proved ([3], Propositions 2,3) that *if* $\delta \in W(X)$ *then*:

(1.6)
$$\|\delta^{k+1} \partial f / \partial x_j\|_\infty \leqslant \sqrt{n} 2^{k+1} \|\delta^k f\|_\infty ,$$

(1.7)
$$\mathscr{H}^{(k)}(X, \delta) \subset \mathcal{O}^{(k+n)}(X, \delta) \quad \textit{and}$$

$$\|\delta^{k+n} f\|_\infty \leqslant 2^{k+n} \tau_n^{-1/2} \|\delta^k f\|_2 ,$$

where τ_n *denotes the volume of the unit ball in* C^n.

We denote by $L^2_{(0,r)}(X, \text{loc})$, $r \in Z_+$, the space of all forms $u = \sum_I{}' u_I d\bar{p}_I$ of type $(0, r)$ with locally square-integrable coefficients, $\sum'$ means that the sum is taken only over strictly increasing multi-indexes $I = (i_1, \ldots, i_r)$, $d\bar{p}_I = d\bar{p}_{i_1} \wedge \ldots \wedge d\bar{p}_{i_r}$. We put $|u|^2 = \sum_I{}' |u_I|^2$. Let

$$\mathscr{F}^{(k)}_{(0,r)}(X, \delta) = \{u \in L^2_{(0,r)}(X, \text{loc}) \colon \|\delta^k u\|_2 = (\int_X |u|^2 \delta^{2k} d\mu_X)^{1/2} < +\infty\} , \ k \geqslant 0, \ r \in Z_+ .$$

In the sequel an important role will be played by the following version of the generalized Hörmander's theorem on the solvability of the $\bar{\partial}$-problem ([3], Theorem 2):

(1.8) *If* X *is a Stein domain and* $-\log\delta$ *is plurisubharmonic on* X *then for every* $u \in \mathscr{F}^{(k)}_{(0,r+1)}(X, \delta)$ *with* $\bar{\partial} u = 0$ ($\bar{\partial}$ *is taken in the sense of the distribution theory*) *there exists* $v \in \mathscr{F}^{(k+2)}_{(0,r)}(X, \delta)$ *such that* $\bar{\partial} v = u$ *and* $\|\delta^{k+2} v\|_2 \leqslant \|\delta^k u\|_2$.

Now we pass to the formulation of the problem of holomorphic continuation which will be investigated in this paper.

Let (X, p) be a connected Riemann—Stein domain over C^n. Let $\delta \in W(X)$ be such that $-\log\delta$ is plurisubharmonic and let $G_1, \ldots, G_m \in \mathcal{O}^{(s_0)}(X, \delta)$. Let us put $\Gamma = \bigcap_{j=1}^m G_j^{-1}(0)$. We always assume that $\phi \neq \Gamma \neq X$. Set $c_0 = \max\{\|\delta^{s_0} G_j\|_\infty \colon j = 1, \ldots, m\}$.

Let us consider the following general problem:

(1.9) Given a normed space $\mathscr{E}$ of holomorphic functions on Γ; whether there exist $l \geqslant 0$ and a linear continuous operator

$$T: \mathscr{E} \to \mathscr{H}^{(l)}(X, \delta)$$

such that $T(f) = f$ on Γ, $f \in \mathscr{E}$.

In Section 2 we shall consider the case when $\mathscr{E} = \mathscr{H}^{(k)}(U, \delta)$, where U is an open neighbourhood of Γ. The main result of this section is the following:

THEOREM 1. *Let (X, p), δ, $G_1, \ldots, G_m$, Γ, U be as above. Assume that there exist an open neighbourhood V of Γ $(V \subset U)$, a function $\phi \in C(X, [0, 1])$ and constants $s_1, s_2 \geqslant 0$, $c_1, c_2 > 0$ such that:*

(1.10) $\phi = 1$ on V, $\operatorname{supp}\phi \subset U$,

(1.11) $\bar{\partial}\phi \in L^2_{(0,1)}(X, \mathrm{loc})$, $|\bar{\partial}\phi|\delta^{s_1} \leqslant c_1$,

(1.12) $|G(x)| \geqslant c_2 \delta^{s_2}(x)$, $x \in U \setminus V$, *where* $|G|^2 = \sum_{j=1}^{m} |G_j|^2$.

Then there exist constants α, $c > 0$ (depending only on n, m, s_j, c_j, $j = 0, 1, 2$) such that for every $k \geqslant 0$ there exists a linear continuous operator

$$T: \mathscr{H}^{(k)}(U, \delta) \to \mathscr{H}^{(k+\alpha)}(X, \delta)$$

such that $\|T\| \leqslant c$ and $T(f) = f$ on Γ.

In Section 3 we shall consider the case when Γ is a graph and $\mathscr{E} = \mathscr{H}^{(k)}(\Gamma, \delta)$. Theorem 2 of this section is a generalization of Theorem 2 in [2].

2. Proof of Theorem 1. Before the proof we shall make a few remarks in relation to the problem (1.9).

Remark 1. *In view of* (1.7), *the operator T in* (1.9) *may be considered as a linear continuous operator from $\mathscr{E}$ into $\mathcal{O}^{(l+n)}(X, \delta)$.*

Remark 2. *By* Remark 1, *the necessary condition of existence of T in* (1.9) *is:*

(2.1) *there exist $k \geqslant 0$, $c \geqslant 0$ such that*

$$\delta^k(x)|f(x)| \leqslant c\|f\|_{\mathscr{E}}, \quad f \in \mathscr{E},\ x \in \Gamma,$$

where $\| \ \|_{\mathscr{E}}$ denotes the norm of $\mathscr{E}$.

LEMMA 1. *The problem* (1.9) *is equivalent to the following one:*

(2.2) *whether there exist $l \geqslant 0$ and $c \geqslant 0$ such that for every $f \in \mathscr{E}$ there exists $\hat{f} \in \mathscr{H}^{(l)}(X, \delta)$ such that $\|\delta^l \hat{f}\|_2 \leqslant c\|f\|_{\mathscr{E}}$ and $\hat{f} = f$ on Γ.*

Proof. It is clear that (2.2) is apparently weaker than (1.9).

Conversely, let $\mathscr{H} = \mathscr{H}^{(l)}(X, \delta)$ and let $\| \ \|_{\mathscr{H}}$ denote the norm of $\mathscr{H}$. Let us put $\mathscr{S} = \{g \in \mathscr{H}: g = 0 \text{ on } \Gamma\}$. By (1.3), $\mathscr{S}$ is a closed subspace of $\mathscr{H}$. Let π denote the projection of $\mathscr{H}$ onto $\mathscr{S}$.

For $f \in \mathscr{E}$, if $g \in \mathscr{H}$ and $g = f$ on Γ, we put

$$T(f) = g - \pi(g).$$

The operator T is well-defined, in fact, if $g_1 = f = g_2$ on Γ then $g_1 - g_2 \in \mathscr{S}$ and hence $\pi(g_1 - g_2) = g_1 - g_2$. It is seen that $T\colon \mathscr{E} \to \mathscr{H}$ is linear. If we put $g = \hat{f}$, we get $\|T(f)\|_{\mathscr{H}} \leqslant 2c\|f\|_{\mathscr{E}}$, which finishes the proof.

Notice that Lemma 1 may be applied to some other problems like (1.9); we have only used the fact that $\mathscr{H}$ is a natural Hilbert space in $\mathcal{O}(X)$.

We pass to the proof of Theorem 1.

LEMMA 2. *There exist* $\alpha', c' > 0$ (*depending only on* $n, m, s_j, c_j, j = 0, 1, 2$) *such that for every* $f \in \mathscr{H}^{(k)}(U, \delta)$ *there exist forms* $u_1, \ldots, u_m \in \mathscr{F}^{(k+\alpha')}_{(0,1)}(X, \delta)$ *such that*:

(2.3) $\quad \bar{\partial} u_j = 0, \quad j = 1, \ldots, m,$

(2.4) $\quad \sum\limits_{j=1}^{m} G_j u = f \bar{\partial}\phi,$

(2.5) $\quad \|\delta^{k+\alpha'} u_j\|_2 \leqslant c' \|\delta^k f\|_2, \quad j = 1, \ldots, m.$

Assuming this lemma for a moment we shall finish the main proof.

Fix $k \geqslant 0$, $f \in \mathscr{H}^{(k)}(U, \delta)$ and let $u_1, \ldots, u_m$ be associated with f accordingly to Lemma 2. By (1.8), there exist $v_1, \ldots, v_m \in \mathscr{F}^{(k+\alpha'+2)}_{(0,0)}(X, \delta)$ such that

$$\bar{\partial} v_j = u_j \quad \text{and} \quad \|\delta^{(k+\alpha'+2)} v_j\|_2 \leqslant c' \|\delta^k f\|_2, \quad j = 1, \ldots, m.$$

Let us put

$$\hat{f} = f\phi - \sum_{j=1}^{m} G_j v_j,$$

where we mean that $f\phi = 0$ in $X \setminus U$.

Since $\bar{\partial}\hat{f} = f\bar{\partial}\phi - \sum\limits_{j=1}^{m} G_j \bar{\partial} v_j$, so in view of (2.4), $\hat{f} \in \mathcal{O}(X)$. It is clear that $\hat{f} = f$ on Γ. It is easy to prove that

$$\|\delta^{k+\alpha}\hat{f}\|_2 \leqslant c\|\delta^k f\|_2, \quad \text{where} \quad \alpha = \alpha' + 2s_0, \quad c^2 = 2[1 + (mc_0 c')^2].$$

By Lemma 1, this finishes the proof of Theorem 1.

Proof of Lemma 2. The general idea of the proof is the same as in the proof of Theorem 1 in [1]. For simplicity of notations we shall write l (resp. c) instead of all the (usually different) constants of the form $k+\alpha$, where α (resp. c) depends only on $n, m, s_j, c_j, j = 0, 1, 2$ (as we shall see, all these constants may be effectively calculated, but this is not essential for our proof).

Let us put

$$\Delta_r^0 = \{h \in L_{(0,r)}(X, \mathrm{loc})\colon \exists l, c \geqslant 0\colon \|\delta^l h\|_2, \|\delta^l \bar{\partial} h\|_2 \leqslant c\|\delta^k f\|_2\}, \quad r \in Z_+,$$

$$\Delta_r^\nu = \{h = (h_I)_I\colon I = (i_1, \ldots, i_\nu),\ 1 \leqslant i_1, \ldots, i_\nu \leqslant m,\ h_I \in \Delta_r^0,$$

$$\text{the system } (h_I)_I \text{ is skew-symmetrical with respect to } I\}.$$

For $h = (h_I)_I \in \Delta_r^\nu$, we put

$$|h|^2 = \sum_I |h_I|^2, \ \|\delta^l h\|_2 = \Big(\int_X |h|^2 \delta^{2l} d\mu_X\Big)^{1/2}, \quad \bar\partial h = (\bar\partial h_I)_I.$$

Observe that the operator $\bar\partial\colon \Delta_r^\nu \to \Delta_{r+1}^\nu$ is well-defined.

For $h \in \Delta_r^{\nu+1}$, we put

$$(Ph)_I = \sum_{j=1}^m G_j h_{I,j},$$

where $I, j = (i_1, \ldots, i_\nu, j)$.

It is easy to prove that the operator $P\colon \Delta_r^{\nu+1} \to \Delta_r^\nu$ is well-defined, $P \circ \bar\partial = \bar\partial \circ P$ (as the mapping from $\Delta_r^{\nu+1}$ into Δ_{r+1}^ν) and $P \circ P = 0$ (as the mapping from $\Delta_r^{\nu+2}$ into Δ_r^ν).

By dint of (1.10), (1.11), $f\bar\partial\phi \in \Delta_1^0$. Let us put

$$h_j^1 = \begin{cases} |G|^{-2}\bar G_j f \bar\partial\phi & \text{on } X\setminus\Gamma \\ 0 & \text{on } V \end{cases}, \ j = 1, \ldots, m,$$

and let $h^1 = (h_1^1, \ldots, h_m^1)$. In view of (1.10), h^1 is well-defined and $\operatorname{supp} h^1 \subset U$. By (1.12), $\|\delta^l h^1\|_2 \leqslant c\|\delta^k f\|_2$. Since $\bar\partial h_j^1 = \bar\partial(|G|^{-2}\bar G_j) \wedge f\bar\partial\phi$, so in view of (1.6) and (1.12), $\|\delta^l \bar\partial h^1\|_2 \leqslant c\|\delta^k f\|_2$. Hence $h^1 \in \Delta_1^1$. It is seen that $Ph^1 = f\bar\partial\phi$.

Unfortunately, if $m > 1$, then $\bar\partial h^1 \neq 0$, so h^1 must be modified.

Firstly, by an increasing induction over ν, we shall construct a sequence $(h_\nu)_{\nu=1}^{m+1}$ such that

$$h^\nu \in \Delta_\nu^\nu, \quad h^\nu = 0 \text{ on } V, \quad \operatorname{supp} h^\nu \subset U, \quad \nu = 1, \ldots, m+1,$$
$$Ph^{\nu+1} = \bar\partial h^\nu, \quad \nu = 1, \ldots, m.$$

h^1 has already been constructed. Suppose that $h^1, \ldots, h^{\nu-1}$ are already constructed $(2 \leqslant \nu \leqslant m)$. It is easy to check that h^ν given by the formula:

$$h_I^\nu = \begin{cases} |G|^{-2} \sum_{j=1}^{\nu} (-1)^{\nu-j} \bar G_{i_j} \bar\partial h^{\nu-1}_{(i_1, \ldots, i_{j-1}, i_{j+1}, \ldots, i_\nu)} & \text{on } X\setminus\Gamma \\ 0 & \text{on } V \end{cases}$$

satisfies all the required conditions.

Notice that by skew-symmetry, $h^{m+1} = 0$.

Now, by a decreasing induction over ν, we shall construct a sequence $(g^\nu)_{\nu=1}^m$ such that

$$g^\nu \in \Delta_\nu^{\nu+1}, \quad \nu = 1, \ldots, m,$$
$$\bar\partial g^{\nu-1} = h^\nu - Pg^\nu, \quad \nu = 2, \ldots, m.$$

Let us put $g^m = 0$ and suppose that $g^m, \ldots, g^\nu$ are already constructed $(2 \leqslant \nu \leqslant m)$. Since $\bar\partial(h^m - Pg^m) = \bar\partial Ph^{m+1} = 0$ and $\bar\partial(h^\nu - Pg^\nu) = \bar\partial h^\nu - P\bar\partial g^\nu = \bar\partial h^\nu - P(h^{\nu+1} - Pg^{\nu+1}) = 0$, $\nu < m$, so $g^{\nu-1}$ may be obtained by componentwise application of (1.8).

Finally, let $u = (u_1, \ldots, u_m) = h^1 - Pg^1$. The conditions (2.5) follows directly from the definition. Since $\bar\partial(h^1 - Pg^1) = \bar\partial h^1 - Ph^2 = 0$, so (2.3) is fulfilled. Since $Pu = Ph^1 = f\bar\partial\phi$, so (2.4) is also fulfilled.

The proof of Lemma 2 is finished.

3. Applications of Theorem 1. Let us consider the case when Γ is a graph.

Let (Y, q) be a connected Riemann domain over C^{n-m} $(1\leqslant m\leqslant n-1)$, let

$$F = (F_1, \ldots, F_m) \in [\mathcal{O}^{(r)}(Y, \delta_Y)]^m$$

and let Γ denote the graph of F. Set $A = \max\{\|\delta_Y^r F_j\|_\infty,\ j = 1, \ldots, m\}$.

Let X be a connected open subset of $Y\times C^m$ containing Γ and let $p = (q\otimes id_{C^m})|_X$. Assume that X is a Stein domain.

Let $\delta \in W(X)$ be such that $-\log\delta$ is plurisubharmonic on X and let $\eta(x) = \delta(x, F(x))$, $x \in Y$.

We pose the following question:

(3.1) Given $k\geqslant 0$, whether there exist $l\geqslant 0$ and a linear continuous operator

$$T\colon \mathscr{H}^{(k)}(Y, \eta) \to \mathscr{H}^{(l)}(X, \delta)$$

such that $(Tf)(x, F(x)) = f(x)$, $x \in Y$.

It is clear that (3.1) may be interpreted as a particular case of (1.9).

In the case $Y\subset C^{n-m}$, $q = id_Y$, a similar problem was investigated by I. Cnop in [2]. The result of this section will be the generalization of Theorem 2 in [2] (notice that our methods of the proof are independent of [2]).

The first idea is to try to put $(Tf)(x, t) = \hat{f}(x, t) = f(x)$, $(x, t)\in X$. Unfortunately, $\hat{f}$ so defined may lie outside of the space $\bigcup_{l\geqslant 0} \mathscr{H}^{(l)}(X, \delta)$. For, let us consider the following example:

$$n = 2,\quad m = 1,\quad Y = \{z\in C\colon 0<|z|<1\},\quad q = id_Y,\quad F(z) = 1/z$$

(since $\varrho_Y(z)\leqslant|z|$, so $F\in\mathcal{O}^{(1)}(Y, \delta_Y)$),

$X = \{(z, t)\in Y\times C\colon |z|<|t|\}$ (it is seen that the graph of F is contained in X and that X is a domain of holomorphy),

$\delta = \delta_X$ (since X is a domain of holomorphy, so $-\log\delta$ is plurisubharmonic),

$f = F$ (since $\varrho_X(z, t)\leqslant\varrho_Y(z)$, so $\delta_X(z, t)\leqslant\delta_Y(z)$ and in particular $\|\eta f\|_2<+\infty$).

Observe that $\frac{1}{2}(|t|-|z|)\leqslant\varrho_X(z, t)\leqslant\sqrt{2}(|t|-|z|)$, $(z, t)\in X$. Hence for $|z|<|t|<1/2$ we have $\delta_X(z, t)\geqslant\frac{1}{2}(|t|-|z|)$, so for every $l\geqslant 0$:

$$\int_X |\hat{f}|^2\delta^{2l}d\mu_X\geqslant 4^{-l}\int_{|z|<|t|<\frac{1}{2}} |z|^{-2}(|t|-|z|)^{2l}d\lambda_2(z, t) = +\infty,$$

where λ_n denote the Lebesgue measure in C^n.

The main result of this section is the following:

THEOREM 2. *Let (Y, q), F, (X, p), δ, η be as in* (3.1). *Then there exist constants $\alpha, c>0$ (depending only on n, m, r, A) such that for every $k\geqslant 0$ there exists a linear continuous operator*

$$T\colon \mathscr{H}^{(k)}(Y, \eta) \to \mathscr{H}^{(k'+\alpha)}(X, \delta)$$

such that $\|T\| \leqslant 2^{k'}c$ *and* $(Tf)(x, F(x)) = f(x)$, $x \in Y$, *where*

$$k' = \begin{cases} 0 & \text{if } 0 \leqslant k \leqslant m, \\ k-m & \text{if } k > m. \end{cases}$$

Proof. We shall show that Theorem 2 is a particular case of Theorem 1. Let $G_j(x, t) = t_j - F_j(x)$, $(x, t) \in X$, $j = 1, \ldots, m$. Since $\delta_X(x, t) \leqslant \delta_Y(x)$, so by (1.1), $G_1, \ldots, G_m \in \mathcal{O}^{(s_0)}(X, \delta)$, where $s_0 = \max\{1, r\}$ $(c_0 = 1 + A)$.

LEMMA 3. *Let* $U = \{(x, t) \in Y \times C^m \colon |t - F(x)| < \eta(x)\}$. *Then*

(3.2) *for every* $(x, t) \in U$: $(x, t) \in \hat{\Delta}(x) = \hat{B}_X((x, F(x)), \eta(x))$

(*notice that* $\eta(x) \leqslant \varrho_X(x, F(x))$, *so the "ball"* $\hat{\Delta}(x)$ *is well-defined*), *in particular* $U \subset X$;

(3.3) $\delta(x, t) < 2\eta(x)$, $\quad (x, t) \in U$;

(3.4) $\int\limits_U |f|^2 \delta^{2k'} d\mu_X \leqslant \tau_m 4^{k'} \int\limits_Y |f|^2 \eta^{2k} d\mu_Y$, $\quad f \in \mathscr{H}^{(k)}(Y, \eta)$.

Proof of Lemma 3.

Ad (3.2). Let us fix $(x, t) \in U$ and consider the mapping:

$[0, 1] \ni \tau \xrightarrow{\gamma} (x, F(x) + \tau[t - F(x)]) \in Y \times C^m$.

It is seen that γ is continuous, $\gamma(0) \in \hat{\Delta}(x)$ and $(q \otimes id_{C^m}) \circ \gamma \colon [0, 1] \to p(\hat{\Delta}(x))$. Hence there exists a continuous curve $\hat{\gamma} \colon [0, 1] \to \hat{\Delta}(x)$ such that $\hat{\gamma}(0) = \gamma(0)$ and

$$p \circ \hat{\gamma} = (q \otimes id_{C^m}) \circ \gamma .$$

Since such a lifting is uniquely determined, so $\hat{\gamma} \equiv \gamma$ and therefore $\gamma(1) = (x, t) \in \hat{\Delta}(x)$.

Ad (3.3). In view of (1.2) and (3.2), for $(x, t) \in U$ we have:

$$\delta(x, t) \leqslant \delta(x, F(x)) + |p(x, t) - p(x, F(x))| = (x) + |t - F(x)| < 2\eta(x) .$$

Ad (3.4). Since $\mu_X = (\mu_Y \otimes \lambda_m)|_X$, so in view of (3.3), by the Fubini theorem we have:

$$\int\limits_U |f|^2 \delta^{2k'} d\mu_X \leqslant 4^{k'} \int\limits_Y |f(x)|^2 \eta^{2k'}(x) \lambda_m(B(F(x), \eta(x))) d\mu_Y(x) \leqslant \tau_m 4^{2k'} \int\limits_Y |f|^2 \eta^{2k} d\mu_Y .$$

The proof of Lemma 3 is finished.

We return to the proof of Theorem 2. Let U be as in Lemma 3. The property (3.4) shows that the natural embedding of $\mathscr{H}^{(k)}(Y, \eta)$ into $\mathscr{H}^{(k')}(U, \delta)$ is well-defined and continuous. Hence it is sufficient to prove that the assumptions of Theorem 1 are fulfilled.

Let us put $V = \{(x, t) \in X \colon |t - F(x)| < \frac{1}{2}\eta(x)\}$. By (3.3), for V so defined, the condition (1.12) is fulfilled. We only need to construct the function ϕ.

Let $\psi \in C_0^\infty(C^m, [0, 1])$ be such that $\psi(z) = 1$ if $|z| \leqslant 1/2$, $\psi(z) = 0$ if $|z| \geqslant 3/4$ and let us put

$$\phi(x, t) = \psi\left(\frac{t - F(x)}{\eta(x)}\right), \quad (x, t) \in X .$$

It is seen that (1.10) holds true. Note that $\bar{\partial}\phi = 0$ in $V \cup (X \setminus \bar{U})$, so the estimate (1.11) is essential only in $U \setminus V$.

Obviously

$$\frac{\partial\phi}{\partial\bar{t}_j} = \frac{\partial\psi}{\partial\bar{z}_j}\left(\frac{t-F}{\eta}\right)\frac{1}{\eta},$$

so

$$\left|\frac{\partial\phi}{\partial\bar{t}_j}\right|\delta \leqslant 2a_0 \text{ in } U,$$

where

$$a_0 = \max\left\{\left\|\frac{\partial\psi}{\partial\bar{z}_j}\right\|_\infty : \quad j = 1, \ldots, m\right\}.$$

By dint of (1.2), the function δ is locally Lipschitz with the constant 1. Hence ϕ is absolutely continuous. In view of (1.6) (applied to $F_1, \ldots, F_m$), using the inequality $\eta \leqslant \delta_Y$ and (3.3), by direct calculation we get:

$$\left|\frac{\partial\phi}{\partial\bar{x}_j}\right|\delta^{r+2} \leqslant a_0 c,$$

where c depends only on n, m, r, A.

The proof of Theorem 2 is completed.

In view of (1.4) (comp. also (1.5)) and Remark 1, from Theorem 2 we get:

COROLLARY 1 (Generalized Cnop's theorem). *There exists a constant $x > 0$ such that if for some $k_0 \geqslant 0$, $\|\eta^{k_0}\|_2 < +\infty$, then for every $k \geqslant 0$ there exist a linear continuous operator*

$$T\colon \mathcal{O}^{(k)}(Y, \eta) \to \mathcal{O}^{(k+k_0+\alpha)}(X, \delta)$$

such that $(Tf)(x, F(x)) = f(x)$, $x \in Y$.

Added in Proof. After this paper has been submitted for publication, the author learnt that recently, basing on the same general ideas, some results in the case of $X \subset C^n$ were earlier obtained in [4]; our Theorem 1 in the case $X \subset C^n$ and

$$U = \{x \in X\colon |G(x)| < \varepsilon\delta^N(x)\}$$

may be deduced from Theorem 1 of [4] and from our Lemma 1.

References

[1] I. Cnop, *Spectral study of holomorphic functions with bounded growth*, Ann. Inst. Fourier XXII (2) (1972).

[2] —, *Extending holomorphic functions with bounded growth from certain graphs*, Value Distribution Theory, M. Dekker (1975).

[3] M. Jarnicki, *Holomorphic functions with bounded growth on Riemann domains over* C^n, Zeszyty Naukowe UJ, 20 (1979).

[4] B. Jennane, *Extension d'une fonction définie sur une sous-variété avec contrôle de la croissance*, Lectures Notes in Math. vol. 694.

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