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Homomorphic Continuation of Holomorphic Functions with Bounded Growth

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Abstract. Let X be a Stein manifold and let M be an analytic subset of X. Let A, A_0 be subalgebras of $\mathcal{O}(X)$ and $\mathcal{O}(M)$, respectively, endowed with some topologies. In the paper we present some relations between continuous algebra-homomorphisms $T: A_0 \to A$ satisfying the condition $(Tf)|_M = f, f \in A_0$, and holomorphic retractions of X onto M.

Introduction. Let X be a connected n-dimensional Stein manifold, let M be an analytic subset of X and let R denote the restriction operator: $\mathcal{O}(X) \ni F \to F|_M \in \mathcal{O}(M)$, where, as usually, $\mathcal{O}(M)$ denotes the algebra of all continuous functions $f\colon M \to \mathbf{C}$ such that for every $a \in M$ there exist an open neighbourhood $U \in \operatorname{top} X$ of the point a and a function $g \in \mathcal{O}(U)$ for which $f|_{M\cap U} = g|_{M \cap U}$.

It is clear that R is an algebra-homomorphism of $\mathcal{O}(X)$ onto $\mathcal{O}(M)$ (R is surjective in virtue of Cartan's Theorem B — cf. [4], p. 177, Th. 5.11, [6], p. 245, Th. 18). Obviously R is also continuous if we endow the algebras $\mathcal{O}(X)$ and $\mathcal{O}(M)$ with the topologies of almost uniform convergence (i.e. uniform convergence on compact subsets) on X and M, respectively.

In view of the theory of holomorphic continuation it is interesting to extend holomorphic functions on M to holomorphic functions on X accordingly to the algebraical and topological structures of $\mathcal{O}(M)$ and $\mathcal{O}(X)$, in other words — to characterize the situations in which there exists a continuous algebra-homomorphism $T\colon \mathcal{O}(M) \to \mathcal{O}(X)$ such that $R \circ T = \mathrm{id}_{\mathcal{O}(M)}$.

Observe that if X is holomorphically retractible on M, i.e. there exists a holomorphic mapping $\pi\colon X \to M$ with $\pi|_M = \mathrm{id}_M$, then such an operator may be given by the formula:

$$\mathcal{O}(M) \ni f \overset{\pi^*}{\to} f \circ \pi \in \mathcal{O}(X).$$

The converse is also true, namely we have the following slightly more general theorem:

THEOREM 1. *For every algebra-homomorphism $T\colon \mathcal{O}(M) \to \mathcal{O}(X)$ with $R \circ T = \mathrm{id}_{\mathcal{O}(M)}$ there exists (uniquely determined) holomorphic retraction $\pi\colon X \to M$ such that $T = \pi^*$. In particular T has to be continuous.*

The theorem may be easily deduced from the well-known more general classical results — cf. [1], p. 141, Th. 8, [2]. However, since our situation is very special, in the sequel we shall present an independent short proof.

We want to generalize the above results in the following sense:

Let A (resp. A_0) be a subalgebra of $\mathcal{O}(X)$ (resp. $\mathcal{O}(M)$) endowed with a topology. We assume that:

— *the topologies of A and A_0 are stronger than the topologies of pointwise convergence on X and M, respectively,*

— *$R|_A$ maps continuously A into A_0,*

— *A separates points in X,*

— *M is determined by functions from A, i.e. there exists a family $\mathcal{F} \subset A$ such that $M = \bigcap_{F \in \mathcal{F}} F^{-1}(0)$.*

We shall consider the following two problems:

— given a holomorphic retraction $\pi: X \to M$, when $\pi^*|_{A_0}$ maps continuously A_0 into A?

— whether for every continuous algebra-homomorphism $T: A_0 \to A$ with $R \circ T = \mathrm{id}_{A_0}$ there exists a holomorphic retraction $\pi: X \to M$ such that $T = \pi^*|_{A_0}$ (observe that, under our assumptions, π is uniquelly determined by T).

Homomorphic continuation of holomorphic functions in the general case.

Remark 1. The existence of an algebra-homomorphism $T: A_0 \to A$ with $R \circ T = \mathrm{id}_{A_0}$ is algebraically equivalent to the existence of a decomposition $A = I(M, A) + B$, where $I(M, A) = A \cap \mathrm{Ker} R$, B is a subalgebra of A and $B \cap I(M, A) = \{0\}$. In particular, since X is connected, if such a decomposition exists then the ideal $I(M, A)$ is prime. Note that $I(M, A)$ is prime if and only if M is irreducible in the following sense: there are no analytic subsets M_1, M_2 determined by functions from A for which $M = M_1 \cup M_2$ and $M_j \neq M$, $j = 1, 2$.

Remark 2. The condition saying that the ideal $I(M, A)$ is prime is only necessary for existence of the homomorphism T. In [10], Prop. 5.3 the authors present examples of connected Stein manifolds Y such that, if $\Phi: Y \to \mathbf{C}^N$ is a Remmert embedding of Y (cf. [7], Th. 5.3.9), then, for $X = \mathbf{C}^N$, $M = \Phi(Y)$, there are no linear (only linear!) continuous operators $L: \mathcal{O}(M) \to \mathcal{O}(X)$ with $R \circ L = \mathrm{id}_{\mathcal{O}(M)}$, in spite of the ideal $I(M, \mathcal{O}(X))$ is obviously prime.

By the way, in view of the results of [10], it seems to be interesting to try to characterize algebra-homomorphisms among linear operators $L: \mathcal{O}(M) \to \mathcal{O}(X)$ with $R \circ L = \mathrm{id}_{\mathcal{O}(M)}$.

PROPOSITION 1. *Let $L: \mathcal{O}(M) \to \mathcal{O}(X)$ be a linear operator such that $R \circ L = \mathrm{id}_{\mathcal{O}(M)}$. Assume that L is continuous in the topologies of almost uniform convergence on M and* *pointwise convergence on X. Then the following conditions are equivalent:*

(i) *L is multiplicative;*

(ii) *$(Lf)(X) = f(M)$, $f \in \mathcal{O}(M)$;*

(iii) *$L(1) = 1$ and $L(\mathcal{O}^*(M)) \subset \mathcal{O}^*(X)$, where $\mathcal{O}^*(M)$ (resp. $\mathcal{O}^*(X)$) denotes the group of all invertible elements in $\mathcal{O}(M)$ (resp. $\mathcal{O}(X)$);*

(iv) *$L(1) = 1$ and $L(e^f) \in \mathcal{O}^*(X)$, $f \in \mathcal{O}(M)$;*

(v) *$L(e^f) = e^{L(f)}$, $f \in \mathcal{O}(M)$;*

(vi) *$L(\Phi \circ f) = \Phi \circ (Lf)$, $f \in \mathcal{O}(M)$, $\Phi \in \mathcal{O}(\boldsymbol{C})$.*

Proof. The plan of the proof: (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) ⇒ (i)

$$\begin{array}{ccccc} (i) & & & & (iv) \\ & \searrow & & \nearrow & \\ & & (vi) \Rightarrow (v) & & \end{array}$$

The implications (ii) ⇒ (iii) ⇒ (iv), (vi) ⇒ (v) ⇒ (iv) are obvious. For the implication (i) ⇒ (ii) suppose that for some $f \in \mathcal{O}(M)$ there exists a point $a \in (Lf)(X) \setminus f(M)$. Then $1 \equiv (Lf-a)L\left(\frac{1}{f-a}\right)$, so we get the contradiction. For the implication (i) ⇒ (vi), assume that $\phi(z) = \sum_{k=0}^{\infty} a_k z^k$. Then $L(\phi \circ f) = L \sum_{k=0}^{\infty} a_k f^k) = \sum_{k=0}^{\infty} a_k (Lf)^k = \phi \circ (Lf)$. The only non-evident implication is (iv) ⇒ (i).

LEMMA 1. *Every linear continuous operator $\xi\colon \mathcal{O}(M) \to \boldsymbol{C}$ such that $\xi(1) = 1$ and $\xi(e^f) \neq 0$, $f \in \mathcal{O}(M)$, is a character (i.e. non-zero homomorphism).*

Proof. The lemma is strictly connected to the well-known theorem on characterization of characters in Banach algebras — cf. [12], Th. 10.9. The proof will be analogous.

It is sufficient to prove that $f \in \operatorname{Ker}\xi \Rightarrow f^2 \in \operatorname{Ker}\xi$ (comp. [12]). For $f \in \operatorname{Ker}\xi$, let $\varphi(z) \doteq \xi(e^{zf})$, $z \in \boldsymbol{C}$. It is clear that $\varphi \in \mathcal{O}^*(\boldsymbol{C})$, $\varphi(0) = \xi(1) = 1$, $\varphi'(0) = \xi(f) = 0$ and, since ξ is continuous, $|\varphi(z)| \leqslant e^{\alpha+\beta|z|}$, $z \in \boldsymbol{C}$. Now, by Lemma 10.8 from [12] (after evident modifications) $\varphi \equiv 1$. In particular $\varphi''(0) = \xi(f^2) = 0$, what finishes the proof of the lemma.

Now, for the proof of the implication (iv) ⇒ (i) we only need to apply Lemma 1 to the mappings: $\mathcal{O}(M) \ni f \to (Lf)(x)$, $x \in X$.

The proof of Proposition 1 is completed.

Let $\operatorname{sp} A$ (resp. $\operatorname{sp} A_0$) denote the set of all continuous characters on A (resp. A_0). Let $E(X, A)$ (resp. $E(M, A_0)$) denote the set of all evaluations on A (resp. A_0) determined by points of X (resp. M), i.e. of all operators of the form:

$$A \ni F \to F(x) \in \boldsymbol{C}, \quad x \in X,$$

$$(\text{resp. } A_0 \ni f \to f(x) \in \boldsymbol{C},\ x \in M).$$

Note that $E(X, A) \subset \operatorname{sp} A$, $E(M, A_0) \subset \operatorname{sp} A_0$.

LEMMA 2. *If $\operatorname{sp} A = E(X, A)$ and $R(A) = A_0$ then $\operatorname{sp} A_0 = E(M, A_0)$.*

Proof. Let us take $\xi \in \mathrm{sp} A_0$. Then $\xi \circ R \in \mathrm{sp} A$. Hence there exists exactly one point $x \in X$ such that $(\xi \circ R)(F) = F(x)$, $F \in A$. In particular, for $F \in \mathscr{F}$ ($\mathscr{F}$ is a family determining M) we have $F(x) = \xi(0) = 0$. Thus $x \in M$. Now let $f \in A_0$ and let $F \in A$ be chosen such that $RF = f$. Then $f(x) = F(x) = \xi(RF) = \xi f$.

The proof is finished.

THEOREM 2. *Assume that* $\mathrm{sp} A = E(X, A)$, $R(A) = A_0$ *and there exist* $N \in \boldsymbol{N}$, $U \in \mathrm{top}\, \boldsymbol{C}^N$ *and* $\phi \in A^N$ *such that* ϕ *is an embedding of* X *onto a submanifold of* U. *Then for every continuous algebra-homomorphism* $T: A_0 \to A$ *with* $R \circ T = \mathrm{id}_{A_0}$ *there exists uniquely determined holomorphic retraction* $\pi: X \to M$ *such that* $T = \pi^*|_{A_0}$.

Proof. Let us take $x \in X$ and consider the functional

$$A_0 \ni f \xrightarrow{\xi} (Tf)(x) \in \boldsymbol{C}.$$

X is connected so $T(1) = 1$. Hence $\xi \in \mathrm{sp} A_0$. By Lemma 2 there exists a point $\pi(x) \in M$ such that $\xi f = f(\pi(x))$, $f \in A_0$, i.e. $(Tf)(x) = f(\pi(x))$, $f \in A_0$. Since A_0 separates points in M so $\pi|_M = \mathrm{id}_M$. Observe that $F \circ \pi = (T \circ R)(F)$, $F \in A$. In particular $\phi \circ \pi \in A^N$. Since $\pi = \phi^{-1} \circ (\phi \circ \pi)$, so π is holomorphic.

The proof is finished.

Remark 3. *Theorem 2 remains true without the assumption on existence of the embedding* ϕ *if we assume, for instance, that* A *is dense in* $\mathcal{O}(X)$ *and* T *is continuous in the topologies of almost uniform convergence on* M *and* X.

Proof. Analogously as in the proof of Theorem 2 we get a mapping $\pi: X \to M$ such that $\pi|_M = \mathrm{id}_M$ and $Tf = f \circ \pi$, $f \in A_0$.

Fix $a \in X$ and let $a_k \to a$ as $k \to +\infty$. Put $K = \{a, a_1, a_2, \ldots\}$. Since K is compact, there exists a compact $L \subset M$ such that $||Tf||_K \leqslant ||f||_L$, $f \in A_0$. In particular $\pi(a_k) \in \hat{L}_{A_0} \subset \hat{L}_A = \hat{L}_{\mathcal{O}(X)} \subset\subset X$. Let $b = \lim\limits_{l \to +\infty} \pi(a_{k_l})$. Then $f(b) = \lim\limits_{l \to +\infty} f(\pi(a_{k_l})) = \lim\limits_{l \to =\infty} (Tf)(a_{k_l}) = (Tf)(a)$, $f \in A_0$. Thus $b = \pi(a)$ and therefore π is continuous.

Let $a \in X$ and let $U, V \in \mathrm{top}\, X$, $F_1, \ldots, F_n \in A$ be such that $a \in U$, $\pi(U) \subset V$ and $(F_1|_V, \ldots, F_n|_V)$ is a coordinate system on V. Since $F_j \circ \pi = (T \circ R)(F_j)$, $j = 1, \ldots, n$, so π is holomorphic.

The proof is completed.

Remark 4. In Theorem 2 the assumption $\mathrm{sp} A = E(X, A)$ cannot be omitted — a counterexample will be given below.

Let $H^\infty(X)$ (resp. $H^\infty(M)$) denote the algebra of all bounded holomorphic functions on X (resp. M) endowed with the topology generated by the supremum-norm. Obviously for every holomorphic retraction $\pi: X \to M$, $\pi^*|_{H^\infty(M)}$ maps continuously $H^\infty(M)$ into $H^\infty(X)$. However, even under very restrictive assumptions on X and M, there exist continuous algebra-homomorphisms $T: H^\infty(M) \to H^\infty(X)$ with $R \circ T = \mathrm{id}_{H^\infty(M)}$ which are not given by any holomorphic retraction. This will be shown in the following construction based on some results of [13].

Let Δ be a connected domain of holomorphy in $\boldsymbol{D}^2$ ($\boldsymbol{D} = \{z \in C: |z| < 1\}$) such that $\Delta \neq \boldsymbol{D}^2$ and the restriction operator

$$H^\infty(\boldsymbol{D}^2) \ni \varphi \xrightarrow{R_\Delta} \varphi|_\Delta \in H^\infty(\Delta)$$

is an isomorphism and an isometry (for the construction of such a domain Δ see [13]). Let

$$U = \{(z_1, z_2, z_3) \in \Delta \times \boldsymbol{D}: |z_3| < d_\Delta(z_1, z_2)\},$$

where d_Δ denotes the distance function to $\boldsymbol{C}^2 \setminus \Delta$ in the sense of the polydiscal norm.

One can prove that U is an H^∞ — domain of holomorphy (cf. [13]).

Let us fix $a \in \Delta$, $b \in \boldsymbol{C}^2$ such that $a+b \in \boldsymbol{D}^2 \setminus \Delta$ and let $c = \frac{2}{d_\Delta(a)} b$. Put $\psi(z_1, z_2, z_3) = (z_1 + c_1 z_3, z_2 + c_2 z_3, z_3)$. It is clear that ψ is an authomorphism of $\boldsymbol{C}^3$, hence $\psi(U)$ is also an H^∞-domain of holomorphy. Observe that $\psi(U) \cap \{z_3 = 0\} = \Delta \times \{0\}$. Let X denote the connected component of $\psi(U) \cap \boldsymbol{D}^3$ containing $\Delta \times \{0\}$ and let $M = X \cap \{z_3 = 0\}$. Clearly $M = \Delta \times \{0\}$. Note that $\left(a_1, a_2, \frac{\theta}{2} d_\Delta(a)\right) \in U$, $0 \leqslant \theta \leqslant 1$, hence $\left(a_1 + \theta b_1, a_2 + \theta b_2, \frac{\theta}{2} d_\Delta(a)\right) \in \psi(U) \cap \boldsymbol{D}^3$, $0 \leqslant \theta \leqslant 1$. In particular $\left(a_1 + b_1, a_2 + b_2, \frac{1}{2} d_\Delta(a)\right) \in X$.

Let

$$\boldsymbol{D}^3 \ni (z_1, z_2, z_3) \xrightarrow{\pi_0} (z_1, z_2, 0) \in \boldsymbol{D}^2 \times \{0\}.$$

Note that $(a_1 + b_1, a_2 + b_2, 0) \in \pi_0(X)$, so $\pi_0(X) \neq M$.

Now let

$$Tf = [(\pi_0^* \circ R_\Delta^{-1})(f)]|_X, \; f \in H^\infty(M)$$

(we identify M with Δ and $\boldsymbol{D}^2 \times \{0\}$ with $\boldsymbol{D}^2$).

Obviously T: $H^\infty(M) \to H^\infty(X)$ is a continuous homomorphism with $R \circ T = \mathrm{id}_{H^\infty(M)}$ but T is not given by any holomorphic retraction of X onto M.

One can easily prove that $X \cup (\boldsymbol{D}^2 \times \{0\}) \subset \mathrm{sp}\, H^\infty(X)$.

Observe that X is holomorphically retractible on M, for instance by the retraction

$$X \ni (z_1, z_2, z_3) \to (z_1 - c_1 z_3, z_2 - c_2 z_3, z_3) \in M.$$

The proof of Theorem 1. For the proof we only need to apply Theorem 2 and the following classical results:

— $M = \bigcap_{F \in \mathrm{Ker}\, R} F^{-1}(0)$ — cf. [6], p. 245, Th. 18,

— every character on $\mathcal{O}(X)$ is continuous and $\mathrm{sp}\,\mathcal{O}(X) = E(X, \mathcal{O}(X))$ — cf. [5], p. 177, Th. 2,

— there exists a Remmert embedding $\phi \in [\mathcal{O}(X)]^{2n+1}$ of X onto a submanifold of $\boldsymbol{C}^{2n+1}$ — cf. [7], Th. 5.3.9.

Homomorphic continuation of holomorphic functions with bounded growth. Let X and M be as in the previous sections. Suppose that $\delta\colon X \to (0, 1]$ is a continuous function and put:

$$\mathcal{O}^{(k)}(X, \delta) = \{F \in \mathcal{O}(X)\colon \|\delta^k F\|_X < +\infty\}, \quad \mathcal{O}(X, \delta) = \bigcup_{k \geqslant 0} \mathcal{O}^{(k)}(X, \delta),$$

analogously

$$\mathcal{O}^{(k)}(M, \delta) = \{f \in \mathcal{O}(M)\colon \|\delta^k f\|_M < +\infty\}, \quad \mathcal{O}(M, \delta) = \bigcup_{k \geqslant 0} \mathcal{O}^{(k)}(M, \delta).$$

It is seen that $\mathcal{O}^{(k)}(X, \delta)$ (resp. $\mathcal{O}^{(k)}(M, \delta)$) is a vector space normed by the function $F \to \|\delta^k F\|_X$ (resp. $f \to \|\delta^k f\|_M$) and that the topology generated by this norm is stronger than the topology of almost uniform convergence. Obviously $\mathcal{O}(X, \delta)$ and $\mathcal{O}(M, \delta)$ are complex algebras.

Let us recall some definitions (cf. [3]). A pair $(E, (E_k)_k)$ is said to be *a polynormed vector space* if any E_k is a normed space, $E_k \subset E_l$ for $k \leqslant l$, id_{E_k} is a continuous mapping of E_k into E_l for $k \leqslant l$ and $E = \bigcup_k E_k$. We say that a linear operator $\xi\colon E \to \boldsymbol{C}$ is *continuous* if, for every k, $\xi|_{E_k}$ maps continuously E_k into $\boldsymbol{C}$. A linear operator $L\colon E \to F$, where $(F, (F_l)_l)$ is also a polynormed space, is said to be *continuous* if for every k there exists l such that $L|_{E_k}$ is a continuous mapping of E_k into F_l.

Observe that $R|_{\mathcal{O}(X,\delta)}$ is a continuous mapping (in the sense of the above definition) of $\mathcal{O}(X, \delta)$ into $\mathcal{O}(M, \delta)$ (in the general case is not known whether this operator is surjective).

Let $\pi\colon X \to M$ be a holomorphic retraction satisfying the inequality $\delta^{\varkappa} \leqslant c\delta \circ \pi$ for some $\varkappa, c > 0$. Then

$$\|\delta^{\varkappa k} \pi^*(f)\|_X \leqslant c^k \|\delta^k f\|_M, \quad f \in \mathcal{O}^k(M, \delta).$$

Hence $\pi^*|_{\mathcal{O}(M,\delta)}$ is a continuous operator of $\mathcal{O}(M, \delta)$ into $\mathcal{O}(X, \delta)$.

It is natural to ask whether this is the universal form of continuous algebra-homomorphisms $T\colon \mathcal{O}(M, \delta) \to \mathcal{O}(X, \delta)$ with $R \circ T = \mathrm{id}_{\mathcal{O}(M,\delta)}$. Below we shall present a partial answer to this question.

Assume additionally that X is a Riemann domain and let $p\colon X \to \boldsymbol{C}^n$ denote its locally biholomorphic projection into $\boldsymbol{C}^n$. Let us introduce some notations (cf. [8], [9]). Let $\varrho_X(x)$ denote the maximal number $r > 0$ such that there exists an open neighbourhood $\hat{B}(x, r)$ of the point x which is mapped homeomorphically by p onto the Euclidean ball $B(p(x), r) \subset \boldsymbol{C}^n$. A function $\delta\colon X \to (0, 1]$ is called *a weight function on* X if $\delta \leqslant \delta_X = \min\{(1+|p|^2)^{-1/2}, \varrho_X\}$ and $|\delta(x) - \delta(x')| \leqslant |p(x) - p(x')|$ for every $x \in X$, $x' \in \hat{B}(x, \varrho_X(x))$.

LEMMA 3. *Let (X, p) be a connected Riemann-Stein domain over $\boldsymbol{C}^n$, let M be an analytic subset of X, let δ be a weight function on X such that $-\log\delta$ is plurisubharmonic. Suppose that for a holomorphic retraction $\pi\colon X \to M$, $\pi^*|_{\mathcal{O}(M,\delta)}$ maps continuously $\mathcal{O}(M, \delta)$ into $\mathcal{O}(X, \delta)$. Then there exist constants $\varkappa, c > 0$ such that $\delta^{\varkappa} \leqslant c\delta \circ \pi$.*

Proof. Using the methods of the proof of Th. 1 in [8] we get: for every $a \in X$ there exists a function $F_a \in \mathcal{O}^{(6n+1)}(X, \delta)$ such that $\delta(a) F_a(a) = 1$ and $\|\delta^{6n+1} F_a\|_X \leqslant c(n)$, where $c(n)$ depends only on n. Since $\pi^*|_{\mathcal{O}(M,\delta)}$ is continuous so for some $\varkappa, c>0$ we have: $\delta^{\varkappa}(x)|F_a(\pi(x))| \leqslant c, a, \ x \in X$. Putting $a = \pi(x)$ we get the thesis.

Remark 5. By a remark of P. Pflug (cf. [11]), from the proof of Th. 3 in [8] follows that, under the assumptions of Lemma 3, $\mathcal{O}^{(4n)}(X, \delta)$ separates points in X. Hence $\mathcal{O}(X, \delta)$ is dense in $\mathcal{O}(X)$ — cf. [8], Th. 4.

THEOREM 3. *Let $X = D$ be a connected domain of holomorphy in $\mathbf{C}^n$. Let $\delta\colon \mathbf{C}^n \to [0, 1]$ satisfy the conditions*

$$\delta(x) \leqslant (1+|x|^2)^{-1/2}, \ |\delta(x)-\delta(x')| \leqslant |x-x'|, \ x, x' \in \mathbf{C}^n, \ D = \{x \in \mathbf{C}^n\colon \delta(x)>0\}$$

and $-\log\delta$ is plurisubharmonic on D. Let M be an analytic subset of D determined by functions from $\mathcal{O}(D, \delta)$. Then for every continuous algebra-homomorphism $T\colon \mathcal{O}(M, \delta) \to \mathcal{O}(D, \delta)$ with $R \circ T = \mathrm{id}_{\mathcal{O}(M,\delta)}$ there exists uniquelly determined holomorphic retraction $\pi\colon D \to M$ such that $T = \pi^*|_{\mathcal{O}(M,\delta)}$ and $\delta^{\varkappa} \leqslant c\delta \circ \pi$ for some $\varkappa, c>0$.

Proof. By Th. 6, p. 52 in [3], $\mathrm{sp}\,\mathcal{O}(D, \delta) = E(D, \mathcal{O}(D, \delta))$. Obviously $\mathrm{id}_D \in [\mathcal{O}(D, \delta)]^n$. Thus our theorem follows from Theorem 2, Lemma 3 and Remark 5.

Remark 6. The example given in [9] shows that, even under the assumptions of Theorem 3, there exist holomorphic retractions $\pi \in [\mathcal{O}(D, \delta)]^n$ for which π^* does not map $\mathcal{O}(M, \delta)$ into $\mathcal{O}(D, \delta)$. However basing on the methods of [9] one can prove the following:

THEOREM 4. *Let D be a connected domain of holomorphy in $\mathbf{C}^n$, let δ be a weight function on D such that $-\log\delta$ is plurisubharmonic and let M be an analytic subset of D determined by a finite number of functions from $\mathcal{O}(D, \delta)$. Assume that there exists a holomorphic retraction $\pi\colon D \to M$ such that $\pi \in [\mathcal{O}(D, \delta)]^n$. Then $R(\mathcal{O}(D, \delta)) = \mathcal{O}(M, \delta)$, more exactly: there exist constants $\alpha, A>0$ such that for every $k \geqslant 0$ there exists a linear continuous operator $L_k\colon \mathcal{O}^{(k)}(M, \delta) \to \mathcal{O}^{(k+\alpha)}(D, \delta)$ such that $R \circ L_k = \mathrm{id}_{\mathcal{O}^{(k)}(M,\delta)}$ and $\|L_k\| \leqslant 4^k A$.*

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