A Note on Harmonic Envelopes of Holomorphy

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Abstract. Basing on [1], an effective construction of the envelope of holomorphy of the ball with respect to some spaces of separately harmonic functions will be presented.

Throughout the paper we shall denote by $\Delta_N$ the Laplace operator in $\mathbb{R}^N$,

$$\Delta_N = \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2},$$

and by $P_{2N}$ — the system of pluriharmonic operators in $\mathbb{R}^{2N}$,

$$P_{2N} = \left( \frac{\partial^2}{\partial x_{2j-1} \partial x_{2k-1}} \right)_{j,k=1,\ldots,N}.$$

Let us fix $n_1, \ldots, n_s \in \mathbb{N} \cap [2, +\infty)$. Put $n = n_1 + \ldots + n_s$ and let $\mathcal{L} = (\mathcal{L}_1, \ldots, \mathcal{L}_s)$ be a system of differential operators in $\mathbb{R}^n = \mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_s}$ such that:

1. $\mathcal{L}_j$ depends only on the variables belonging to $\mathbb{R}^{n_j}$,

2. $\mathcal{L}_j$ considered as a differential operator in $\mathbb{R}^{n_j}$ is either the Laplace operator $\Delta_{n_j}$ or the pluriharmonic system $P_{n_j}$ (obviously the last case is admissible only if $n_j$ is an even number).

For a domain $D \subset \mathbb{R}^n$ let $\mathcal{L}(D)$ denote the space of all $f \in C^2(D, \mathbb{C})$ for which $\mathcal{L}_j(f) = 0$, $j = 1, \ldots, s$. It is seen that all classical spaces of separately harmonic or pluriharmonic functions may be described in such a way. One may prove that $\mathcal{L}(D)$ is a closed (in the topology of uniform convergence on compact subsets of $D$) subspace of the space $H(D)$ of all harmonic functions in $D$.

We shall consider the problem of the effective construction of the envelope of holomorphy of the Euclidean ball

$$B_0(r) = \{x \in \mathbb{R}^n : ||x|| < r\}$$

with respect to the space $\mathcal{L}(B_0(r))$; shortly: the $\mathcal{L}$-envelope of holomorphy of $B_0(r)$.

In some particular cases the problem was earlier solved in [5], [6], [3] ($s = 1$, $\mathcal{L} = \Delta_n$) and [4] ($s = 1$, $\mathcal{L} = P_{2n}$). Unfortunately, the methods used in these papers cannot be adopted to the general case. Recently, theoretical foundations for the construction of the
\(L\)-envelopes of holomorphy were given in [1]. Our note may be regarded as a trial of a practical application of Th. 6 from [1].

We shall use the following notations:

\[ \langle z, w \rangle = \sum_{j=1}^{N} z_j w_j, \quad \|z\| = \langle z, z \rangle^{1/2}, \quad z = (z_1, \ldots, z_N), \quad w = (w_1, \ldots, w_N) \in \mathbb{C}^N; \]

\[ B(r) = \{ z \in \mathbb{C}^n : \|z\| < r \}; \]

\[ \mathcal{L}(B(r)) = \text{the space of all functions } f \text{ holomorphic in } B(r) \text{ such that } f|_{B_0(r)} \in \mathcal{L}(B_0(r)); \]

it may be easily verified that \(\mathcal{L}(B(r))\) is a closed subspace of the space of all holomorphic functions in \(B(r)\);

\[ T(\Delta_N) = \{ z = (z_1, \ldots, z_N) \in \mathbb{C}^N : z_1^2 + \ldots + z_N^2 = 0 \}; \]

\[ T(\mathcal{P}_{2N}) = \{ z \in \mathbb{C}^{2N} : z_{2j-1}z_{2k-1} + z_{2j}z_{2k} = 0, j, k = 1, \ldots, N \}; \]

\[ T(\mathcal{L}) = T(\mathcal{L}_1) \times \ldots \times T(\mathcal{L}_s); \]

\[ q_{x_j}(z) = \max \{ \|z, w\| : w \in T(\mathcal{L}_j), \|w\| = 1 \}, \quad z \in \mathbb{C}^n, \quad j = 1, \ldots, s; \]

\[ q_{x}(z) = \max \{ \|z, w\| : w \in T(\mathcal{L}), \|w\| = 1 \}, \quad z \in \mathbb{C}^n; \quad \text{note that } q_{x} \text{ is a seminorm}; \]

\[ \bar{B}_{x}(r) = \{ z \in \mathbb{C}^n : q_{x}(z) < r \}. \]

Directly from Th. 6 in [1] we get

**THEOREM I.** The domain \(\text{int}(\bar{B}_{x}(r))\) is the \(\mathcal{L}(B(r))\)-envelope of holomorphy of \(B(r)\).

Let \( t : \mathbb{C}^N \to [0, +\infty) \) denote the complexification of the Euclidean norm in \(\mathbb{R}^N\). Recall that (comp. [2], [6]):

\[ t(z) = \|z\|^2 + (\|z\|^4 - \langle z, z \rangle^2)^{1/2}; \]

or, in the real coordinates,

\[ t(x + iy) = \|x\|^2 + \|y\|^2 + 2(\|x\|^2 \|y\|^2 - \langle x, y \rangle^2)^{1/2}. \]

Let \( u : \mathbb{C}^{2N} \to [0, +\infty) \) denote the norm given by the formula:

\[ u(z) = \max \left\{ \left( \sum_{j=1}^{N} |z_{2j-1} - iz_{2j}|^2 \right)^{1/2}, \left( \sum_{j=1}^{N} |z_{2j-1} + iz_{2j}|^2 \right)^{1/2} \right\}. \]

Observe that \( u = t \) in \(\mathbb{C}^2\) (comp. [3]), \( u \leq t, u \not\equiv t \) in \(\mathbb{C}^{2N}, N \geq 2 \) and \( u(x) = \|x\|, x \in \mathbb{R}^{2N}. \)

The purpose of this note is to prove the following

**THEOREM 1.**

(i) \( q_{\Delta}(z) = \left[ \sum_{i=1}^{N} q_{\Delta_j}(z^i) \right]^{1/2}, \quad z = (z^1, \ldots, z^N) \in \mathbb{C}^n, \quad \mathbb{C}^{n_1} \times \ldots \times \mathbb{C}^{n_k}; \)

(ii) \( q_{\Delta_N} = \frac{1}{\sqrt{2}} t; \)

(iii) \( q_{\Delta_{2N}} = \frac{1}{\sqrt{2}} u. \)
Remark 1. $q_x$ is a norm and it may be effectively calculated.

Hence, in view of Th. 1 we get

**Remark 2.** $\mathcal{B}_x(r)$ is the $\mathcal{L}(B(r))$-envelope of holomorphy of $B(r)$.

**Theorem 2.** $\mathcal{B}_x(r/\sqrt{2})$ is the $\mathcal{L}$-envelope of holomorphy of $B_0(r)$.

**Proof.** $\mathcal{L}(B_0(r)) \subseteq \mathcal{H}(B_0(r))$, so by classical properties of harmonic functions (comp. [6], Th. B), for every $f \in \mathcal{L}(B_0(r))$ there exists $f_\mathcal{L} \in \mathcal{L}(B(r/\sqrt{2}))$ such that $f_\mathcal{L} = f$ on $B_0(r/\sqrt{2})$. In view of Th. 1, $q_x(x) = \frac{1}{\sqrt{2}} ||x||$, $x \in \mathbb{R}^n$. In particular, $B_0(r) \subseteq \mathcal{B}_x(r/\sqrt{2})$.

Thus the $\mathcal{L}(B(r/\sqrt{2}))$-envelope of holomorphy of $B(r/\sqrt{2})$ is the $\mathcal{L}$-envelope of $B_0(r)$.

The proof is completed.

**Corollary 1.** ([5], [6], [3]). The Lie ball $\mathcal{B} = \{z \in \mathbb{C}^n : t(z) < r\}$ is the harmonic envelope of holomorphy of $B_0(r)$.

**Corollary 2** ([4]). The domain $\mathcal{B} \{z \in \mathbb{C}^2 : u(z) < r\}$ is the pluriharmonic envelope of holomorphy of $B_0(r)$.

**Corollary 3.** Let $\mathcal{L}, \mathcal{L}'$ be two systems satisfying (1), (2) with the same decomposition $n = n_1 + \ldots + n_s$. Then $\mathcal{B}_x(r) = \mathcal{B}_x(r) \iff \mathcal{L} = \mathcal{L}'$.

**Proof of Theorem 1.** The proof of (i) follows directly from the fact that $T(\mathcal{L}_1), \ldots, T(\mathcal{L}_s), T(\mathcal{L})$ are closed $C$-cones, namely

$$q_x(z^1, \ldots, z^s) = \max \{t_1 q_x(z^1) + \ldots + t_s q_x(z^s) : t_1, \ldots, t_s \geq 0, t_1^2 + \ldots + t_s^2 = 1 \} = \left[ \sum_{j=1}^{s} q_x^2(z^j) \right]^{1/2}.$$

(ii) Let us fix $z \in \mathbb{C}^n$, $z \neq 0$. Since $q_{A_\mathcal{L}}$ and $t$ are seminorms, it is sufficient to consider the case when $||z|| = 1$ and $\langle z, z \rangle \in \mathbb{R}$, which in the real coordinates $z = x + iy$ means that $\langle x, y \rangle = 0$.

In view of the definition of $q_{A_\mathcal{L}}$, there exists a point $w = \xi + i\eta \in T(A_\mathcal{L})$, $||w|| = 1$ such that $q_{A_\mathcal{L}}(z) = \langle z, w \rangle$. Hence $q_{A_\mathcal{L}}(z) = \langle x, \xi \rangle - \langle y, \eta \rangle$ and $||\xi|| = ||\eta|| = \frac{1}{\sqrt{2}}$. In consequence $q_{A_\mathcal{L}}(z) \leq \frac{1}{\sqrt{2}} (||x|| + ||y||) = \frac{1}{\sqrt{2}} t(z)$. Now it remains to find $w_0 \in T(A_\mathcal{L})$, $||w_0|| = 1$ such that $\langle z, w_0 \rangle = \frac{1}{\sqrt{2}} (||x|| + ||y||)$. We shall distinguish three cases:

a) $x, y \neq 0$ — we put $w_0 = \frac{1}{\sqrt{2}} \left( \frac{x}{||x||} - i \frac{y}{||y||} \right)$. 

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b) \( x \neq 0, \ y = 0 \rightarrow w_0 = \frac{1}{\sqrt{2}} (x + i\eta_0) \), where \( \eta_0 \in \mathbb{R}^N \) is chosen in such a way that \( \langle x, \eta_0 \rangle = 0, ||\eta_0|| = 1 \).

c) \( x = 0, \ y \neq 0 \rightarrow w_0 = \frac{1}{\sqrt{2}} (\xi_0 - i\eta_0) \), where \( \langle y, \xi_0 \rangle = 0, ||\xi_0|| = 1 \).

For the proof of (iii) let us remark that

\[
T(\mathcal{D}_{2N}) = \{ (v_1, \tau v_1, \ldots, v_N, \tau v_N) : (v_1, \ldots, v_N) \in \mathbb{C}^N, \tau = \pm i \}.
\]

Hence

\[
q_{\mathcal{D}_{2N}}(z) = \max \left\{ \left| \sum_{j=1}^{N} (z_{2j-1} + \tau z_{2j}) v_j \right| : \begin{array}{l}
v = (v_1, \ldots, v_N) \in \mathbb{C}^N, ||v|| = \frac{1}{\sqrt{2}}, \tau = \pm i \end{array} \right\} = \frac{1}{\sqrt{2}} u(z).
\]

The proof is finished.

Remark 3 (due to J. Siciak). The equality (ii) in Theorem 1 may be interpreted as a new description of the crossnorm in complexified unitary spaces (see the condition (iv) below).

Let \( E \) be a unitary space over \( \mathbb{R} \) and let \( \langle , \rangle_E \) denote its scalar product. Put \( ||x||_E = (x, x)_E^{1/2} \). Let \( \bar{E} = E + iE \) denote the complexification of \( E \) and let

\[
||x + iy||_E = (||x||_E^2 + ||y||_E^2)^{1/2}.
\]

Define \( \langle , \rangle_{\bar{E}} : \bar{E} \times \bar{E} \to \mathbb{C} \) by the formula

\[
\langle x + iy, u + iv \rangle_{\bar{E}} = [(x, u)_E - (y, v)_E] + i[(x, v)_E + (y, u)_E].
\]

It is easily seen that \( \langle , \rangle_{\bar{E}} \) is a \( \mathbb{C} \)-bilinear symmetric continuous mapping such that \( \langle x, u \rangle_{\bar{E}} = (x, u)_E, x, u \in E \). For \( z = x + iy \in \bar{E} \) let us put \( \bar{z} = x - iy \) and define \( (z, w)_E = \langle z, \bar{w} \rangle_{\bar{E}} \), \( z, w \in \bar{E} \). Clearly \( ( , )_{\bar{E}} \) is a complex scalar product on \( \bar{E} \) such that \( (x, u)_{\bar{E}} = (x, u)_E, x, u \in E \) and \( ||z||_{\bar{E}} = ||z, z||_{\bar{E}} \).

For a normed nonzero complex vector space \( (F, || \cdot ||_F) \) we shall denote by \( P^k(\bar{E}, F) \) the space of all continuous homogeneous polynomials of degree \( k \) from \( \bar{E} \) into \( F \), \( k \in \mathbb{N} \).

The crossnorm \( t \) on \( \bar{E} \) is defined as follows:

(i) \( t(z) = \inf \{ \sum_{j=1}^{N} |\lambda_j| ||x_j||_E : \ N \in \mathbb{N}, \ \lambda_j \in \mathbb{C}, \ x_j \in E, \ j = 1, \ldots, N, \ z = \sum_{j=1}^{N} \lambda_j x_j, \ z \in \bar{E}. \)

It is known (see [2]) that

(ii) \( t(z) = \sup \{ ||f(z)||_{\bar{E}}^{1/k} : f \in P^k(\bar{E}, F), (x, E, ||x||_E \leq 1) \Rightarrow ||f(x)||_F \leq 1 \}, \ k \in \mathbb{N}; \)

(iii) \( t(x + iy) = (||x||_{\bar{E}}^2 + ||y||_{\bar{E}}^2 + 2||x||_{\bar{E}}||y||_{\bar{E}} - (x, y)_{\bar{E}}^2)^{1/2})^{1/2}, \)

which in the "complex" coordinates gives

\[
t(z) = \left[ ||z||_{\bar{E}}^2 + (||z||_{\bar{E}}^4 - ||z, z||_{\bar{E}}^2)^{1/2} \right]^{1/2}.
\]
Repeating (with only formal changes) the proof of Theorem 1 (ii) we get
(iv) $t(z) = \max \{ |\langle z, w \rangle|_{E^*} : w \in \tilde{E}, \langle w, w \rangle_{E^*} = 0, ||w||_{E^*} = \sqrt{2} \}$,
or, equivalently, in the “real coordinates”

$$t(x + iy) = \max \{ |\langle x + iy, u + iv \rangle|_{E^*} : u, v \in E, ||u||_{E^*} = ||v||_{E^*} = 1, (u, v)_{E^*} = 0 \}.$$ 

References


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