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A Note on Harmonic Envelopes of Holomorphy

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Abstract. Basing on [1], an effective construction of the envelope of holomorphy of the ball with respect to some spaces of separately harmonic functions will be presented.

Throughout the paper we shall denote by Δ_N the Laplace operator in $\boldsymbol{R}^N$,

$$\Delta_N = \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2},$$

and by P_{2N} — the system of pluriharmonic operators in $\boldsymbol{R}^{2N}$,

$$\mathscr{P}_{2N} = \left(\frac{\partial^2}{\partial x_{2j-1}\partial x_{2k-1}} + \frac{\partial^2}{\partial x_{2j}\partial x_{2k}}, \frac{\partial^2}{\partial x_{2j-1}\partial x_{2k}} - \frac{\partial^2}{\partial x_{2j}\partial x_{2k-1}}\right)_{j,k=1,\ldots,N}.$$

Let us fix $n_1, \ldots, n_s \in \boldsymbol{N} \cap [2, +\infty)$. Put $n = n_1 + \ldots + n_s$ and let $\mathscr{L} = (\mathscr{L}_1, \ldots, \mathscr{L}_s)$ be a system of differential operators in $\boldsymbol{R}^n = \boldsymbol{R}^{n_1} \times .. \times \boldsymbol{R}^{n_s}$ such that:

(1) $\mathscr{L}_j$ depends only on the variables belonging to $\boldsymbol{R}^{n_j}$,

(2) $\mathscr{L}_j$ considered as a differential operator in $\boldsymbol{R}^{n_j}$ is either the Laplace operator Δ_{n_j} or the pluriharmonic system $\mathscr{P}_{n_j}$ (obviously the last case is admissible only if n_j is an even number).

For a domain $D \subset \boldsymbol{R}^n$ let $\mathscr{L}(D)$ denote the space of all $f \in C^2(D, \boldsymbol{C})$ for which $\mathscr{L}_j(f) = 0$, $j = 1, \ldots, s$. It is seen that all classical spaces of separately harmonic or pluriharmonic functions may be described in such a way. One may prove that $\mathscr{L}(D)$ is a closed (in the topology of uniform convergence on compact subsets of D) subspace of the space $\mathscr{H}(D)$ of all harmonic functions in D.

We shall consider the problem of the effective construction of the envelope of holomorphy of the Euclidean ball

$$B_0(r) = \{x \in \boldsymbol{R}^n\colon ||x|| < r\}$$

with respect to the space $\mathscr{L}(B_0(r))$; shortly: *the $\mathscr{L}$-envelope of holomorphy of* $B_0(r)$.

In some particular cases the problem was earlier solved in [5], [6], [3] ($s = 1$, $\mathscr{L} = \Delta_n$) and [4] ($s = 1$, $\mathscr{L} = \mathscr{P}_{2n}$). Unfortunately, the methods used in these papers cannot be adopted to the general case. Recently, theoretical foundations for the construction of the

$\mathscr{L}$-envelopes of holomorphy were given in [1]. *Our note may be regarded as a trial of a practical application of Th.* 6 *from* [1].

We shall use the following notations:

$$\langle z, w\rangle = \sum_{j=1}^{N} z_j w_j, \quad \|z\| = \langle \bar{z}, z\rangle^{1/2}, \; z = (z_1, \ldots, z_N), \; w = (w_1, \ldots, w_N) \in \boldsymbol{C}^N;$$

$$B(r) = \{z \in \boldsymbol{C}^n: \|z\| < r\};$$

$\mathscr{L}(B(r))$ = the space of all functions f holomorphic in $B(r)$ such that $f|_{B_0(r)} \in \mathscr{L}(B_0(r))$; it may be easily verified that $\mathscr{L}(B(r))$ is a closed subspace of the space of all holomorphic functions in $B(r)$;

$T(\Delta_N) = \{z = (z_1, \ldots, z_N) \in \boldsymbol{C}^N: z_1^2 + \ldots + z_N^2 = 0\}$;

$T(\mathscr{P}_{2N}) = \{z \in \boldsymbol{C}^{2N}: z_{2j-1}z_{2k-1} + z_{2j}z_{2k} = 0, \; z_{2j-1}z_{2k} - z_{2j}z_{2k-1} = 0, \; j, k = 1, \ldots, N\}$;

$T(\mathscr{L}) = T(\mathscr{L}_1) \times \ldots \times T(\mathscr{L}_s)$;

$q_{\mathscr{L}_j}(z) = \max\{|\langle z, w\rangle|: w \in T(\mathscr{L}_j), \|w\| = 1\}, \; z \in \boldsymbol{C}^{n_j}, \; j = 1, \ldots, s$;

$q_{\mathscr{L}}(z) = \max\{|\langle z, w\rangle|: w \in T(\mathscr{L}), \|w\| = 1\}, \; z \in \boldsymbol{C}^n$; note that $q_{\mathscr{L}}$ is a seminorm;

$\tilde{B}_{\mathscr{L}}(r) = \{z \in \boldsymbol{C}^n: q_{\mathscr{L}}(z) < r\}$.

Directly from Th. 6 in [1] we get

THEOREM I. *The domain* $\operatorname{int}(\tilde{B}_{\mathscr{L}}(r))$ *is the* $\mathscr{L}(B(r))$*-envelope of holomorphy of* $B(r)$.

Let $\mathrm{t}: \boldsymbol{C}^N \to [0, +\infty)$ denote the complexification of the Euclidean norm in $\boldsymbol{R}^N$. Recall that (comp. [2], [6]):

$$\mathrm{t}(z) = [\|z\|^2 + (\|z\|^4 - |\langle z, z\rangle|^2)^{1/2}]^{1/2}$$

or, in the real coordinates,

$$\mathrm{t}(x+iy) = [\|x\|^2 + \|y\|^2 + 2(\|x\|^2\|y\|^2 - \langle x, y\rangle^2)^{1/2}]^{1/2}.$$

Let $\mathrm{u}: C^{2N} \to [0, +\infty)$ denote the norm given by the formula:

$$\mathrm{u}(z) = \max\{(\sum_{j=1}^{N} |z_{2j-1} - iz_{2j}|^2)^{1/2}, \; (\sum_{j=1}^{N} |z_{2j-1} + iz_{2j}|^2)^{1/2}\}.$$

Observe that $\mathrm{u} = \mathrm{t}$ in $\boldsymbol{C}^2$ (comp. [3]), $\mathrm{u} \leqslant \mathrm{t}$, $\mathrm{u} \not\equiv \mathrm{t}$ in $\boldsymbol{C}^{2N}$, $N \geqslant 2$ and $\mathrm{u}(x) = \|x\|$, $x \in \boldsymbol{R}^{2N}$.

The purpose of this note is to prove the following

THEOREM 1. (i) $q_{\mathscr{L}}(z) = [\sum_{j=1}^{s} q_{\mathscr{L}_j}^2(z^j)]^{1/2}, \; z = (z^1, \ldots, z^s) \in \boldsymbol{C}^n = \boldsymbol{C}^{n_1} \times \ldots \times \boldsymbol{C}^{n_s}$;

(ii) $q_{\Delta_N} = \frac{1}{\sqrt{2}}\mathrm{t}$;

(iii) $q_{\mathscr{P}_{2N}} = \frac{1}{\sqrt{2}}\mathrm{u}$.

Remark 1. *$q_{\mathscr{L}}$ is a norm and it may be effctively calculated.*

Hence, in view of Th. I we get

Remark 2. *$\tilde{B}_{\mathscr{L}}(r)$ is the $\mathscr{L}(B(r))$-envelope of holomorphy of $B(r)$.*

THEOREM 2. *$\tilde{B}_{\mathscr{L}}(r/\sqrt{2})$ is the $\mathscr{L}$-envelope of holomorphy of $B_0(r)$.*

Proof. $\mathscr{L}(B_0(r))\subset\mathscr{H}(B_0(r))$, so by classical properties of harmonic functions (comp. [6], Th. B), for every $f\in\mathscr{L}(B_0(r))$ there exists $\tilde{f}\in\mathscr{L}(B(r/\sqrt{2}))$ such that $\tilde{f}=f$ on $B_0(r/\sqrt{2})$. In view of Th. 1, $q_{\mathscr{L}}(x) = \frac{1}{\sqrt{2}}||x||$, $x\in \boldsymbol{R}^n$. In particular, $B_0(r)\subset\tilde{B}_{\mathscr{L}}(r/\sqrt{2})$. Thus the $\mathscr{L}(B(r/\sqrt{2}))$-envelope of holomorphy of $B(r/\sqrt{2})$ is the $\mathscr{L}$-envelope of $B_0(r)$. The proof is completed.

COROLLARY 1. ([5], [6], [3]). *The Lie ball $\tilde{B} = \{z\in \boldsymbol{C}^n\colon \mathrm{t}(z)<r\}$ is the harmonic envelope of holomorphy of $B_0(r)$.*

COROLLARY 2 ([4]). *The domain $\hat{B} = \{z\in \boldsymbol{C}^{2n}\colon \mathrm{u}(z)<r\}$ is the pluriharmonic envelope of holomorphy of $B_0(r)$.*

COROLLARY 3. *Let $\mathscr{L}$, $\mathscr{L}'$ be two systems satisfying* (1), (2) *with the same decomposition $n = n_1+\ldots+n_s$. Then $\tilde{B}_{\mathscr{L}}(r) = \tilde{B}_{\mathscr{L}'}(r) \Leftrightarrow \mathscr{L} = \mathscr{L}'$.*

Proof of Theorem 1. The proof of (i) follows directly from the fact that $T(\mathscr{L}_1),\ldots,T(\mathscr{L}_s),T(\mathscr{L})$ are closed $\boldsymbol{C}$-cones, namely

$$q_{\mathscr{L}}(z^1,\ldots,z^s) = \max\{t_1q_{\mathscr{L}_1}(z^1)+\ldots+t_sq_{\mathscr{L}_s}(z^s)\colon t_1,\ldots,t_s\geqslant 0,\ t_1^2+\ldots+t_s^2 = 1\} = [\sum_{j=1}^{s} q^2_{\mathscr{L}_j}(z^j)]^{1/2}.$$

(ii) Let us fix $z\in \boldsymbol{C}^N$, $z\neq 0$. Since q_{Δ_N} and t are seminorms, it is sufficient to consider the case when $||z|| = 1$ and $\langle z,z\rangle\in \boldsymbol{R}$, which in the real coordinates $z = x+iy$ means that $\langle x,y\rangle = 0$.

In view of the definition of q_{Δ_N}, there exists a point $w = \xi+i\eta\in T(\Delta_N)$, $||w|| = 1$ such that $q_{\Delta_N}(z) = \langle z,w\rangle$. Hence $q_{\Delta_N}(z) = \langle x,\xi\rangle-\langle y,\eta\rangle$ and $||\xi|| = ||\eta|| = \frac{1}{\sqrt{2}}$. In consequence $q_{\Delta_N}(z)\leqslant \frac{1}{\sqrt{2}}(||x||+||y||) = \frac{1}{\sqrt{2}}\mathrm{t}(z)$. Now it remains to find $w_0\in T(\Delta_N)$, $||w_0|| = 1$ such that $\langle z,w_0\rangle = \frac{1}{\sqrt{2}}(||x||+||y||)$. We shall distinguish three cases:

a) $x,y\neq 0$ — we put $w_0 = \frac{1}{\sqrt{2}}\left(\frac{x}{||x||}-i\frac{y}{||y||}\right)$,

b) $x \neq 0,\ y = 0$ — $w_0 = \frac{1}{\sqrt{2}}(x+i\eta_0)$, where $\eta_0 \in \boldsymbol{R}^N$ is chosen in such a way that $\langle x, \eta_0\rangle = 0$, $\|\eta_0\| = 1$,

c) $x = 0, y \neq 0$ — $w_0 = \frac{1}{\sqrt{2}}(\xi_0 - iy)$, where $\langle y, \xi_0\rangle = 0$, $\|\xi_0\| = 1$.

For the proof of (iii) let us remark that

$$T(\mathscr{P}_{2N}) = \{(v_1, \tau v_1, \ldots, v_N, \tau v_N): (v_1, \ldots, v_N) \in \boldsymbol{C}^N, \tau = \pm i\}.$$

Hence

$$q_{\mathscr{P}_{2N}}(z) = \max\left\{\left|\sum_{j=1}^{N} (z_{2j-1}+\tau z_{2j})v_j\right|: v = (v_1, \ldots, v_N) \in \boldsymbol{C}^N, \|v\| = \frac{1}{\sqrt{2}}, \tau = \pm i\right\} = \frac{1}{\sqrt{2}}\mathrm{u}(z).$$

The proof is finished.

Remark 3 (due to J. Siciak). *The equality* (ii) *in Theorem* 1 *may be interpreted as a new description of the crossnorm in complexified unitary spaces* (see the condition (iv) below).

Let E be a unitary space over $\boldsymbol{R}$ and let $(\ ,\ )_E$ denote its scalar product. Put $\|x\|_E = (x, x)_E^{1/2}$. Let $\tilde{E} = E+iE$ denote the complexification of E and let

$$\|x+iy\|_{\tilde{E}} = (\|x\|_E^2+\|y\|_E^2)^{1/2}.$$

Define $\langle\ ,\ \rangle_E$: $\tilde{E}\times\tilde{E} \to \boldsymbol{C}$ by the formula

$$\langle x+iy, u+iv\rangle_{\tilde{E}} = [(x, u)_E-(y, v)_E]+i[(x, v)_E+(y, u)_E].$$

It is easily seen that $\langle\ ,\ \rangle_{\tilde{E}}$ is a $\boldsymbol{C}$-bilinear symmetric continuous mapping such that $\langle x, u\rangle_{\tilde{E}} = (x, u)_E, x, u \in E$. For $z = x+iy \in \tilde{E}$ let us put $\bar{z} = x-iy$ and define $(z, w)_{\tilde{E}} = \langle z, \bar{w}\rangle_{\tilde{E}}$, $z, w \in \tilde{E}$. Clearly $(\ ,\ )_{\tilde{E}}$ is a complex scalar product on $\tilde{E}$ such that $(x, u)_{\tilde{E}} = (x, u)_E$, $x, u \in E$ and $\|z\|_{\tilde{E}}^2 = (z, z)_{\tilde{E}}$.

For a normed nonzero complex vector space $(F, \|\ \|_F)$ we shall denote by $P^k(\tilde{E}, F)$ the space of all continuous homogeneous polynomials of degree k from $\tilde{E}$ into F, $k \in \boldsymbol{N}$.

The crossnorm t on $\tilde{E}$ is defined as follows:

(i) $\mathrm{t}(z) = \inf\{\sum_{j=1}^{N}|\lambda_j|\,\|x_j\|_E\colon N \in \boldsymbol{N},\ \lambda_j \in \boldsymbol{C},\ x_j \in E,\ j = 1, \ldots, N,\ z = \sum_{j=1}^{N}\lambda_j x_j\}$, $z \in \tilde{E}$.

It is known (see [2]) that

(ii) $\mathrm{t}(z) = \sup\{\|f(z)\|_F^{1/k}\colon f \in P^k(\tilde{E}, F), (x \in E, \|x\|_E \leqslant 1) \Rightarrow \|f(x)\|_F \leqslant 1\}$, $k \in \boldsymbol{N}$;

(iii) $\mathrm{t}(x+iy) = \{\|x\|_E^2+\|y\|_E^2+2[\|x\|_E^2\|y\|_E^2-(x, y)_E^2]^{1/2}\}^{1/2}$,

which in the "complex" coordinates" gives

$$\mathrm{t}(z) = [\|z\|_{\tilde{E}}^2+(\|z\|_{\tilde{E}}^4-|\langle z, z\rangle_{\tilde{E}}|^2)_{\tilde{E}}^{21/2}]^{1/2}.$$

Repeating (with only formal changes) the proof of Theorem 1 (ii) we get

(iv) $\mathfrak{t}(z) = \max\{|\langle z, w\rangle|_{\tilde{E}}\colon w \in \tilde{E}, \langle w, w\rangle_{\tilde{E}} = 0, \|w\|_{\tilde{E}} = \sqrt{2}\}$,

or, equivalently, in the "real coordinates"

$$\mathfrak{t}(x+iy) = \max\left\{|\langle x+iy, u+iv\rangle_{\tilde{E}}|\colon u, v \in E, \|u\|_E = \|v\|_E = 1, (u, v)_E = 0\right\}.$$

References

[1] A. Andreotti, M. Nacinovich, *On the Envelope of Regularity for Solutions of Homogeneous Systems of Linear Partial Differential Operators*, Ann. Sc. Norm. Sup. Pisa, (IV), VI(1), (1979), 69-141.

[2] L. Drużkowski, *Effective Formula for the Crossnorm in the Complexified Unitary Spaces*, Zeszyty Naukowe UJ, 16, (1974), 47-53.

[3] M. Jarnicki, *Analytic Continuation of Harmonic Functions*, ibid., 17, (1975), 93-104.

[4] —, *Analytic Continuation of Pluriharmonic Functions*, ibid., 18, (1977), 45-51.

[5] C. O. Kiselman, *Prolongement des solutions d'une équation aux dérivées partielles à coefficients constants*, Bull. Soc. Math. France, 97, (1969), 329-356.

[6] J. Siciak, *Holomorphic continuation of harmonic functions*, Ann. Pol. Math., XXIX, (1974), 67-73.

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