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A Note on Holomorphic Continuation with Restricted Growth

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The central problem of the theory of holomorphic continuation with restricted growth is the following one (comp. for example [1], [2], [4], [5], [6], [7]).

Given an open set $D \subset C^n$ and a complex analytic submanifold $M \subset D$, denote by R the restriction operator

$$\mathcal{O}(D) \ni f \to f|_M \in \mathcal{O}(M).$$

Let $\delta\colon D \to (0, 1]$ be *a weight function on* D, that is (comp. [3]):

(i) $$\delta \leqslant \delta_D = \min\{\varrho_D,\ (1+\|z\|^2)^{-1/2}\}.$$

where ϱ_D denotes the distance to ∂D taken with respect to the Euclidean norm,

(ii) $$|\delta(z_1)-\delta(z_2)| \leqslant \|z_1-z_2\|,\ z_1, z_2 \in D.$$

For $k \geqslant 0$, let $\mathcal{O}^{(k)}(D, \delta)$ (resp. $\mathcal{O}^{(k)}(M, \delta)$) denote the space of all functions f holomorphic on D (resp. on M) for which the function $\delta^k f$ is bounded.

It may easily be verified that $\mathcal{O}^{(k)}(D, \delta)$ (resp. $\mathcal{O}^{(k)}(M, \delta)$) endowed with the norm

$$f \to \|\delta^k f\|_\infty$$

is a Banach space. Put

$$\mathcal{O}(D, \delta) = \bigcup_{k \geqslant 0} \mathcal{O}^{(k)}(D, \delta)$$

and analogously

$$\mathcal{O}(M, \delta) = \bigcup_{k \geqslant 0} \mathcal{O}^{(k)}(M, \delta).$$

One may easily prove that $\mathcal{O}(D, \delta)$ (resp. $\mathcal{O}(M, \delta)$) is a complex algebra.

Observe that $R(\mathcal{O}^{(k)}(D, \delta)) \subset \mathcal{O}^{(k)}(M, \delta)$.

The problem is as follows: *Under which additional assumptions*

(a) $$R(\mathcal{O}(D, \delta)) = \mathcal{O}(M, \delta)?,$$

i.e. for every $f \in \mathcal{O}(M, \delta)$ *there exists* $\hat{f} \in \mathcal{O}(D, \delta)$ *such that* $R(\hat{f}) = f$.

It is natural to try to estimate the growth of $\hat{f}$ in terms of the growth of f. This leads to the following (slightly more restrictive) conditions.

(b_1) *For every* $k \geqslant 0$ *there exists* $\hat{k} \geqslant 0$ *such that: for every* $f \in \mathcal{O}^{(k)}(M, \delta)$ *there exists* $\hat{f} \in \mathcal{O}^{(\hat{k})}(D, \delta)$ *such that* $R(\hat{f}) = f$.

(b_2) *For every* $k\geqslant 0$ *there exist* $\hat{k}\geqslant 0$, $b>0$ *such that: for every* $f\in \mathcal{O}^{(k)}(M,\delta)$ *there exists* $\hat{f}\in \mathcal{O}^{(\hat{k})}(D,\delta)$ *such that* $R(\hat{f})=f$ *and* $\|\delta^{\hat{k}}_{\hat{f}}\|_\infty \leqslant b\|\delta^k f\|_\infty$.

(b_3) *For every* $k\geqslant 0$ *there exist* $\hat{k}\geqslant 0$ *and a linear continuous extension-operator* $L_{k\hat{k}}$: $\mathcal{O}^{(k)}(M,\delta)\to \mathcal{O}^{(\hat{k})}(D,\delta)$ *such that* $R\circ L_{k\hat{k}} = \mathrm{id}_{\mathcal{O}^{(k)}(M,\delta)}$.
It is seen that $(b_3)\Rightarrow(b_2)\Rightarrow(b_1)\Rightarrow(a)$.

Let $H^{(k)}(D,\delta)=\{f\in\mathcal{O}(D)\colon \|\delta^k f\|_2 := (\int_D |f|^2\delta^{2k}d\lambda)^{1/2}<+\infty\}$, $k\geqslant 0$, where λ denotes the Lebesgue measure in $\boldsymbol{C}^n$.

It is easily seen that $H^{(k)}(D,\delta)$ is a complex Hilbert space. In view of Prop. 2, § 1.3 in [3].

$$\bigcup_{k\geqslant 0} H^{(k)}(D,\delta)=\mathcal{O}(D,\delta) \text{ and}$$

$$\mathcal{O}^{(k)}(D,\delta)\subset H^{(k+n+\varepsilon)}(D,\delta),\quad \|\delta^{k+n+\varepsilon}f\|_2\leqslant c(n,\varepsilon)\|\delta^k f\|_\infty,\quad \varepsilon>0,$$

$$H^{(k)}(D,\delta)\subset \mathcal{O}^{(k+n)}(D,\delta),\quad \|\delta^{k+n}f\|_\infty\leqslant c(n,k)\|\delta^k f\|_2 .$$

In consequence, the conditions (b_1), (b_2), (b_3) may be equivalently formulated as:

(c_1) ($\Leftrightarrow$(b_1)) *For every* $k\geqslant 0$ *there exists* $l\geqslant 0$ *such that: for every* $f\in\mathcal{O}^{(k)}(M,\delta)$ *there exists* $\hat{f}\in H^{(l)}(D,\delta)$ *such that* $R(\hat{f})=f$.

(c_2) ($\Leftrightarrow$(b_2)) *For every* $k\geqslant 0$ *there exist* $l\geqslant 0$, $c>0$ *such that: for every* $f\in\mathcal{O}^{(k)}(M,\delta)$ *there exists* $\hat{f}\in H^{(l)}(D,\delta)$ *such that* $R(\hat{f})=f$ *and* $\|\delta^l f\|_2\leqslant c\|\delta^k f\|_\infty$.

(c_3) ($\Leftrightarrow$(b_3)) *For every* $k\geqslant 0$ *there exist* $l\geqslant 0$ *and a linear continuous extension-operator* L_{kl}: $\mathcal{O}^{(k)}(M,\delta)\to H^{(l)}(D,\delta)$ *such that* $R\circ L_{kl}=\mathrm{id}_{\mathcal{O}^{(k)}(M,\delta)}$. The main result of this note is the following

THEOREM 1. *All the conditions* (a), (b_1), (b_2), (b_3), (c_1), (c_2), (c_3) *are equivalent.*

Proof. Since $H^{(l)}(D,\delta)$ is a Hilbert space, so by Lemma 1 in [4], $(c_2)\Rightarrow(c_3)$. It remains to show that $(a)\Rightarrow(b_2)$.

Let us fix $k\geqslant 0$ and let

$$C_N=\{f\in\mathcal{O}^{(k)}(M,\delta)\colon \|\delta^k f\|_\infty\leqslant N,\ \exists\hat{f}\in\mathcal{O}^{(N)}(D,\delta)\colon R(\hat{f})=f,\ \|\delta^N\hat{f}\|_\infty\leqslant N\},\ N\in\boldsymbol{N}.$$

It is seen that C_N is absolutely convex and, since $\mathcal{O}^{(k)}(M,\delta)\subset R(\mathcal{O}(D,\delta))$, so $\mathcal{O}^{(k)}(M,\delta) = \bigcup_{N=1}^{\infty} C_N$.

The space $\mathcal{O}^{(k)}(M,\delta)$ has the Baire property, hence there exists $N\in\boldsymbol{N}$ such that $\mathrm{int}(\bar{C}_N)\neq\emptyset$ (the closure and the interior are taken with respect to the space $\mathcal{O}^{(k)}(M,\delta)$).

In consequence, $\bar{C}_N$ is a neighbourhood of zero in $\mathcal{O}^{(k)}(M,\delta)$ and hence

$$(\S)\qquad \bar{C}_N\subset C_N+\tfrac{1}{2}\bar{C}_N .$$

Let $r>0$ be such that the ball $\{f\in\mathcal{O}^{(k)}(M,\delta)\colon \|\delta^k f\|_\infty\leqslant r\}$ is contained in $\bar{C}_N$.

Fix $f\in\mathcal{O}^{(k)}(M,\delta)$, $f\neq 0$ and let $f_0=r(\|\delta^k\hat{f}\|)_\infty^{-1}f$. Observe that $f_0\in\bar{C}_N$, so in view of (§), there exist sequences $(F_m)_{m=1}^{\infty}\subset\mathcal{O}^{(N)}(D,\delta)$, $(f_m)_{m=1}^{\infty}\subset\bar{C}_N$ such that

$$f_{m-1}=R(F_m)+\tfrac{1}{2}f_m,\ m\geqslant 1,$$

$$\|\delta^N F_m\|_\infty\leqslant N,\ m\geqslant 1 .$$

In particular,

$$f_0 = [\sum_{p=1}^{\infty} 2^{-p+1} R(F_p)] + 2^{-m} f_m, \quad m \geqslant 1 .$$

Note that in view of the definition of C_N, $\|\delta^k f_m\|_\infty \leqslant N$. Hence $2^{-m} f_m \to 0$ in $\mathcal{O}^{(k)}(M, \delta)$ as $m \to +\infty$. This shows that

$$f_0 = \sum_{p=1}^{\infty} 2^{-p+1} R(F_p) .$$

On the other hand, since $\|\delta^N F_m\|_\infty \leqslant N, m \geqslant 1$, the series

$$\sum_{p=1}^{\infty} 2^{-p+1} F_p$$

is absolutely convergent in $\mathcal{O}^{(N)}(D, \delta)$ to a function F_0; note that $\|\delta^N F_0\|_\infty \leqslant 2N$. Hence $f_0 = R(F_0)$ and in consequence, $f = R(\hat{f})$, where

$$\hat{f} = \|\delta^k f\|_\infty r^{-1} F_0 .$$

Thus we get (b_2) with $\hat{k} = N$ and $b = \frac{2N}{r}$. The proof is completed.

References

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