A Note on Holomorphic Continuation with Restricted Growth

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The central problem of the theory of holomorphic continuation with restricted growth is the following one (comp. for example [1], [2], [4], [5], [6], [7]).

Given an open set \( D \subset \mathbb{C}^n \) and a complex analytic submanifold \( M \subset D \), denote by \( R \) the restriction operator

\[
\mathcal{O}(D) \ni f \rightarrow f|_M \in \mathcal{O}(M).
\]

Let \( \delta : D \rightarrow (0, 1] \) be a weight function on \( D \), that is (comp. [3]):

(i) \[
\delta \leq \delta_D = \min \{ q_D, (1 + |z|^2)^{-1/2} \}.
\]

where \( q_D \) denotes the distance to \( \partial D \) taken with respect to the Euclidean norm,

(ii) \[
|\delta(z_1) - \delta(z_2)| \leq ||z_1 - z_2||, \quad z_1, z_2 \in D.
\]

For \( k \geq 0 \), let \( \mathcal{O}^{(k)}(D, \delta) \) (resp. \( \mathcal{O}^{(k)}(M, \delta) \)) denote the space of all functions \( f \) holomorphic on \( D \) (resp. on \( M \)) for which the function \( \delta^kf \) is bounded.

It may easily be verified that \( \mathcal{O}^{(k)}(D, \delta) \) (resp. \( \mathcal{O}^{(k)}(M, \delta) \)) endowed with the norm

\[
|f|_{\infty} \rightarrow ||\delta^k f||_{\infty}
\]

is a Banach space. Put

\[
\mathcal{O}(D, \delta) = \bigcup_{k \geq 0} \mathcal{O}^{(k)}(D, \delta)
\]

and analogously

\[
\mathcal{O}(M, \delta) = \bigcup_{k \geq 0} \mathcal{O}^{(k)}(M, \delta).
\]

One may easily prove that \( \mathcal{O}(D, \delta) \) (resp. \( \mathcal{O}(M, \delta) \)) is a complex algebra. Observe that \( R(\mathcal{O}^{(k)}(D, \delta)) \subset \mathcal{O}^{(k)}(M, \delta) \).

The problem is as follows: Under which additional assumptions

(a) \[
R(\mathcal{O}(D, \delta)) \subset \mathcal{O}(M, \delta),
\]

i.e. for every \( f \in \mathcal{O}(M, \delta) \) there exists \( \hat{f} \in \mathcal{O}(D, \delta) \) such that \( R(\hat{f}) = f \).

It is natural to try to estimate the growth of \( \hat{f} \) in terms of the growth of \( f \). This leads to the following (slightly more restrictive) conditions.

(b) For every \( k \geq 0 \) there exists \( \widetilde{k} \geq 0 \) such that: for every \( f \in \mathcal{O}^{(k)}(M, \delta) \) there exists \( \hat{f} \in \mathcal{O}^{(\widetilde{k})}(D, \delta) \) such that \( R(\hat{f}) = f \).
(b₂) For every $k \geq 0$ there exist $\tilde{k} \geq 0$, $b > 0$ such that: for every $f \in \mathfrak{C}^{(k)}(M, \delta)$ there exists $\tilde{f} \in \mathfrak{C}^{(\tilde{k})}(D, \delta)$ such that $R(\tilde{f}) = f$ and $\|\delta^k f\|_\infty \leq b\|\delta^\tilde{k} f\|_\infty$.

(b₃) For every $k \geq 0$ there exist $\tilde{k} \geq 0$ and a linear continuous extension-operator $L_{\tilde{k}}$: $\mathfrak{C}^{(k)}(M, \delta) \to \mathfrak{C}^{(\tilde{k})}(D, \delta)$ such that $R \circ L_{\tilde{k}} = \text{id}_{\mathfrak{C}^{(k)}(M, \delta)}$.

It is seen that (b₃) $\Rightarrow$ (b₂) $\Rightarrow$ (b₁) $\Rightarrow$ (a).

Let $H^{(k)}(D, \delta) = \{ f \in \mathfrak{C}(D): \|\delta^k f\|_2 : = (\int_D |f|^2 \delta^2 d\lambda)^{1/2} < +\infty\}$, $k \geq 0$, where $\lambda$ denotes the Lebesgue measure in $C^n$.

It is easily seen that $H^{(k)}(D, \delta)$ is a complex Hilbert space. In view of Prop. 2, § 1.3 in [3],

$$\bigcup_{k \geq 0} H^{(k)}(D, \delta) = \mathfrak{C}(D, \delta)$$

$$\mathfrak{C}^{(k)}(D, \delta) \subset H^{(k+n+\varepsilon)}(D, \delta), \quad \|\delta^{k+n+\varepsilon} f\|_2 \leq c(n, \delta)\|\delta^k f\|_\infty, \quad \varepsilon > 0;$$

$$H^{(k)}(D, \delta) \subset \mathfrak{C}^{(k+n)}(D, \delta), \quad \|\delta^{k+n} f\|_\infty \leq c(n, k)\|\delta^k f\|_2.$$}

In consequence, the conditions (b₁), (b₂), (b₃) may be equivalently formulated as:

(c₁) $\iff$ (b₁)) For every $k \geq 0$ there exists $l \geq 0$ such that: for every $f \in \mathfrak{C}^{(k)}(M, \delta)$ there exists $\tilde{f} \in H^{(l)}(D, \delta)$ such that $R(\tilde{f}) = f$.

(c₂) $\iff$ (b₂)) For every $k \geq 0$ there exists $l \geq 0$, $c > 0$ such that: for every $f \in \mathfrak{C}^{(k)}(M, \delta)$ there exists $\tilde{f} \in H^{(l)}(D, \delta)$ such that $R(\tilde{f}) = f$ and $\|\delta^k f\|_2 \leq c\|\delta^k f\|_\infty$.

(c₃) $\iff$ (b₃)) For every $k \geq 0$ there exist $l \geq 0$ and a linear continuous extension-operator $L_{kl}$: $\mathfrak{C}^{(k)}(M, \delta) \to H^{(l)}(D, \delta)$ such that $R \circ L_{kl} = \text{id}_{\mathfrak{C}^{(k)}(M, \delta)}$. The main result of this note is the following

**Theorem 1.** All the conditions (a), (b₁), (b₂), (b₃), (c₁), (c₂), (c₃) are equivalent.

**Proof.** Since $H^{(l)}(D, \delta)$ is a Hilbert space, so by Lemma 1 in [4], (c₂) $\Rightarrow$ (c₃). It remains to show that (a) $\Rightarrow$ (b₂).

Let us fix $k \geq 0$ and let

$$C_N = \{ f \in \mathfrak{C}^{(k)}(M, \delta): \|\delta^k f\|_\infty \leq N, \exists \tilde{f} \in \mathfrak{C}^{(N)}(D, \delta): R(\tilde{f}) = f, \|\delta^N \tilde{f}\|_\infty \leq N\}. \quad N \in \mathbb{N}.$$ 

It is seen that $C_N$ is absolutely convex and, since $\mathfrak{C}^{(k)}(M, \delta) \subset R(\mathfrak{C}(D, \delta))$, so $\mathfrak{C}^{(k)}(M, \delta)$

$$= \bigcup_{N = 1}^{\infty} C_N.$$ 

The space $\mathfrak{C}^{(k)}(M, \delta)$ has the Baire property, hence there exists $N \in \mathbb{N}$ such that $\text{int}(\overline{C}_N) \neq \emptyset$ (the closure and the interior are taken with respect to the space $\mathfrak{C}^{(k)}(M, \delta)$).

In consequence, $\overline{C}_N$ is a neighbourhood of zero in $\mathfrak{C}^{(k)}(M, \delta)$ and hence

(§) $\quad \overline{C}_N \subset \mathfrak{C}_N + \frac{1}{2} \overline{C}_N$.

Let $r > 0$ be such that the ball $\{ f \in \mathfrak{C}^{(k)}(M, \delta): \|\delta^k f\|_\infty \leq r\}$ is contained in $\overline{C}_N$.

Fix $f \in \mathfrak{C}^{(k)}(M, \delta), f \neq 0$ and let $f_0 = r(\|\delta^k f\|_\infty)^{-1} f$. Observe that $f_0 \in \overline{C}_N$, so in view of (§), there exist sequences $(F_m)_{m=1}^{\infty} \subset \mathfrak{C}^{(N)}(D, \delta)$, $(f_m)_{m=1}^{\infty} \subset \overline{C}_N$ such that

$$f_{m-1} = R(F_m) + \frac{1}{2} f_m, \quad m \geq 1,$$

$$\|\delta^N f_m\|_\infty \leq N, \quad m \geq 1.$$
In particular,
\[ f_0 = \left[ \sum_{p=1}^{\infty} 2^{-p+1} R(F_p) \right] + 2^{-m} f_m, \quad m \geq 1. \]

Note that in view of the definition of \( C_N, \|\delta^k f_m\|_\infty \leq N \). Hence \( 2^{-m} f_m \to 0 \) in \( \mathcal{O}(k)(M, \delta) \) as \( m \to +\infty \). This shows that
\[ f_0 = \sum_{p=1}^{\infty} 2^{-p+1} R(F_p). \]

On the other hand, since \( \|\delta^N f_m\|_\infty \leq N, m \geq 1 \), the series
\[ \sum_{p=1}^{\infty} 2^{-p+1} F_p \]
is absolutely convergent in \( \mathcal{O}(N)(D, \delta) \) to a function \( F_0 \); note that \( \|\delta^N F_0\|_\infty \leq 2N \). Hence \( f_0 = R(F_0) \) and in consequence, \( f = R(\tilde{f}) \), where \( \tilde{f} = \|\delta^k f\|_\infty r^{-1} F_0 \).

Thus we get (b) with \( \hat{k} = N \) and \( b = \frac{2N}{r} \). The proof is completed.

References


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