UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA, FASCICULUS XXVIII

1991

A note on blow-up points for a semilinear parabolic equation

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Let D be a bounded domain in $\mathbf{R}^n$, $0 < T < +\infty$. Let us consider the following problem:

$$\begin{aligned} u'_t - \Delta u &= f(u) \text{ in } D \times (0, T), \\ u(\xi, t) &= 0, \quad (\xi, t) \in \partial D \times [0, T), \\ u(\xi, 0) &= \Phi(\xi), \quad \xi \in \bar{D}, \end{aligned} \tag{1}$$

where $f \in C^1(\mathbf{R})$, $f(\eta) > 0$, $f'(\eta) > 0$, $\eta \in (0, +\infty)$, $\Phi \in C^1(\bar{D})$, $\Phi \geqslant 0$, $\Phi|_{\partial D} = 0$.

Suppose that u is a classical solution of (1); note that u is uniquely determined and, in view of the maximum principle, $u \geqslant 0$.

A point $\xi^\circ \in \bar{D}$ is said to be a blow-up point for u $(\xi^\circ \in B(u))$ if

$$\limsup_{(\xi, t) \to (\xi^0, T)} u(\xi, t) = +\infty .$$

The problem of existence of blow-up points was considered, for instance, in [3]. For some classes of strictly convex domains D, the structure of the set $B(u)$ was studied, for example, in [2], [3].

In this note (being inspired by the methods of [2]) we will consider the problem of characterization of the set $B(u)$ for the case where

$$D = P := \{\xi \in \mathbf{R}^n \colon R_1 < ||\xi|| < R_2\} \quad (0 < R_1 < R_2 < +\infty)$$

and

$$\Phi(\xi) = \varphi(||\xi||), \quad \xi \in \bar{P} .$$

In this case, problem (1) may be reduced to the following one:

$$u'_t - \frac{n-1}{r} u'_r - u''_r = f(u) \text{ in } (R_1, R_2) \times (0, T),$$

$$u(R_1, t) = u(R_2, t) = 0, \quad 0 \leqslant t < T,$$

$$u(r, 0) = \varphi(r), \quad R_1 \leqslant r \leqslant R_2 .$$

In fact, we will consider a more general problem, namely:

$$u'_t - g(x)u'_x - u''_{xx} = f(u) \text{ in } (a, b)\times(0, T) \quad (-\infty < a < b < +\infty), \tag{2}$$
$$u(a, t) = u(b, t) = 0, \quad 0 \leqslant t < T,$$
$$u(x, 0) = \varphi(x), \quad a \leqslant x \leqslant b.$$

If u is a solution of (2), then $B(u)$ will denote the set of all $c \in [a, b]$ such that $\limsup_{(x,t)\to(c,T)} u(x, t) = +\infty$.

The main result of the paper is the following theorem.

THEOREM 1. *Assume that*

g: $[a, b]\to[0, +\infty)$ (*resp.* g: $[a, b]\to(-\infty, 0]$) *is a real-analytic function*;

(3) f: $\mathbf{R}\to\mathbf{R}$, $f|_{[0,+\infty)}$ *is real-analytic*, $f(\eta) > 0$, $f'(\eta) > 0$, $\eta \in (0, +\infty)$;

(4) *there exists a* C^2-*function* F: $[0, +\infty)\to[0, +\infty)$ *such that*

$$F(\eta) > 0,\ F'(\eta) > 0,\ F''(\eta) \geqslant 0,\ \eta \in (0, +\infty),$$

$$\int_1^{\infty} \frac{d\eta}{F(\eta)} < +\infty.$$

$$f'F - fF' \geqslant FF';$$

(5) φ: $[a, b]\to[0, +\infty)$ *is a* C^1-*function such that* $\varphi(a) = \varphi(b) = 0$ *and for some* $x_0 \in (a, b)$:

$$\varphi'(x) > 0, \quad a < x < x_0,$$
$$\varphi'(x) < 0, \quad x_0 < x < b.$$

Let u be a solution of (2) *such that* $B(u) \neq \emptyset$. *Then there exists a point* $c \in \left[a, \frac{x_0+b}{2}\right]$ $\left(\textit{resp. } c \in \left[\frac{a+x_0}{2} 0, b\right]\right)$ *such that* $B(u) = \{c\}$.

As an immediate consequence, we get

COROLLARY 1. *Assume that the functions* f, φ *satisfy* (3), (4), (5) *with* $a = R_1$, $b = R_2$ $(0 < R_1 < R_2 < +\infty)$. *Let u be a solution of* (1) *with* $D = P$, $\Phi(\xi) = \varphi(||\xi||)$, $\xi \in \overline{P}$, *such that* $B(u) \neq \emptyset$. *Then there exists* $R_0 \in \left[R_1, \frac{x_0+R_2}{2}\right]$ *such that* $B(u) = \{\xi \in \mathbf{R}^n\colon ||\xi|| = R_0\}$.

Remark 1. Theorem 1 (and, consequently, Corollary 1) may be extended to more general classes of functions g, f, φ.

Remark 2. Observe that, for a large class of functions f satisfying (3), condition (4) is automatically fulfilled. For instance:

If there exists $0<p<1$ such that

$$ff''\geqslant(1-p)(f')^2 \text{ in } [0,+\infty),$$

$$\int_1^\infty \frac{d\eta}{[f(\eta)]^p}<+\infty,$$

then for every $q>0$, the function

$$F(\eta):=\frac{1-p}{p}[f(0)+q]^{1-p}[f(\eta)+q]^p, \quad \eta\geqslant 0,$$

satisfies condition (4).

Standard examples: $f(\eta)=e^\eta$, $f(\eta)=\eta^\alpha \quad (\alpha>1)$.

Proof of Theorem 1. The proof will be divided into five steps.

1°. By the same methods as in [2], Lemma 5.2 and Remark 5.1, one can prove that there exists a continuous function $s\colon [0,T)\to(a,b)$ such that $s(0)=x_0$ and

$$u_x'(x,t)>0,\ a<x<s(t),\ 0<t<T,$$

$$u_x'(x,t)<0,\ s(t)<x<b,\ 0<t<T.$$

Put

$$s^-:=\liminf_{t\to T} s(t), \qquad s^+:=\limsup_{t\to T} s(t).$$

2°. (cf. [1], Lemma 2.1). If $s^-<s^+$ then

$$\lim_{t\to\infty} u(x,t)=+\infty,\ s^-<x<s^+.$$

3°. $[(a,s^-)\cup(s^+,b)]\cap B(u)=\emptyset$.

Proof of 3° (cf. [2], the proof of Theorem 3.3). Suppose that $a<s^-$ and fix $a<x_2<s^-$. We will prove that $(a,x_2)\cap B(u)=\emptyset$.

Define $N_0=\|g\|_\infty$, $N_1=\|g'\|_\infty$, and let

$$d(\zeta):=e^{\lambda_2\xi}-e^{\lambda_1\xi},\ \zeta\in \mathbf{R},$$

where $\lambda_1<\lambda_2$ are such that $(\lambda-\lambda_1)(\lambda-\lambda_2)=\lambda^2-N_0\lambda-N_1$; d is a solution of $y''-N_0y'-N_1=0$. Observe that

$$d(0)=0,\ d(\zeta)>0,\ d'(\zeta)>0,\ \zeta>0.$$

Set

$$c(x):=\varepsilon d(x_2-x), \qquad x\in\mathbf{R},$$

$$J(x,t):=-u_x'(x,t)+c(x)F(u),\ a\leqslant x\leqslant b,\ 0<t<T,$$

where $\varepsilon>0$ will be chosen in the sequel (F is as in (4)).

In view of 1°, there exists $0<t_0<T$ such that

$$u_x'(x_2,t)>0,\ t_0\leqslant t<T,$$

and hence

$$J(x_2, t) < 0, \ t_0 \leqslant t < T.$$

If ε is sufficiently small then

$$J(x, t_2) < 0, \ a \leqslant x \leqslant x_2 .$$

Comparing u with the solution of the problem

$$v'_t - g(x) v'_x - v''_{xx} = 0 \text{ in } (a, b) \times (0, T),$$

$$v(a, t) = v(b, t) = 0, \ 0 \leqslant t < T,$$

$$v(x, 0) = \varphi(x), \ a \leqslant x \leqslant b .$$

we conclude that for small ε,

$$J(a, t) < 0, \ t_0 \leqslant t < T.$$

In view of (2), we get

$$J'_t - g(x) J'_x - J''_{xx} \leqslant C(x, t) J - c(x) H_1(x, t) - F(u) H_2(x), \ a < x < b, \ 0 < t < T,$$

where

$$C(x, t) = f'(u) + g'(x) + c(x) F'(u)$$

(note that C is continuous on $[a, b] \times [0, T)$),

$$H_1(x, t) = f'(u) F(u) - f(u) F'(u) + 2c'(x) F(u) F'(u),$$

$$H_2 = cg' + gc' + c'' .$$

By (4), if ε is sufficiently small, then

$$H_1(x, t) \geqslant 0, \ a < x < x_2, \ t_0 < t < T.$$

By the definition of the function d,

$$H_2(x) \geqslant 0, \ a < x < x_{..} .$$

Thus, by the maximum principle,

$$(*) \qquad J(x, t) \leqslant 0, \ a \leqslant x \leqslant x_2, \ t_0 \leqslant t < T.$$

Let

$$G(s) := \int_s^\infty \frac{d\eta}{F(\eta)}, \ s > 0,$$

(observe that $G(s) \to 0$ as $s \to +\infty$). In view of $(*)$,

$$(G \circ u)'_x(x, t) \leqslant -c(x), \ a < x \leqslant x_2, \ t_0 \leqslant t < T.$$

Hence

$$G(u(x, t)) \geqslant \int_x^{x_2} c(\zeta) d\xi > 0, \ a < x < x_2, \ t_0 \leqslant t < T,$$

and therefore

$$\limsup_{(x,t)\to(c,T)_-} u(x,t) < +\infty, \; a < c < x_2 .$$

The proof for the interval (s^+, b) is analogous $(s^+ < x_1 < b,\ c(x) := \varepsilon d(x-x_1),\ J(x,t) := u'_x(x,t)+c(x)F(u))$.

4°. If $g \geqslant 0$ then $s^+ \leqslant \dfrac{x_0+b}{2}$ and $b \notin B(u)$ (resp. if $g \leqslant 0$ then $s^- \geqslant \dfrac{a+x_0}{2}$ and $a \notin B(u)$).

Proof of 4°. Assume that $g \geqslant 0$, let $\alpha := \dfrac{x_0+b}{2}$ and define

$$w(x,t) = u(x,t)-u(2\alpha-x,t), \; \alpha \leqslant x \leqslant b, \; 0 \leqslant t < T .$$

Observe that

$$w(\alpha,t) = 0, \; w(b,t) < 0, \; 0 \leqslant t < T,$$

$$w(x,0) \leqslant 0, \; \alpha \leqslant x \leqslant b .$$

Put

$$D(x,t) := u'_x(x,t)+u'_x(2\alpha-x,t),$$

$$B(x,t) := g(2\alpha-x)u'_x(2\alpha-x,t)-g(x)u'_x(x,t), \; \alpha < x < b, \; 0 < t < T,$$

and let

$$A(x,t) := \begin{cases} \dfrac{B(x,t)}{D(x,t)} & \text{if } D(x,t) \neq 0 \\ 0 & \text{if } D(x,t) = 0 . \end{cases}$$

Note that $B(x,t) \geqslant 0$ if $D(x,t) = 0$. In view of (2),

$$w'_t + Aw'_x - w''_{xx} = Cw - B + AD \leqslant Cw, \; \alpha < x < b, \; 0 < t < T,$$

where $C\colon (\alpha, b)\times(0,T) \mapsto (0, +\infty)$ is a function such that, for any $0 < \bar{t} < T$, C is bounded in $(\alpha, b)\times(0, \bar{t}]$. Consequently, by the maximum principle,

$$w \leqslant 0 \text{ in } [\alpha, b]\times[0,T),$$

and hence, in view of 1°,

$$s(t) \leqslant \alpha, \; 0 < t < T .$$

In particular, $s^+ \leqslant \alpha$ and $b \notin B(u)$.

If $g \leqslant 0$ then the proof is similar $\left(\alpha := \dfrac{a+x_0}{2},\ w(x,t) := u(x,t)-u(2\alpha-x,t),\ a \leqslant x \leqslant \alpha,\ 0 \leqslant t < T\right)$.

5°. $s^- = s^+$.

Proof of 5°. Assume that $g \geqslant 0$ and suppose that $s^- < s^+$. Let $s^- < x_1 < s^+ < x_2 < b$ be such that $\alpha := \frac{x_1 + x_2}{2} < s^+$ (cf. 4°). In view of 3°, there exists $M > 0$ such that

$$u(x_2, t) < M, \ 0 \leqslant t < T.$$

By 2°, there exists $0 < t_0 < T$ such that $x_1 = s(t_0)$ and

$$u(x_1, t) > M, \ t_0 \leqslant t < T.$$

Put

$$w(x, t) := u(x, t) - u(2\alpha - x, t), \ \alpha \leqslant x \leqslant x_2, \ t_0 \leqslant t < T.$$

Now, after formal changes only, we can apply the method of the proof of 4° which leads to the inequality $s^+ \leqslant \alpha$ — contradiction.

The case $g \leqslant 0$ is analogous (we start with $a < x_1 < s^- < x_2 < s^+$).

The proof of Theorem 1 is completed.

References

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Received October 1, 1987