VECTOR BUNDLES ON REAL ALGEBRAIC CURVES

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Abstract. We prove that any topological real line bundle on a compact real algebraic curve $X$ is isomorphic to an algebraic line bundle. The result is then generalized to vector bundles of an arbitrary constant rank. As a consequence we prove that any continuous map from $X$ into a real Grassmannian can be approximated by regular maps.

1. Introduction. Throughout this paper $X$ denotes a compact real algebraic curve, that is, a compact 1-dimensional algebraic subset of $\mathbb{R}^d$ for some $d \in \mathbb{N}$. We refer to [1] for terminology and background material on real algebraic geometry. In this paper all vector bundles are real vector bundles. Recall that algebraic vector bundles on $X$ correspond to finitely generated projective modules over the ring of real-valued regular functions on $X$, cf. [1], p. 302. Our main goal is the following:

**Theorem 1.1.** Any topological line bundle on $X$ is isomorphic to an algebraic line bundle.

Theorem 1.1 is proved in section 2. It can be easily generalized.

**Corollary 1.2.** Any topological constant rank vector bundle on $X$ is isomorphic to an algebraic vector bundle.

**Proof.** Any topological vector bundle on $X$ of constant rank $r \geq 1$ splits off a trivial vector bundle of rank $r - 1$, since $\dim(X) = 1$. Hence it suffices to apply Theorem 1.1.

2000 Mathematics Subject Classification. 14P05, 14P25.

Key words and phrases. Real algebraic curve, algebraic vector bundle, approximation of continuous maps by regular maps.
As a consequence of Corollary 1.2, we obtain a counterpart of the classical Weierstrass approximation theorem for maps from $X$ into the Grassmann variety $\mathbb{G}_{n,k}$ of $k$-dimensional vector subspaces of $\mathbb{R}^n$.

**Corollary 1.3.** Let $f : X \rightarrow \mathbb{G}_{n,k}$ be a continuous map. Each neighborhood of $f$ in the compact-open topology contains a regular map.

**Proof.** It suffices to show that the pullback vector bundle $f^*\gamma_{n,k}$ on $X$, where $\gamma_{n,k}$ is the tautological vector bundle on $\mathbb{G}_{n,k}$, is isomorphic to an algebraic vector bundle, cf. [1, Theorem 13.3.1]. This however follows from Corollary 1.2.

Since the real variety $\mathbb{G}_{2,1}$ is biregularly isomorphic to the unit circle

$$\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\},$$

we immediately get:

**Corollary 1.4.** Let $f : X \rightarrow \mathbb{S}^1$ be a continuous map. Each neighborhood of $f$ in the compact-open topology contains a regular map.

All the results above are proved in [1] under the assumption that the curve $X$ is nonsingular. The arguments presented in [1] do not directly generalize to yield Theorem 1.1.

**Corollary 1.5.** For every cohomology class $u$ in $H^1(X; \mathbb{Z}/2)$, there exists a regular map $f : X \rightarrow \mathbb{S}^1$ such that $f^*(s_1) = u$, where $s_1$ is the unique generator of the cohomology group $H^1(\mathbb{S}^1; \mathbb{Z}/2) \cong \mathbb{Z}/2$.

**Proof.** There is a one-to-one correspondence between the homotopy classes of continuous maps from $X$ into $\mathbb{S}^1$ and the cohomology classes in $H^1(X; \mathbb{Z})$, cf., [2, p. 300]. Since the reduction modulo 2 homomorphism $H^1(X; \mathbb{Z}) \rightarrow H^1(X; \mathbb{Z}/2)$ is surjective, it follows that each cohomology class in $H^1(X; \mathbb{Z}/2)$ is of the form $f^*(s_1)$ for some continuous map $f : X \rightarrow \mathbb{S}^1$. According to Corollary 1.4, the map $f$ can be assumed to be regular.

Let us note that Corollary 1.5 implies Theorem 1.1. Indeed, let $\xi$ be a topological line bundle on $X$. The first Stiefel–Whitney class $w_1(\gamma_{2,1})$ of the tautological line bundle $\gamma_{2,1}$ on $\mathbb{G}_{2,1}$ generates the cohomology group $H^1(\mathbb{G}_{2,1}; \mathbb{Z}/2)$. According to Corollary 1.5, there exists a regular map $f : X \rightarrow \mathbb{G}_{2,1}$ satisfying $w_1(\xi) = f^*(w_1(\gamma_{2,1})) = w_1(f^*\gamma_{2,1})$. Since topological line bundles are classified by the first Stiefel-Whitney class (cf. [3, Proposition 3.10]), it follows that $\xi$ is isomorphic to the algebraic line bundle $f^*\gamma_{2,1}$. However, we do not know how to prove Corollary 1.5 without making use of Theorem 1.1.
2. Line bundles on real algebraic curves. We first recall a useful construction of algebraic line bundles on an arbitrary affine real algebraic variety \( V \). Lemma 2.1 below is a special case of [1] Theorem 12.1.11.

**Lemma 2.1.** Let \( \{U_1,\ldots,U_r\} \) be a Zariski open cover of \( V \) and let \( h_{ij} : U_j \longrightarrow \mathbb{R} \) be a regular function satisfying \( h_{ij}(U_i \cap U_j) \subset \mathbb{R} \setminus \{0\} \) for \( 1 \leq i,j \leq r \). Assume that \( h_{ij} \cdot h_{jk} = h_{ik} \) on \( U_j \cap U_k \) for all \( i,j,k \), and \( h_{ii}(x) = 1 \) for all \( i \) and \( x \) in \( U_i \). Let

\[
E = \{(x, (v_1,\ldots,v_r)) \in V \times \mathbb{R}^r : v_i = h_{ij}(x)v_j \text{ for } x \in U_j, 1 \leq i,j \leq r\}
\]
and let \( p : E \longrightarrow V \) be defined by \( p(x, (v_1,\ldots,v_r)) = x \). Then \( \xi = (E, p, V) \) is an algebraic line subbundle of the product vector bundle on \( V \) with total space \( V \times \mathbb{R}^r \), and the map

\[
U_i \times \mathbb{R} \longrightarrow p^{-1}(U_i), (x, v) \mapsto (h_{i1}(x)v,\ldots,h_{ir}(x)v)
\]
is an algebraic trivialization of \( \xi \) over \( U_i \) for \( 1 \leq i \leq r \).

For any vector bundle \( \eta \) and any global section \( s \) of \( \eta \), let \( Z(s) \) denote the zero locus of \( s \).

The set \( \text{Reg}(X) \) of nonsingular points of \( X \) in dimension 1 is a Zariski open subset of \( X \), cf. [1] p. 69]. Furthermore, \( \text{Reg}(X) \) is a 1-dimensional \( C^\infty \) manifold.

**Lemma 2.2.** Let \( x_0 \) be a point in \( \text{Reg}(X) \). There exists an algebraic line bundle \( \xi = (E, p, X) \) on \( X \) which admits an algebraic section \( s : X \longrightarrow E \) such that \( Z(s) = \{x_0\} \) and the restriction of \( s \) to \( \text{Reg}(X) \) is transverse to the zero section of \( \xi \).

**Proof.** Let \( \mathcal{R}_X \) be the sheaf of real-valued regular functions on \( X \). For any point \( x \) on \( X \), we identify the stalk \( \mathcal{R}_{X,x} \) with the localization of the ring \( \mathcal{R}_X(X) \) at the maximal ideal

\[
m_x = \{f \in \mathcal{R}_X(X) : f(x) = 0\},
\]

cf. [1] Proposition 3.2.3]. Since the point \( x_0 \) is in \( \text{Reg}(X) \), the stalk \( \mathcal{R}_{X,x_0} \) is a regular local ring of dimension 1 and thus a principal ideal domain. In particular, the ideal \( m_{x_0} \mathcal{R}_{X,x_0} \) of the ring \( \mathcal{R}_{X,x_0} \) is principal. Thus we can find a regular function \( f_1 \) in \( m_{x_0} \) and a Zariski open neighborhood \( U_1 \) of \( x_0 \) in \( \text{Reg}(X) \) such that

\[
m_{x_0} \mathcal{R}_X(U_1) = (f_1) \mathcal{R}_X(U_1).
\]

In particular, \( f_1|_{U_1} : U_1 \longrightarrow \mathbb{R} \) is a \( C^\infty \) function for which 0 in \( \mathbb{R} \) is a regular value and \( (f_1|_{U_1})^{-1}(0) = \{x_0\} \).

Let \( f_2 \) be any regular function in \( m_{x_0} \) with \( f_2^{-1}(0) = \{x_0\} \), e.g., a polynomial given by the formula \( \|x - x_0\|^2 \), where \( \| \cdot \| \) denotes the euclidean metric
in \( \mathbb{R}^d \). We have

\[ f_2|_{U_1} = h_{21}f_1|_{U_1} \]

for some regular function \( h_{21} : U_1 \to \mathbb{R} \). If \( U_2 = X \setminus \{x_0\} \), then

\[ h_{12} = \frac{f_1}{f_2} : U_2 \to \mathbb{R} \]

is a regular function on \( U_2 \). By construction, the sets \( h_{21}(U_1 \cap U_2) \) and \( h_{12}(U_1 \cap U_2) \) are contained in \( \mathbb{R} \setminus \{0\} \). Define \( h_{11} : U_1 \to \mathbb{R} \) and \( h_{22} : U_2 \to \mathbb{R} \) to be constant functions identically equal to 1. Let \( \xi = (E, p, X) \) be the algebraic line bundle on \( X \) determined, as in Lemma 2.1, by the Zariski open cover \( \{U_1, U_2\} \) of \( X \) and the regular functions \( h_{ij} \). Note that

\[ s : X \to E, \quad s(x) = (x, (f_1(x), f_2(x))) \]

is an algebraic section of \( \xi \) with \( Z(s) = \{x_0\} \). On the set \( U_1 \), the section \( s \) is represented by the map

\[ U_1 \to U_1 \times \mathbb{R}, \quad x \mapsto (x, f_1(x)) \]

and hence the restriction of \( s \) to \( \text{Reg}(X) \) is transverse to the zero section of \( \xi \).

We will now give a convenient description of the first cohomology group \( H^1(X; \mathbb{Z}/2) \) of the curve \( X \). The subset \( X \setminus \text{Reg}(X) \) of \( X \) is finite. If \( X \) has nonsingular connected components, we choose one arbitrary point in each of those and denote the set of such points by \( Z \). The curve \( X \) can be regarded as a graph (1-dimensional CW complex) with \( (X \setminus \text{Reg}(X)) \cup Z \) as the set of vertices. This assertion is a straightforward consequence of the triangulation theorem for semi-algebraic sets, cf. \[1\] Theorem 9.2.1.

**Lemma 2.3.** There exist subgraphs \( X_1, \ldots, X_n \) of \( X \) such that each \( X_i \) is homeomorphic to the unit circle \( S^1 \), and the inclusion maps \( X_i \hookrightarrow X \) induce an isomorphism

\[ \varphi : H^1(X; \mathbb{Z}/2) \to \bigoplus_{i=1}^{n} H^1(X_i; \mathbb{Z}/2) \]

**Proof.** Let \( K \) be a connected 1-dimensional component of \( X \) and let \( T \) be a maximal tree of the graph \( K \). The quotient map \( q : K \to K/T \) is a homotopy equivalence and the quotient space \( K/T \) is homeomorphic to the wedge sum of a finite number of pointed circles, \[2\] p. 153. Each such pointed circle corresponds to a subset of \( K/T \) of the form \( q(C) \), where \( C \) is a subgraph of \( K \) homeomorphic to the unit circle. The inclusion maps \( q(C) \hookrightarrow K/T \) induce an isomorphism

\[ \psi : H^1(K/T; \mathbb{Z}/2) \to \bigoplus_C H^1(q(C); \mathbb{Z}/2) \]
If \( q_C : C \to q(C) \) is the restriction of the map \( q \), then the homomorphism
\[
\alpha = \bigoplus_C q_C^* : \bigoplus_C H^1(q(C) ; \mathbb{Z}/2) \to \bigoplus_C H^1(C ; \mathbb{Z}/2)
\]
is an isomorphism. The homomorphism
\[
q^* : H^1(K/T ; \mathbb{Z}/2) \to H^1(K ; \mathbb{Z}/2)
\]
is an isomorphism, the quotient map being a homotopy equivalence. Finally, the inclusion maps \( C \hookrightarrow K \) induce a homomorphism
\[
\varphi_K : H^1(K ; \mathbb{Z}/2) \to \bigoplus_C H^1(C ; \mathbb{Z}/2)
\]
satisfying \( \varphi_K \circ q^* = \alpha \circ \psi \). Consequently, \( \varphi_K \) is an isomorphism.

The assertion of the lemma follows, because \( X \) has finitely many connected components.

**Proof of Theorem 1.1.** The isomorphism classes of topological line bundles on \( X \) form a group, denoted \( \text{Vect}^1(X) \), with tensor product as the group operation. The first Stiefel–Whitney class gives a group isomorphism between \( \text{Vect}^1(X) \) and the first cohomology group \( H^1(X ; \mathbb{Z}/2) \), cf. [3, Proposition 3.10]. Also, note that the isomorphism classes of algebraic vector bundles form a subgroup of \( \text{Vect}^1(X) \). Hence, in view of Lemma 2.3, it remains to construct for each \( i = 1, \ldots, n \) an algebraic line bundle \( \xi_i \) on \( X \) with \( w_1(\xi_i|_{X_i}) \neq 0 \) and \( w_1(\xi_i|_{X_j}) = 0 \) for all \( j \neq i \) (note that \( H^1(X_i ; \mathbb{Z}/2) \cong \mathbb{Z}/2 \)). Such a line bundle \( \xi_i \) can be obtained as follows.

Let \( x_i \) be a point in
\[
(X_i \cap \text{Reg}(X)) \setminus \bigcup_{j \neq i} X_j
\]
and let \( \xi = (E, p, X) \) be an algebraic line bundle on \( X \) as in Lemma 2.2 with \( x_0 = x_i \). There exists an algebraic section \( s : X \to E \) such that \( Z(s) = \{ x_i \} \) and the restriction of \( s \) to \( \text{Reg}(X) \) is transverse to the zero section of \( \xi \). It follows that the line bundle \( \xi|_{X_j} \) is trivial and \( w_1(\xi|_{X_j}) = 0 \) for \( j \neq i \).

Suppose for a moment that the line bundle \( \xi|_{X_i} \) is trivial, and let
\[
\theta : p^{-1}(X_i) \to X_i \times \mathbb{R}
\]
be a topological trivialization of \( \xi|_{X_i} \). Then \( \theta(s(x)) = (x, f(x)) \) for each \( x \) in \( X_i \), where \( f : X_i \to \mathbb{R} \) is a continuous function. By construction, \( f^{-1}(0) = \{ x_i \} \).

The function \( f \) does not change sign on \( X_i \setminus \{ x_i \} \), the set \( X_i \setminus \{ x_i \} \) being homeomorphic to \( \mathbb{R} \). This however is impossible since \( s \) is transverse to the zero section of \( \xi \) in a neighborhood of \( x_i \). Consequently, the line bundle \( \xi|_{X_i} \) is nontrivial and \( w_1(\xi|_{X_i}) \neq 0 \).

We complete the proof by setting \( \xi_i = \xi \).
References


Received August 7, 2012

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