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Andrzej Zajtz

On the algebraic operations on the geometric objects

In the present paper we give some remarks on the algebra of geometric objects and we define the notion of Kronecker multiplication of linear homogeneous objects. Theoretically, the problem of algebra is not much considered; only for some special classes of objects as, for instance, for tensors or densites. This problem has been dealt with by J. Aczél, M. Hossú, S. Gołąb and M. Kucharzewski — M. Kuczma (the complete references are given in [4]).

1. Under *q-nary operation* on the objects (Ω_i), $i = 1, \ldots, q \geqslant 2$ is understood ([1], p. 92) any system of functions

$$f_j(\Omega_1, \ldots, \Omega_q), \quad j = 1, \ldots, m \tag{1}$$

such that the values of these functions are the components of an geometric object. This object—the result of the operation (1), can be considered as an algebraic concomitant of the objects (Ω_i) or, what means the same, as a concomitant of one object

$$\Omega = (\Omega_1, \ldots, \Omega_q) \tag{2}$$

— the union of these objects ([1], p. 13). Thus the problem of finding all the operations (1) (that is all the algebras for the objects (Ω_i)) reduces to the problem of determining all the algebraic concomitants of the object (2). Basing on this fact we shall give now some other (more algebraic) formula for the above definition.

Let F denote a family of homologeous geometric objects, i.e. of the objects, which are defined in the same point of a manifold and related to the same group of local transformations. An operation on F will be called a composition law, according to which to some ordered subsets of F there corresponds exactly one geometric object and this correspondence is independent of the choice of the coordinate system. (That means that any operation on F allows to build from the objects of F some new objects—concomitants of theirs.) An *algebra* will be called a system of F and some number on F defined ope-

rations. This notion touches the families of objects. fields too. The result of an operation will be called a *composition object.*

If we treat any geometric object as a structure $\Omega = (X, \Phi)$, where X denote the domain of values of object (the fibre) and Φ its transformation formula, then, for instance, the object (2) can be considered as result of an algebraic composition and namely as the direct product of the given structures $\Omega_i = (X_i, \Phi_i)$.

Let object $\Omega = (X, \Phi)$. Any operation π defined on the direct product (Ω, Ω) such that the composition object is of the same type, with the fibre X' contained in X, is called an *interior operation* on the object Ω. The operation π determines a mapping

$$X \times X \to X$$

(that is, π is an algebraic operation on the set X), which is invariant under the transformations Φ. A system of Ω and some number of interior operations on Ω will be called an *interior algebra* of this object. In Section 3 we determine the complete interior algebra of a symmetric tensor a_{ij} (with the fibre $X = R^{\frac{n(n+1)}{2}}$). In the paper [4] (p. 56) is given, in fact, the complete interior algebra of a contravariant vector v^i. The interior algebras of any object with one component can be deduced from the corresponding results in [1], p. 112.

2. Now we shall give some suggestions to the problem of classification of operations on geometric objects.

If Ω is a composition object under an operation α on the objects (Ω_i), then each its concomitant θ may be treated as a new composition object of these objects and namely under the operation β: $(\Omega_i) \to \theta$ which is a superposition of the above operation α: $(\Omega_i) \to \Omega$ and the mapping h: $\Omega \to \theta$, i.e. it holds

$$\beta = h \circ \alpha . \tag{3}$$

An operation β will be called a *prolongation* of the operation α if β has the form (3). Two operations are called *equivalent* if each of them is a prolongation of the other one.

In the above sense, every operation is a prolongation of the direct product. In the sequel we shall ignore this fact.

Let be given a family F of geometric objects. An operation be called *elementar* (on F) if it is not a prolongation of some other operation on F, with accuracy to equivalence. A system of operations will be called a *fundamental system* of operations on F if it contains only the elementar operations and any operation on F may be obtained as an iteration of the finite number of operations of this system.

In general, the problem of finding of a fundamental system on a given F is very difficult. From the practical point of view the most interesting are the algebraic operations (the functions (1) are algebraic) and their fundamental

systems. As example, we give below the fundamental system of algebraic operations on tensors.

Let F denote the family of all the tensors the indices of which run by the same set of numbers. The fundamental system of algebraic operations on F consists of the following five operations viz. the addition and multiplication of tensors, the contradiction and permutation of indices and the multiplication by scalars.

That follows from the fundamental theorem of the theory of invariants (Cramlet 1928, [3], p. 182) and namely, that any algebraic invariant of tensors is an algebraic function of expressions; each of them may be obtained by the above mentioned operations.

3. Let a_{ij} $(i, j = 1, \dots, n)$ be a symmetric tensor with values in the whole space $R^{\frac{n(n+1)}{2}}$. Any interior operation of this object determines a concomitant of a pair $(b_{ij}, c_{ij}) \in X \times X$, where X is the fibre of a_{ij}, i.e. a concomitant of a pair of symmetric tensors, which itself represents a symmetric tensor ω_{ij}. Such concomitants were determined in [5], p. 33. We construct them in the following way:

We define a binom

$$\lambda b_{ij} + \mu c_{ij} \tag{4}$$

(λ, μ—parameters). If the determinant $\det(\lambda b_{ij} + \mu c_{ij})$ vanishes identically, then any concomitant ω_{ij} has the form

$$\omega_{ij} = \lambda_0 b_{ij} + \mu_0 c_{ij} .$$

In the opposite case there exist values λ, μ such that the corresponding tensor (4) is regular and we can build the inverse contravariant tensor, and then the mixed tensor s^i_j. We put

$$\underset{(1)}{s^i_j} = s^i_j , \quad \underset{(p)}{s^i_j} = s^{k_1}_j s^{k_2}_{k_1} \dots s^i_{k_{p-1}} , \quad p = 1, \dots, N$$

where N is the degree of minimal polynom of components matrix $[s^i_j]$. With the aid of the above power tensors of s^i_j we construct the following N linear independent covariant tensors

$$a_{ij}, \underset{(1)}{a_{ik} s^k_j}, \dots, \underset{(N-1)}{a_{ik} s^k_j} .$$

Any concomitant ω_{ij} is a linear combination (with scalar coefficients) of these tensors. It follows from the above facts that any interior operation of a_{ij} reduces to the superposition of the following elementar operations viz. the addition and multiplication of tensors, the contradiction, the inversion of tensor and the multiplication by scalars.

4. Let be given two linear homogeneous geometric objects Ω_1, Ω_2 with p and q components, respectively and with the transformation formulae

$$\underset{(1)}{\omega^{i'}} = a_i^{i'} \underset{(1)}{\omega^i}, \qquad \underset{(2)}{\omega^{k'}} = b_k^{k'} \underset{(2)}{\omega^k} \tag{5}$$

$(i, i' = 1, \ldots, p;\ k, k' = 1, \ldots, q)$.

We denote by $C = A \times B$ the Kronecker product of the transformation matrices $A = [a_i^{i'}]$, $B = [b_k^{k'}]$ of order p and q respectively (cf. [2], p. 15). The elements $c_m^{m'}$ $(m, m' = 1, \ldots, pq)$ of C have the form

$$c_{(ik)}^{(i'k')} = a_i^{i'} b_k^{k'},$$

where the indices pairs $(i'k')$ and (ik) are ordered lexicographish.

The Kronecker multiplication of matrices have the following properties:

$$(A_1 A_2) \times (B_1 B_2) = (A_1 \times B_1)(A_2 \times B_2)$$

$$(A \times B)^{-1} = A^{-1} \times B^{-1}$$

$$E^p \times E^q = E^{pq} \qquad (E^s \text{ — the unit matrix of degree } s)$$

$$(A \times B)^T = A^T \times B^T.$$

From the above relations follows that if G_1, G_2 are two linear groups then is such the set $G_1 \times G_2$ too (the set of all products $A \times B$ where $A \in G_1$ and $B \in G_2$).

We put in an arbitrary coordinate system

$$\omega^{ik} = \underset{(1)\,(2)}{\omega^i \omega^k} \qquad (i = 1, \ldots, p;\ k = 1, \ldots, q).$$

In view of (5) the numbers ω^{ik} transform by a change of coordinate system as follows

$$\omega^{i'k'} = a_i^{i'} b_k^{k'} \underset{(1)\,(2)}{\omega^i \omega^k}. \tag{6}$$

The formula (6) may be written in the following fashion

$$\omega^{m'} = c_m^{m'} \omega^m \tag{7}$$

where $m = (ik)$ and $m = 1, \ldots, pq$. If G_1, G_2 are the transformation groups of objects Ω_1, Ω_2 respectively, then $G = G_1 \times G_2$ is the transformation group of the sequence of pq numbers (ω^m). Thus a new linear homogeneous geometric object $\Omega = (\omega^1, \ldots, \omega^{pq})$ with the transformation formula (7) is defined. It will be called the *Kronecker product* of objects (5) and denoted by $\Omega_1 \times \Omega_2$. The corresponding operation will be called a *Kronecker multiplication.*

The well-known multiplication of tensors is a particular case of *Kronecker multiplication.*

Finally we remark that the following propositions hold

1. The object $\Omega_1 \times \Omega_2$ is equivalent to the object $\Omega_2 \times \Omega_1$.

2. If $\Omega_1 \sim \Omega_2$, $\Omega_1' \sim \Omega_2'$ then $(\Omega_1 \times \Omega_1') \sim (\Omega_2 \times \Omega_2')$, where the sign "$\sim$" denote the equivalence relation.

3. The direct product $(\Omega_1 \times \Omega_3, \Omega_2 \times \Omega_3)$ is identical with $(\Omega_1, \Omega_2) \times \Omega_3$.

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