ZESZYTY NAUKOWE UNIWERSYTETU JAGIELLOŃSKIEGO

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Gabriela Kleiner

Convex Sets and the Modulus of Continuity

1. By a well-known theorem of Lebesgue [4] a continuous function f defined on a closed set X in R^m possesses a continuous extension to the whole space, bounded by the same bounds. This has been generalized by Tietze [7] to metric spaces and by Urysohn [8] to the normal ones. An explicit formula for a space with distance ϱ has been given by Hausdorff [2],

$$f_H(x) = \inf_{x' \in E}\left[f(x') + \frac{\varrho(x, x')}{\varrho(x, E)} - 1\right]. \tag{1}$$

In the approximation theory an important role is played by the modulus of continuity

$$\omega(\delta) = \omega(\delta, f) = \sup\{|f(x'') - f(x')| : |x'' - x'| \leqslant \delta;\ x', x'' \in E\}. \tag{2}$$

ω is clearly increasing with δ and $\omega(0+) = 0$ if and only if f is uniformly continuous. It is natural to raise the problem of extending f with the same modulus of continuity *). This is impossible in general, so instead of ω some substitutes are being used (e.g., [6]). J. Siciak and W. Kleiner have stated the following conjecture:

Let E be a closed subset of a normed linear space Ω. Then every function defined and uniformly continuous on E can be extended to a function on Ω with the same modulus of continuity if only if E is convex.

I will prove this conjecture for Hilbert spaces, where a simple continuation formula may also be given.

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*) For special ω, e.g. that of Lipschitz or Hölder type, strong results have been obtained by McShane [5], Kirszbraun [3] and Valentine [9].

2. Let E be a closed convex set in the Hilbert space Ω with inner product (x, y) and norm $\|x\| = \sqrt{(x, x)}$. Then, for any $x \in \Omega$ there is a unique point $\mathrm{pr}_E\, x$ in E, called its projection, such that [1]

$$\|\mathrm{pr}_E x - x\| \leqslant \|y - x\| \qquad (y \in E)\,.$$

Lemma. *The operation* pr_E *is contractive.*

Proof. Let $x_1, x_2 \in \Omega$. Setting

$$a = x_1 - \mathrm{pr}_E x_1\,, \qquad b = x_2 - \mathrm{pr}_E x_2\,,$$
$$c = \mathrm{pr}_E x_1 - \mathrm{pr}_E x_2\,, \qquad d = x_1 - x_2\,,$$

we have $(b, c) \leqslant 0$ (if $(b, c) > 0$, then for a small positive λ the point $x = \mathrm{pr}_E x_2 + \lambda c \in E$ would lie nearer to x_2 than $\mathrm{pr}_E\, x_2$) and, similarly ,$(a, -c) \leqslant 0$. So

$$(b, c) = (c + a - d, c) = \|c\|^2 + (a, c) - (d, c) \leqslant 0\,,$$
$$\|c\|^2 \leqslant (d, c) - (a, c) \leqslant (d, c) \leqslant \|d\| \cdot \|c\|\,,$$
$$\|c\| \leqslant \|d\|\,.$$

3. Theorem. *Convex sets E in a Hilbert space Ω are characterized by the following property:*

any mapping f of E into an arbitrary metric space Ω' can be extended to Ω preserving its modulus of continuity.

Proof. Suppose E to be closed and convex, and f a function on E with the modulus of continuity (2). Put, for $x \in \Omega$,

$$f_*(x) = f(\mathrm{pr}_E x)\,. \tag{3}$$

Then $f_*(x) = f(x)$ on E, and for arbitrary x_1, x_2 with $\|x_1 - x_2\| < \delta$ there exist $y_1, y_2 \in E$ — namely their projections — such that

$$|f_*(x_1) - f_*(x_2)| = |f(y_1) - f(y_2)| \tag{4}$$

and $\|y_1 - y_2\| < \delta$ (by the Lemma). So the left-hand side is bounded by $\omega(\delta, f)$, whence its least upper bound $\omega(\delta, f_*) \leqslant \omega(\delta, f)$. The opposite inequality is immediate.

Remark. Let E be not convex, but $E = F \setminus D$ with F convex and closed, $D \subset F^0 =$ the interior of F, and let f_* be defined by (3) in $\Omega \setminus D$ only. Then the conclusion holds as well, since $\mathrm{pr}_E(\Omega \setminus E) \subset F$.

Let now E be not convex, and let a, b be two points in E such that the segment S joining them is not contained in E. Replacing S by a subsegment we may assume that $S^0 = S \setminus \{a, b\} \subset \Omega \setminus E$. Assume also $a = 0$, $\|b\| = 1$. Let for $x \in S^0$, $r(x)$ be a number — say, the largest one — such that $U_x = \{y : \|y - x\| < r(x)\} \subset \Omega \setminus E$. Let $D = \bigcup_{x \in S^0} U_x$.

We shall define a function $f(x)$ for $x \in \Omega \setminus D$ in such a way that extending it onto D necessarily enlarges the modulus of continuity.

Let $\Omega = \Omega_1 \oplus \Omega_2$, where $\Omega_1 = \{\lambda b : -\infty < \lambda < +\infty\}$ and Ω_2 is its orthogonal complement. So any $x \in \Omega$ is uniquely decomposed into $x = x_1 + x_2$, $x_i \in \Omega_i$. Let

(5) $$F = \{x = x_1 + x_2 : x_1 \in S, \|x_2\| \leqslant r\} \quad (r = \sup r(x)) .$$

Put for $x \in F \setminus D$

(6) $$f_0(x) = \|z\| \text{ where } z = x_1 + \frac{x_2}{2},$$

and for $x \in \Omega \setminus D$

(7) $$f(x) = f_0(\mathrm{pr}_F x) .$$

We have to show that f has a modulus of continuity which increases at some points when f is extended in any way to D. Let f^* be any such extension; it is defined in the linear space Ω so for $\omega^*(\delta) = \omega(\delta, f^*)$ we have $\omega^*(\Sigma\delta_i) \leqslant \Sigma\omega^*(\delta_i)$, thus

$$1 = f^*(b) - f^*(0) \leqslant n\omega^*\left(\frac{1}{n}\right),$$

so

(8) $$\omega^*(\delta) \geqslant \delta \text{ for } \delta = \frac{1}{n}, \quad n = 1, 2, \ldots$$

On the other hand, we shall show that

(9) $$\omega(\delta) = \omega(\delta, f) = \omega(\delta, f_0) < \delta \quad \text{for } \delta \in (0, 1) .$$

(The last equality results from the remark above, F being clearly convex and closed).

I claim that

(10) $$|f(x') - f(x'')| < \|x' - x''\| \quad \text{for } x', x'' \in F \setminus D, \quad 0 < \|x' - x''\| < 1.$$

Indeed, we have

(11) $$|\,\|z'\| - \|z''\|\,| \leqslant \|z' - z''\| \leqslant \|x' - x''\|$$

(see (6)) with both equalities only if $z'' = \lambda z'$ (or $z' = \lambda z''$, $\lambda \geqslant 0$) and $x_2' = x_2''$. But $z'' = x_1'' + \frac{1}{2}x_2'' = \lambda x_1' + \frac{1}{2}\lambda x_2'$ then implies either $x_2' = 0 = x_2''$ or $\lambda = 0$. The first contradicts the restrictions on $x's$; $\lambda = 0$ gives $z'' = 0$, $x'' = 0$, now (11) becomes $\|z'\| \leqslant \|z'\| \leqslant x'\|$ and the equality would imply again $x_2' = 0$. So (11) is proved.

F is not compact in general. But, by the symmetry, we may start with a section of F. Take an arbitrary three-dimensional subspace π passing through the segment S. Let $f_0|\pi$ denote the restriction of f_0 to the set $F \cap \pi$ which is compact. So the upper bound $\omega(\delta, f_0|\pi)$ is attained by a pair $x^*, x^{**} \in F \cap \pi$ with $\|x^* - x^{**}\| \leqslant \delta$: if $\delta \in (0, 1)$, we have by (10)

(12) $$\omega(\delta, f_0|\pi) = |f_0(x^*) - f_0(x^{**})| < \|x^* - x^{**}\| \leqslant \delta .$$

Observe that $\omega(\delta, f_0|\pi)$ does not depend on the choice of $\pi \supset S$, since f is invariant under rotations around S. Any x^*, x^{**} belong to a certain π so $\omega(\delta, f_0)$

$= \omega(\delta, f_0|\pi)$. Thus any extension of f to Ω leads to enlarging the values of the modulus of continuity for $\delta = \frac{1}{n}$ $(n = 1, 2, \ldots)$ at least.

4. As to the general case of Banach spaces, let us observe that *our Lemma does not hold there.* E.g., let Ω be the linear space R^2 with a norm making the square $B = (-1, 1) \times (-1, 1)$ to the unit ball, i. e. $\|(u, v)\| = \max\{|u|, |v|\}$. Let E be the segment joining $y_1 = 0$ with $y_2 = (4,2)$. These points are projections of $x_1 = 0$ and $x_2 = (3,3)$, respectively; nevertheless $\|y_1 - y_2\| = 4 > \|x_1 - x_2\| = 3$, thus the projection is not contractive. A similar example of uniformly convex Ω is obtained by changing B slightly to obtain a strictly convex set.

So the extension (3) may well enlarge ω. The same may occur to the Hausdorff extension (1). Indeed, define — in the preceding example — f as a linear function on E, with $f(x_2) = 1$, $f(\mathrm{pr}_E x_1) = 0$. Then $\omega(\delta; f) = \delta$ $(\delta \in [0, 1])$. The Hausdorff extension gives

$$f_H(x_2) - f_H(x_1) = 1 - \inf_{x' \in E}\left[f(x') + \frac{\varrho(x_1, x')}{\varrho(x_1, E)} - 1\right].$$

Since $[\ ] \geqslant 0$ and equals 0 for $x' = \mathrm{pr}_E x_1$, so for $\delta_0 = \|x_2 - x_1\| < 1$ we have

$$\omega(\delta_0, f_H) \geqslant 1 > \delta_0 = \omega(\delta_0, f).$$

The failure of both methods is unpromising for the general conjecture; on the other hand, it is advocated by some geometrical reasonings.

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