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# A Note on the Uniqueness of Two Point Boundary Value Problems II

1. The previous paper [2] dealt with the connections between the existence and the uniqueness of the solutions of the boundary value problem for the equation

$$x'' = f(t, x, x') . \tag{1}$$

It has been assumed that the right-hand side of equation (1) satisfies the following *Condition* $C_1$:

1° $f(t, x, y)$ is continuous on $\Delta \times R \times R$, where $\Delta$ is an open interval and $R$ is real line,

2° for every $a \in \Delta$ and $p, q \in R$ there exists exactly one solution of the initial problem $x(a) = p$, $x'(a) = q$ and it is defined on the whole $\Delta$.

Simultaneously with equation (1) the condition

$$\begin{aligned} \alpha x(a) + \beta x'(a) &= p , \\ \gamma x(b) + \delta x'(b) &= q , \end{aligned} \tag{2}$$

where $\alpha, \beta, \gamma, \delta$ are fixed and $a, b \in \Delta$, $a \neq b$ has been taken into the consideration.

The important particular case of this problem is the two point boundary value problem

$$x(a) = p, \quad x(b) = q . \tag{3}$$

It should be remembered that problem (1), (2) with any fixed $\alpha, \beta, \gamma, \delta$ is said to be *globally unique* if for every $a, b \in \Delta$, $a \neq b$ and $p, q \in R$ there exists at most one solution of equation (1) defined on $\Delta$ satisfying (2). Problem (1), (2) with any fixed $\alpha, \beta, \gamma, \delta$ is said to be *globally solvable* if for every $a, b \in \Delta$,

$a \neq b$ and $p, q \in R$ there exists at least one solution of equation (1) defined on the whole $\Delta$ and such that (2) holds.

We have proved the following:

Theorem 1. *If $f(t, x, y)$ satisfies condition $C_1$ and problem* (1), (2) *with $\beta = 0$ is globally unique, then*

1° *problem* (1), (2) *with the same $\alpha, \beta, \gamma, \delta$ is globally solvable,*

2° *problem* (1), (3) *is globally unique and globally solvable.*

The present paper has been written in order to give some applications of the above theorem. The obtained result seems to be a natural supplement to results of C. Corduneanu [1] and Z. Opial [4].

2. In what follows $f(t, x, y)$ is a continuous function defined on $\Delta \times R \times R$ and $\Delta$ stands for an open interval.

Theorem 2. *If $f(t, x, y)$ has the continuous partial derivatives $f'_x, f'_y$ satisfying the inequalities*

$$|f'_x(t, x, y)| \leqslant M, \quad f'_y(t, x, y) \leqslant K \quad \text{on } \Delta \times R \times R, \tag{4}$$

*where $M, K$ are non-negative constants, then $f(t, x, y)$ satisfies condition $C_1$.*

Proof. The existence of the continuous derivatives $f'_x, f'_y$ evidently implies the uniqueness and the local existence of the solutions of Cauchy problem for equation (1). To complete the proof it is sufficient to show that all solutions of equation (1) and their derivatives are bounded on every compact subinterval $\Delta_0 \subset \Delta$.

Write $u(t) = x(t)^2 + x'(t)^2$; then

$$u'(t) = 2[f(t, x(t), x'(t)) - f(t, 0, 0)]x'(t) + 2x(t)x'(t) + 2f(t, 0, 0)x'(t).$$

By the mean value theorem and by the inequality $2xy \leqslant x^2 + y^2$, we have the estimation

$$u'(t) \leqslant Au(t) + B,$$

where $B = \sup_{t \in \Delta_0} |f(t, 0, 0)|$, $A = M + 2K + B + 1$.

Hence (see e.g. [5], p. 28) it follows the estimation

$$u(t) \leqslant \left(\frac{B}{A} + u(t_0)\right) e^{A|\Delta_0|}, \quad t, t_0 \in \Delta_0$$

which finishes the proof.

Theorem 3. *Assume that the function $f(t, x, y)$ has the continuous partial derivatives $f'_x, f'_y$ satisfying the inequalities*

$$|f'_x(t, x, y)| \leqslant M, \quad |f'_y(t, x, y)| \leqslant K \quad \text{on } \Delta \times R \times R, \tag{5}$$

*where $M$, $K$ are non-negative constants. If, moreover, the length of the interval $\Delta$ satisfies the inequality*

$$|\Delta| \leqslant \int_{-\frac{1}{h}}^{\infty} \frac{du}{u^2+K|u|+M}, \tag{6}$$

*then for $a=1$, $\beta=0$, $\gamma=1$, $\delta=h\neq 0$ problem* (1), (2) *is globally unique.*

Proof. Suppose that problem (1), (2) has two different solutions $x_1(t)$ and $x_2(t)$. Then for $z(t)=x_1(t)-x_2(t)$ we have

$$|z''(t)| \leqslant M|z(t)|+K|z'(t)|, \tag{7}$$

$$z(a)=0, \quad z(b)+hz'(b)=0, \tag{8}$$

$$z'(a)\neq 0, \quad z(b)\neq 0. \tag{9}$$

By shifting the point $a$ towards the point $b$ and replacing eventually $z$ by $-z$, the condition $z(t)>0$ between $a$ and $b$ can be obtained.

Setting $w(t)=z'(t)/z(t)$ from (7), (8), (9) we get

$$|w'(t)| \leqslant w(t)^2+K|w(t)|+M, \tag{10}$$

$$w(a)=+\infty, \quad w(b)=-\frac{1}{h}. \tag{11}$$

When $a>b$, let $\varphi(t)$ be the solution of the equation

$$u'=u^2+K|u|+M \tag{12}$$

with the condition $u(b)=-\frac{1}{h}$. By (10) and by $w(b)=\varphi(b)$, we have $w(t)\leqslant\varphi(t)$ for $b<t<a$. Hence $\lim\limits_{t\to a-}\varphi(t)=+\infty$. Since from (12) it follows that

$$\int_{-\frac{1}{h}}^{\infty} \frac{du}{u^2+K|U|+M}=a-b<|\Delta|, \tag{13}$$

a contradiction with assumption (6) is yielded.

When $a<b$, we majorize $w(t)$ by the solution $\psi(t)$ of the equation $u'=-u^2-K|u|-M$ with the condition $u(b)=-\frac{1}{h}$. Similarly as in the preceding case we obtain inequality (13), which contradicts assumption (6).

Remark. Theorem 3 remains also true when assumptions (5), (6) are replaced by assumption (4) and by two other assumptions

$$b<a, \quad |\Delta| \leqslant \int_{\frac{1}{|h|}}^{\infty} \frac{du}{u^2+Ku+M}.$$

In the proof $|w(t)|$ and its right-hand derivative should be considered instead of $w(t)$ and $w'(t)$.

3. From theorem 2 and 3 owing to theorem 1 it follows immediately

Theorem 4. *Assume that the function $f(t, x, y)$ has the continuous partial derivatives $f'_x, f'_y$ satisfying* (5). *If $\alpha = 1$, $\beta = 0$, $\gamma = 1$, $\delta = h \neq 0$ and the length of interval $\Delta$ satisfies* (6), *then*

1° *problem* (1), (2) *with the same $\alpha, \beta, \gamma, \delta$ is globally solvable,*

2° *two point boundary value problem* (1), (3) *is globally unique and globally solvable.*

In the formulation of problem (1), (3) the constant $h$ does not occur. In inequality (6) $h$ can be arbitrary. Hence, for the global uniqueness and global solvability of problem (1), (3) inequality (6) may be replaced by

$$|\Delta| \leqslant \int_{-\infty}^{\infty} \frac{du}{u^2+K|u|+M}.$$

The uniqueness statement in Theorem 4 is known as Ch. de la Vallée Poussin's [6] result. The existence criterion has been obtained by A. Lasota and Z. Opial [3].

## REFERENCES


[1] C. Corduneanu, *Asupra unei probleme la limită ecuațiile differențiale neliniare de ordinul al doilea*, Analele Stiințifice Iași, Sec. I. 1 (1955), 11—16.

[2] A. Lasota, M. Łuczyński, *A note on the uniqueness of two point boundary value problems I*, Zeszyty Naukowe UJ, Prace matematyczne 12 (1968), 27—29.

[3] A. Lasota, Z. Opial, *L'existence et l'unicité des solutions du problème d'interpolation pour l'équation différentielle ordinaire d'ordre n*, Ann. Polon. Math. 15 (1964), 253—271.

[4] Z. Opial, *Sur un problème aux limites pour l'équation différentielle du second ordre*, Ann. Polon. Math. 7 (1960), 223—231.

[5] J. Szarski, *Differential inequalities*, Warszawa, 1965.

[6] Ch. de la Vallée Poussin, *Sur l'équation différentielle du second ordre*, Journal Math. Pures et Appl. 8 (1929), 125—144.