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The physical meaning of the class of the geometric objects

In the Lagrangian theory of fields the expressions of the properties of the physical system under consideration are appropriate comitants both of the field descriptors $\psi_A(\xi^k)$, by which the system is described, as well as of the Lagrangian

$$L = L(\psi_A, \psi_{A,k}, \ldots) \tag{1}$$

which contains all information concerning the system. The form and the possible numerical values of these comitants depend on the transformation properties of the descriptors ψ_A under those symmetry bioperatives (grupoids, ..., groups) which are connected with the property in question.

In order to explain this more precisely we will consider the charge and the energy of the fields ψ_A.

THE CHARGE AND ITS GROUP

In the Lagrangian formalism the expression of the charge-current vector density j^k is

$$j^k = \frac{\partial L}{\partial \psi_{A,k}} \frac{\delta \psi_A}{\delta \varphi}, \tag{2}$$

where

$$\frac{\delta \psi_A}{\delta \varphi} = ie\psi_A, \tag{3}$$

when the Lagrangian L is invariant under the electric gauge group

$$\overline{\psi}_A = e^{ie\varphi(\xi)}\psi_A \equiv T_A[\psi_B, \varphi, \varphi_{,k}, \ldots], \tag{4}$$

where $\varphi(\xi)$ is the descriptor function on the Lie group. Since, according to (4), the variation of ψ_A with respect to φ is

$$\delta\psi_A = ie\psi_A \delta\varphi \equiv \left.\frac{\partial T_A}{\partial \varphi}\right|_{\varphi=0} \cdot \delta\varphi + \left.\frac{\partial T_A}{\partial \varphi_k}\right|_{\varphi,k=0} + \ldots \tag{5}$$

it is seen from the comparison of (3) and (5) that expression (2) contains the derivatives of the transformator (4) too. Taking this into account one sees from (2) that j^k depends on ψ_A, T_A, and L, i.e. it is a comitant of them.

THE COMITANT OF THE INTERNAL ENERGY

In the special relativistic field theory of subnuclear particles the canonical energy-momentum concomitant has the form

$$T_l^k = \frac{\partial L}{\partial \psi_{A,k}} \frac{\delta \psi_A}{\delta x^l} - \frac{\partial L}{\partial \psi_{A,k}} \psi_{A,l} + \delta_l^k L \,, \tag{6}$$

where $\delta\psi_A$ is the isolocal variation of ψ_A under the infinitesimal Poincaré transformation δx^l of coordinates.

The third term of (6), $\delta_l^k L$, is a pressure, which is a result of the interaction (or self-interaction). Therefore it does not constitute a part of the internal energy.

The second term, $-\dfrac{\partial L}{\partial \psi_{A,k}} \psi_{A,l}$, is the four dimensional path (kinetic) energy, which does not belong to the internal mass either.

Thus the density of the internal energy is either identical with, or is contained in the first term

$$T_0^0(1) = \frac{\partial L}{\partial \psi_{A,0}} \frac{\partial \psi_A}{\partial x^0} \tag{7}$$

Now the questions arise:

1. Is it natural that the subnuclear particles possess a number of charges (electric, barinic, etc) which are internal properties, that they possess spin, which is their internal angular momentum, and that they do not possess internal energy (mass)?

2. Supposing that they have internal mass, by what objects it may be described?

The answer to the last question is given in what follows, in the form of a theorem.

THE ROLE OF THE TRANSFORMATORS

It is a known theoretical physical fact that the different descriptor fields of the particles with different spin have different transformation character (different transformators in our terminology) under the rotation subgroup

of the Poincaré group of coordinate transformations. For example, the scalar fields describe zero-spin particles, the spinor fields describe $\frac{\hbar}{2}$ spin particles, the vector fields describe $\hbar$ spin ones, etc. The spin is thus „determined" by the transformator T of ψ_A

$$\bar{\psi}_A(\bar{\xi}) = T_A\left[\psi_B(\xi(\bar{\xi})), \bar{\xi}^k, \xi^k, \frac{\partial \xi^k}{\partial \bar{\xi}^l}, \ldots\right] \tag{8}$$

under the rotation subgroup.

Similarly the electric (barionic, etc.) charge of a field is determined by the transformator T of ψ_A

$$\bar{\psi}_A = T_A\left[\psi_B, \varphi(\xi), \frac{\partial \varphi}{\partial \xi^k}, \ldots\right] \tag{9}$$

under the electric (barionic, etc.) gauge group. For example, the proton field ψ_A has electric charge, because its transformator under the electric gauge group

$$\bar{\psi}_A = e^{ie\varphi(\xi)}\psi_A \tag{10}$$

is of class zero. The electromagnetic field $A_k(\xi)$ has no charge, because its transformator under the electric gauge group

$$\bar{A}_k = A_k + \varphi_k(\xi) \tag{11}$$

is of class one. $A_k(\xi)$ is namely a gauge connector (connection).

These facts mean that the values of the internal quantities are determined by the transformators T via variations $\delta\psi_A$. According to this rule the internal energy, if it exists, should be also determined by the transformator T of ψ_A. Particles of different internal energy (mass) should have different transformators under the bioperative of coordinate transformations. This is supported by the following

Theorem: Those objects of the Poincaré group of coordinate transformations whose class is greater than zero (whose transformators do not contain the coordinates explicitly), describe particles of zero internal energy (mass). The nonzero internal mass may be described only by objects of class zero.

Proof: The transformators in question are of the form

$$\bar{\psi}_A(\bar{\xi}) = T_A\left[\psi_B(\xi(\bar{\xi})), \frac{\partial \xi^i}{\partial \bar{\xi}^k}\right]. \tag{12}$$

The infinitesimal isolocal variation of ψ_A is therefore

$$\delta\psi_A = -\left.\frac{\partial T_A}{\partial \frac{\partial \xi^i}{\partial \bar{\xi}^k}}\right|_{\xi=\bar{\xi}} \cdot \delta \frac{\partial \xi^i}{\partial \bar{\xi}^k}. \tag{13}$$

Since in the case of the Poincaré group the variations $\delta \frac{\partial \xi^i}{\partial \bar{\xi}^k}$ are infinitesimal constants, which do not depend on $\delta\xi^0$ at all, $\delta \frac{\partial \xi^l}{\partial \bar{\xi}^k}\Big|_{\xi=\bar{\xi}} \delta\xi^0$ is zero. If so, $\delta\psi_A$ is also zero. Thus

$$T_0^0(1) = \frac{\partial L}{\partial \psi_{A,0}} \frac{\delta \psi_A}{\delta^2} = 0 \tag{14}$$

and the first part of the theorem is proved.

In the case of objects of class zero $\delta\psi_A$ depends on $\delta\xi^0$ explicitly. Therefore (14) may be different from zero. QED.

This theorem means that scalars, spinors, vectors, tensors, etc. can describe particles of zero internal mass only. These objects are not suitable to describe particles whose internal mass is different from zero.