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Nr CCXXIII PRACE MATEMATYCZNE. ZESZYT 14 1969

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A remark on the dependence of functions

We need, in many investigations, conditions under which one can decide wether two given functions are independent or not. More exactly, the problem is the following: considering the sets A, B, C and onto mappings

$$\varphi\colon A \to B\,, \quad \psi\colon A \to C\,,$$

which conditions are necessary and sufficient for the existence of a mapping $\chi\colon B \to C$ such that we have the commutative diagram

$$\begin{tikzcd} & A \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ B \arrow[rr, dashed, "\chi"'] & & C \end{tikzcd}$$

i.e. $\psi a = \chi\varphi a$ for every $a \in A$.

Naturally, also this χ must be onto.

It is interesting also the special case where $A = C$ and ψ is the identical mapping. Then χ must be the inverse mapping of φ. For the existence of the inverse it is necessary and sufficient that φ be 1—1, i.e.

$$\varphi a_1 = \varphi a_2 \Rightarrow a_1 = a_2\,.$$

Therefore, for the general case it lies in hand the condition

$$\text{(I)} \qquad \varphi a_1 = \varphi a_2 \Rightarrow \psi a_1 = \psi a_2\,.$$

In fact, this condition is necessary for the dependence of ψ on φ. Conversely, it can be seen immediately that (I) is sufficient to define a mapping χ by

$$\chi b = \psi A_b\,, \quad A_b = \{a \mid a \in A\,,\ \varphi a = b\}$$

and for which we have the following observations:

1) χb is defined for every $b \in B$ (since $\varphi A = B$);

2) $\psi A_b = c$ consists of a single element $c \in C$ for a fixed $b \in B$ [since (I) $\varphi A_b = b \Rightarrow \psi A_b = c$];

3) $\psi a = \{\varphi a\}$ for every $a \in A$ [since $\chi b = \psi a$ for $b = \varphi a$].

If we would like to know more about χ, then we must take stronger conditions. E.g. if we suppose

(II) $\varphi X_1 \subset \varphi X_2 \Rightarrow \psi X_1 \subset \psi X_2$ for the subsets $X_i \subseteq A$,

then the mapping χ defined above will be inclusion preserving:

$$Y_1 \subset Y_2 \Rightarrow \chi Y_1 \subset \chi Y_2 \text{ for the subsets } Y_i \text{ of } B .$$

Another condition is:

(III) $\varphi a_1 = \varphi a_2 \Longleftrightarrow \psi a_1 = \psi a_2$.

This implies that χ is $1-1$. In fact, then we have $\varphi = \varkappa\psi$ and thus both χ and $\varkappa$ must be $1-1$ and each of them is the inverse of the other.

This last condition (III) was used by J. Aczél [1] in order to solve a problem of A. A. J. Marley on connection with the dependence of the functions $f(x, y)$ and $g(x)+h(y)$. There the problem was considered for real functions of two variables and the original condition was

(IV) $f(x_1, y_1) \leqslant f(x_2, y_2) \Longleftrightarrow g(x_1)+h(y_1) \leqslant g(x_2)+h(y_2)$

which has geometrical interpretation by niveau sets.

Naturally, considering the product set A with elements (x, y) and subsets of the real numbers defined by $Y = \{y \mid y \leqslant a\}$, we see that (IV) is a special case of (II).

PROBLEMS

Solve the functional equations

$$f(x+y-xy)+f(xy) = f(x)+f(y) , \tag{1}$$

$$g(x \circ y)+g(x*y) = g(x)+g(y) , \tag{2}$$

$$h(xy)+h(xy^{-1}) = 2h(x)+2h(y) , \tag{3}$$

$$c(xy)+c(yx) = 2c(x)+2c(y) , \tag{4}$$

where f is a real valued function defined on reals, g, h, c are defined on an abstract set G which is a group under multiplication and a semigroup under binary operation $x \circ y$ resp. $x*y$ while the range is an Abelian group such that $2a \neq 0$ for $a \neq 0$.

Partial results: By differentiating, (1) can be reduced to

$$\frac{xf'(x)-yf'(y)}{x-y} = f'(x+y-xy) ,$$

consequently, $f'(x) = \text{const}$.

(3) implies

$$\begin{cases} h(1) = 0 , \ h(y^{-1}) = h(y) , \\ h(xy) = h(yx) , \ h(x^2) = 4h(x) , \\ h(xyxy^{-1}) = h(x^2) , \end{cases}$$

[1] A remark on functional dependence, Journal of Math. Psych. 2 (1965), 125-127.

hence

$$H_1(x, y) \stackrel{\text{def}}{=} h(xy) - h(x) - h(y) = h(x) + h(y) - h(xy^{-1}),$$

$$H_2(x, y) \stackrel{\text{def}}{=} h(xy) - h(xy^{-1}) = 2H_1$$

have the following properties:

$$(3') \qquad \begin{cases} H(x, y) = H(y, x), \quad H(1, x) = 0, \\ H(xy, xy) = H(yx, yx), \\ H(xy, xy^{-1}) = H(yx, xy^{-1}), \\ H(xy, z) + H(xy^{-1}, z) = 2H(x, z) \quad \text{(Jensen)}. \end{cases}$$

Conversely, for every H with properties (3′) we have a solution $h(x) = H(x, x)$ of (3).

This $H(x, y)$ satisfies also

$$H(xy, z) = H(x, z) + H(y, z) \quad \text{(Cauchy)}$$

if and only if

$$H(xy, z) = H(yx, z) \quad [\text{i.e. } h(xyz) = h(yxz)]$$

holds for arbitrary x, y, z (not only for $z = xy^{-1}$).

From (3′/4) it follows that

$$c(x) = H(x, z)$$

satisfies (4) for every fixed z.