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Sets of Covering Mappings

1. Let M and P denote p-dimensional separable topological manifolds; P is assumed to be connected and ϱ will denote a metric function on P. Let $\mathcal{C} = \mathcal{C}(M, P)$ be the set of all continuous mappings from M into P with the so-called C^0-Whitney topology: if $f \in \mathcal{C}$ then the sets

$$V(f, \delta) = \{g \in \mathcal{C} : \varrho(f(x), g(x)) < \delta(x)\}$$

(when $\delta : M \to R_+$ is a positive continuous function) form a neighborhood basis for a C^0-topology (Note that this topology is independent of the metric chosen for P).

Let $\mathfrak{L}$ (respectively $\mathfrak{L}_\sigma$) denote the set of all local homeomorphisms of M into (respectively onto) P, with induced Whitney's topology. Denote by $\nu(A)$ the cardinality of the set A, and $\infty = \nu(N)$.

Recall that $f \in \mathfrak{L}_\sigma$ is said to be a *covering mapping* if for each $y \in P$ there exist a neighborhood V such that $f^{-1}(V)$ is a disjoint union of open neighborhoods W_x of $x \in f^{-1}(y)$ and $f|W_x : W_x \to V$ are homeomorphic mappings of W_x onto V for all x.

The set $\mathcal{K}$ of all covering mappings is a disjoint union of the sets (possibly void)

$$\mathcal{K}_n = \{f \in \mathfrak{L}_\sigma : \nu(f^{-1}(y)) = n, \quad \text{for each } y \in P\} \qquad (n \in N \cup \{\infty\})$$

$\mathcal{K}_n$ is the set of k-tuple covering mappings.

It is easy to verify that if $f \in \bigcup_{i \in N} \mathcal{K}_i$, then f is a proper mapping, i.e. $f^{-1}(F)$ is compact for every compact subset F of P.

Remark 1. The set of all proper mappings is open in $\mathcal{C}$ [2]. Observe that, in general, the sets $\mathfrak{L}_\sigma, \mathcal{K}_1, \mathcal{K}_2, \ldots$ are not open in $\mathcal{C}$. The purpose of this paper is to prove the following

Theorem. *The sets* $\mathfrak{L}_\sigma, \mathcal{K}_1, \mathcal{K}_2, \ldots$ *are open in* $\mathfrak{L}$.

Remark 2. I do not know if $\mathcal{K}_\infty$ is open in $\mathcal{L}$ (obviously it is open in $\mathcal{K}$).

If M and P are differentiable manifolds, if $\mathfrak{D}$ is the set of all differentiable mappings from M into P with C^m-Whithey topology ($m \geqslant 1$) [2], and if $\mathcal{E}_\sigma \subset \mathfrak{D}$ is the set of all immersions of M onto P and $\mathcal{E}_k = \mathcal{E}_\sigma \cap \mathcal{K}_k$, then our theorem together with the inverse function theorem implies the following

Corollary. *The sets* $\mathcal{E}_\sigma, \mathcal{E}_1, \mathcal{E}_2, \ldots$ *are open in* $\mathfrak{D}$.

2. The proof of our Theorem will be preceded by several lemmas.

Lemma 1. $\bigcup_{i \in N} \mathcal{K}_i = \{f \in \mathcal{L}_\sigma : f \text{ is a closed mapping}\}$.

Proof. It is obvious that if $f \in \mathcal{L}_\sigma$ is a closed mapping then $\nu(f^{-1}(y)) < \infty$, for all $y \in P$. Hence the following function is well defined

$$\nu : P \ni y \to \nu(f^{-1}(y)) \in N.$$

It is continuous, i.e. for any $y \in P$ there is a neighborhood of y such that $\nu(f^{-1}(z)) = \nu(f^{-1}(y))$ for any $z \in U$. Really, choose pairwise disjoint open neighborhoods $V_1, \ldots, V_k$ of all points $x_1, \ldots, x_k$ of $f^{-1}(y)$ which are mapped homeomorphically onto neighborhoods $U_1, \ldots, U_k$ of y in P.

Since f is closed, we may take $U = U_1 \cap \ldots \cap U_k \setminus f(M \setminus (V_1 \cup \ldots \cup V_k))$. Now, P being a connected space, ν must be constant function, i.e. $f \in \bigcup_{i \in N} \mathcal{K}_i$.

Conversely, if $g \in \bigcup_{i \in N} \mathcal{K}_i$ then g is proper and, therefore, closed.

Corollary. *If M is compact then* $\mathcal{L}_\sigma = \bigcup_{i \in N} \mathcal{K}_i$.

Let $B_a = \{x \in R^p : ||x|| = \sqrt{x_i^2} \leqslant a\}$, $B_1 = B$ and let S be the unit sphere in R^p.

Lemma 2. *If $f: B \to R^p$ is a continuous mapping and $||f(x) - x|| < \varepsilon$ then* $B_{1-\varepsilon} \subset f(B)$.

Proof. Suppose that there exists $x \in B_{1-\varepsilon} \setminus f(B)$ and let $\lambda : R^p \setminus \{x\} \to S$ be the radial projection from x onto S. Then $g = \lambda \circ f : B \to S$ is a well defined continuous mapping.

It is clear that the inclusion mapping $j : S \to B$ is homotopic to a constant mapping so $g \circ j : S \to S$ is also homotopic to a constant mapping $c : S \to S$. The map

$$S \times [0,1] \ni (n, t) \to \lambda(tu + (1-t)f(u)) \in S$$

realizes a homotopy between $g \circ j$ and the identity map e_S of S onto itself. This implies that the constant mapping c and the identity mapping e_S are homotopic, but this is impossible by the Brouwer fixed-point theorem.

Lemma 3. *If $f \in \mathcal{L}_\sigma$, then there exists a continuous function $\delta_f : M \to R_+$ such that $V(f, \delta_f)$ is a set of surjective mappings.*

Proof (following Munkres). Let $(U_i)_{i \in N}$ be an open, relatively compact and locally finite coverning of M, such that $f|\overline{U}_i$ are injective and there exists coordinate

systems (V_i, k_i) about $f(\overline{U}_i)$ so that $h_i(f(\overline{U}_i))$ equals the unit p-ball B, for all i. For each $i \in N$ there exist $\gamma_i \in (0,1)$ so that $B_{1+\gamma_i} \subset k_i(V_i)$ and the sets $k_i^{-1}(B_{1-\gamma_i})$ cover P. It is easy to verify that for each γ_i there exists $\varepsilon_i > 0$ having the following property: if $g \in \mathcal{C}$ and if, for all $z \in \overline{U}_i$, $\varrho(f(z), g(z)) < \varepsilon_i$, then $\|k_i(f(z)) - k_i(g(z))\| < \gamma_i$ for all $z \in \overline{U}_i$.

Let $\{\varphi_i\}_{i \in N}$ be a partition of unity dominated by $(U_i)_{i \in N}$, $\tilde{\varepsilon}_i = \{\min \varepsilon_i : U_i \cap U_j \neq \emptyset\}$ and $\delta_f(x) = \sum_{i \in N} \tilde{\varepsilon}_i \varphi_i(x)$. We prove that $V(f, \delta_f)$ is a set of surjective mappings. Really, for $g \in V(f, \delta_f)$ consider the map

$$h_i = k_i \circ g \circ (k_i \circ f | \overline{U}_i)^{-1} : B \to R^p.$$

If $p \in B$ then $p = k_i(f(z))$ for some $z \in \overline{U}_i$ and

$$\|h_i(p) - p\| = \|k_i(g(z)) - k_i(f(z))\| < \gamma_i$$

so by Lemma 2, $B_{1-\gamma_i} \subset h_i(B) = k_i(g(\overline{U}_i))$. This implies that $k_i^{-1}(B_{1-\gamma_i}) \subset g(\overline{U}_i)$, so that

$$\bigcup_{i \in N} k_i(B_{1-\gamma_i}) \subset \bigcup_{i \in N} g(\overline{U}_i) = g(M),$$

i.e. $g(M) = P$.

It is known [1] that each p-dimensional topological manifolds can be imbeded in the Euclidean space R^{2p+1} and is the neighborhood retract in this space. This implies the following simple.

Lemma 4. *For each $f \in \mathcal{C}$, there exists a continuous function $\gamma_f : M \to R_+$ such that if $g \in V(f, \gamma_f)$ then g is homotopic to f.*

Proof. Assume that P is imbeded in R^{2p+1} and $r : W \to P$ is a continuous retraction of some neighborhood W of P onto P.

Let $\gamma_f : M \to R_+$ be a continuous function such that for each $x \in M$ the ball centered at $f(x)$ with radius $\gamma_f(x)$ is contained in W. Evidently such a function exists (partition of unity). Then, if $g \in V(f, \gamma_f)$, the map

$$M \times [0,1] \ni (x, t) \to r(tf(x) + (1-t)g(x)) \in P$$

realizes a homotopy between f and g.

3. Proof of the theorem. It is clear by Lemma 3 that $\mathcal{L}_\sigma$ is open in $\mathcal{L}$.

Let $f \in \mathcal{K}_k$ $(k \in N)$ and let $\eta_f : M \to R_+$ be a continuous function such that $V(f, \eta_f)$ is contained in the set of proper mappings. Let $\beta = \min\{\delta_f, \gamma_f, \eta_f\}$, where δ_f and γ_f are the functions defined in Lemmas 3 and 4, respectively.

We want to prove that $V(f, \beta) \subset \mathcal{K}_k$. If $g \in V(f, \beta)$ then, by Lemmas 1 and 3, $g \in \mathcal{K}_s$ for some $s \in N$. We must prove that $k = s$.

It is well known [4] that the covering mapping $f \in \mathcal{K}_k$ induces the monomorphism of fundamental groups

$$f^* : \pi_1(M) \to \pi_1(P)$$

so that the index $[\pi_1(P) : f^*(\pi_1(M))] = k$. By Lemma 4, f is homotopic to g, so $f^* = g^*$ and $[\pi_1(P) : g^*(\pi_1(M))] = k$.

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