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On Mappings Defined by Differential Equations *

1. Given the multi-valued differential equation

$$\text{(ME)} \qquad \dot{x} \in \Gamma(t, x)$$

where $\Gamma: R^1 \times R^p \to 2^{R^1}$ is upper semi-continuous and convex, we can define $\pi(\xi, \tau; T)$ to be $\{\eta \in R^p : \exists x(\cdot)$ solution of (ME) such that $x(\tau) = \xi,\ x(T) = \eta\}$. This is a mapping from R^p into the set of subsets of R^p; restricting our attention to those values of ξ, τ, T such that $\pi(\xi, \tau, T)$ is not empty, it follows from known theorems that it is a closed multi-valued mapping, i.e. that its graph is a closed subset of $R^p \times R^p$.

It is of some interest to investigate whether this mapping has the property of having a fixed point when mapping a compact convex set of R^p into itself. If this is the case in fact we can have results about the existence of periodic solutions or of critical points. When the function $\Gamma(t, x)$ is single-valued and smooth enough to provide uniqueness of solutions, the Schauder Fixed Point Theorem can be used and the existence of periodic solutions or of critical points for ordinary differential equations proved [3], [4]. On the other hand when Γ is only assumed to be upper semi-continuous (or continuous if single-valued), the mapping π is in general multi-valued; moreover images of points need not be acyclic. It is known that for multivalued mappings with images of points not be acyclic, there need not be fixed points. The point of this note is to show that for the mapping π defined by a multi-valued differential equation (and in particular by an ordinary differential equation) when Γ is assumed to be upper semi-continuous (in particular continuous) indeed there are fixed points.

In the proof we shall use a generalization of the Schauder Fixed Point Theorem (Theorem 1) that is of interest in itself.

2. For X a metric space and $x, y \in X$, $d(x, y)$ is the *distance* between x and y and for any $K \subset X$, $d(x, K) = \inf\{d(x, y) : y \in K\}$. For $A, B \in X$ we set $d^*(A, B) = \sup\{d(a, B) : a \in B\}$. We define also an *open ball* of radius r about x, $B[x, r]$,

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to be $\{y \in X: d(x, y) < r\}$. For a mapping f defined on K and $A \subset K$, we set $f(A) = \{y = f(x): x \in A\}$. 2^X is the set of subsets of X. The *graph* of a mapping $\Gamma: S \to 2^X$, $S \subsetneq X$, is defined to be $\{(\xi, \eta) \in X \times X: \eta \in \Gamma(\xi)\}$. R^p is the Euclidean p-dimensional space.

A mapping $\Gamma: S \to 2^Y$ is called *closed* when its graph is closed. Γ is called *upper semi-continuous* when for each point $x \in S$ and every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\Gamma(B[x, \delta]) \subset B[\Gamma(x), \varepsilon]$. When the range of Γ is contained in a compact set, if Γ is closed it is also upper semi-continuous.

Lemma. *Let $S \subset R^p$ be compact; $\Gamma: S \to 2^S$ be closed. Let $\Gamma_n: S \to R^p$ be such that*

$$d^*(G_n, G) \to 0$$

where G_n and G are the graphs of Γ_n and Γ. Let $(\xi_n, \eta_n) \in G_n$, $(\xi_n, \eta_n) \to (\xi_0, \eta_0)$. Then $(\xi_0, \eta_0) \in G$.

The proof follows easily by contradiction.

3. The following theorem will be used in a finite dimensional space, but we shall prove it in a metric locally convex linear topological space (l.c.l.t.s.). In its proof we shall assume that the metric has been chosen so that balls are convex.

Theorem 1. *Let E be a metric l.c.l.t.s., $K \subset E$ be compact and convex and $f: K \to E$ be continuous. Then there exists at Weast one point $P \in K$ such that*

$$d(f(P), P) = d(f(P), K).$$

Proof. Assume that the Theorem is not true. Then there are not fixed points and therefore for each $P \in K$, $d(f(P), P) > 0$. For each $P \in K$, define

$$F(P) = K \cap B[f(P), d(f(P), P)].$$

For each $P \in K$, $F(P)$ is non empty and convex.

Let $x \in F(K)$ and let $P \in F^{-1}(x)$, i.e. $x \in F(P)$. Let $\varepsilon > 0$ such that $d(f(P), x) = d(f(P), P) - \varepsilon$ and let $\delta > 0$ such that $f(B[P, \delta]) \subset B[f(P), \varepsilon/3]$. Set $\delta_1 = \min\{\delta, \varepsilon/3\}$ and let $P' \in B[P, \delta_1]$. Then

$$\begin{aligned} d(f(P'), x) &\leqslant d(f(P), x) + \frac{\varepsilon}{3} = d(f(P), P) - \frac{2}{3}\varepsilon \\ &\leqslant d(f(P'), P') + d(f(P), fP') + d(P', P) - \frac{2}{3}\varepsilon \\ &\leqslant d(f(P'), P'), \end{aligned}$$

i.e. $P' \in F^{-1}(x)$, i.e. $F^{-1}(x)$ is open.

By Theorem 1 of [1] F has a fixed point P^0 in K, but this is impossible since, for all $P \in K$,

$$P \notin B[f(P), d(f(P)), d(f(P)), P].$$

Theorem 2. *Let S be a convex compact subset of R^p. Let $\Gamma: R^1 \times R^p \to 2^{R^p}$ be upper semi-continuous and such that $\Gamma(t, x)$ is compact and convex for all $(t, x) \in R^1 \times R^p$. Assume that all solutions $x(\cdot)$ of (ME) for all initial conditions $x(0) = \xi \in S$*

exist on $[0, T]$ *and that they are contained in a bounded open set* $K \subset R^1 \times R^p$. *Let* $\pi(\xi) = \pi(\xi, 0; T)$ *be such that* $\pi(S) \subset S$. *Then there exists a* $\xi_0 \in S$ *such that* $\xi_0 \in \pi(\xi_0)$.

Proof. Following [2], let $g_n(t, x)$ be a sequence of Lipschitz-continuous functions converging to $\Gamma(t, x)$ in the sense that $d^*(G_n, G) \to 0$, where G_n and G are the graphs of $g_n : \bar{K} \to R^p$ and G is the graph of the restriction of Γ to $\bar{K}$. Consider the differential equations

$$(\mathrm{E}_n) \qquad x = g_n(t, x).$$

For n sufficiently large solutions of (E_n) with initial conditions $\xi \in S$ exist on $[0, T]$. Set $\gamma_n(\xi) = x(T)$, $x_n(\cdot)$ solution of (E_n) such that $x_n(0) = \xi$. Let F_n be the graphs of the mappings $\gamma_n : S \to R^p$, and F be the graph of $\pi : S \to 2^S$. Then by a contradiction argument, using a convergence theorem proved in [2], it follows that $d^*(F_n, F) \to 0$.

Set $\varepsilon_n = d^*(F_n, F)$. The single-valued function γ_n maps S into $B[S, \varepsilon_n]$ and therefore need not have a fixed point. Nevertheless by Theorem 1 for each n there exists $\xi_n \in S$ such that $d(\gamma_n(\xi_n), \xi_n) = d(\gamma(\xi_n), S) < \varepsilon_n$. Taking a subsequence, if necessary, we can assume that $\xi_n \to \xi_0 \in S$. It follows then that

$$d(\gamma_n(\xi_n), \xi_0) \leqslant d(\gamma_n(\xi_n), \xi_n) + d(\xi_n, \xi_0)$$

and since the right-hand side converges to zero, $\gamma_n(\xi_n) \to \xi_0$. By Lemma 1 $(\xi_0, \xi_0) \in F$, i.e. $\xi_0 \in \pi(\xi_0)$.

Remark. The condition $\pi(S) \subset S$ seems somewhat too strong for many applications. A more natural one would be $\pi(\xi) \cap S \neq \emptyset$, for all $\xi \in S$. The author does not know whether Theorem 2 would be true in this case.

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