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A Metric Connection on the Null Hypersurface and Discussion of Treder's Parallel Transport

I. INTRODUCTION

Let $L(M_n)$ be the principal fibre bundle of linear frames over a connected, n-dimensional metric manifold [1]) M_n of the class C^∞. Components $\Gamma^\alpha_{\beta\gamma}$ of the linear metric connection in the bundle $L(M_n)$ with respect to a local coordinate system on the M_n satisfy locally the equations [14], [16]

$$\nabla_\alpha g_{\beta\gamma} \equiv \partial_\alpha g_{\beta\gamma} - \Gamma^\delta_{\alpha\beta} g_{\delta\gamma} - \Gamma^\delta_{\alpha\gamma} g_{\beta\delta} = 0 \tag{1}$$
$$\alpha, \beta, \gamma, \delta, \varepsilon = 1, \ldots, n,$$

where $g_{\alpha\beta}$ is the metric tensor of the M_n.

Equations (1) permit the components $\Gamma^\alpha_{\beta\gamma}$ to be calculated explicitly for a given symmetric tensor $g_{\alpha\beta}$ and, therefore, a metric connection Γ in the bundle $L(M_n)$ [14]. We can transform them [24] to the more convenient form

$$\Gamma^\alpha_{\beta\gamma} g_{\delta\alpha} = [\beta\gamma, \delta] - g_{\gamma\alpha} S^\alpha_{\delta\beta} - g_{\beta\alpha} S^\alpha_{\delta\gamma} + g_{\alpha\delta} S^\alpha_{\beta\gamma}, \tag{2}$$

where $S^\alpha_{\beta\gamma}$ is the torsion tensor.

If the metric of M_n is non-singular, i.e. if M_n is V_n then there exists a tensor $g^{\alpha\beta}$ such that

$$g^{\alpha\beta} g_{\beta\delta} = \delta^\alpha_\delta. \tag{3}$$

The contraction of $g^{\delta\varepsilon}$ with the equations (2) leads immediately to the solution

$$\Gamma^\varepsilon_{\beta\gamma} = \left\{{\varepsilon \atop \beta\gamma}\right\} + S^\varepsilon_{\beta\gamma} - g^{\delta\varepsilon}(g_{\gamma\alpha} S^\alpha_{\delta\beta} + g_{\beta\alpha} S^\alpha_{\delta\gamma}) \tag{4}$$

[1]) We restrict ourselves to connected manifolds and fields both of class $k = \infty$ (exception: a null hypersurface immersed in space-time and fields on it). However, the results for $n = 3$ are true for the values of k demanded in general relativity. This is immediately evident from Section II and from Section V.

We see from (4) that symmetric, non-degenerate metric and torsion determine a unique metric connection in the bundle $L(V_n)$. We conclude also that a symmetric and non-singular metric $g_{\alpha\beta}$ determines uniquely a metric and symmetric linear connection Γ in this bundle. The components of this connection with respect to a local coordinate system in the V_n are Christoffel symbols of the second kind. In this paper we shall express a linear connection and related concepts mostly in terms of the local coordinate system (exception: Section III).

n-dimensional metric manifold with a metric tensor $g_{\alpha\beta}$ such that $\det\|g_{\alpha\beta}\| = 0$ is called a degenerate or singular Riemannian manifold. The number r such that $(n-r)$ is the constant rank of the matrix $\|g_{\alpha\beta}\|$ determines the degree of degeneracy. In r-time degenerate, n-dimensional Riemannian manifold, which will be denoted by $V_n^{(r)}$, there exist r linearly independent vector fields $\underset{A}{k}^{\alpha}$, $A = (n-r+1), \ldots, n$ such that at every point of the manifold

$$g_{\alpha\beta}\underset{A}{k}^{\alpha} = 0\,. \tag{5}$$

We shall call the vectors $\underset{A}{k}^{\alpha}(p)$, at the point p, the null vectors of the metric $g_{\alpha\beta}$ at the point p. In the manifold $V_n^{(r)}$ the tensor $g^{\alpha\beta}$, which satisfies the equations (3), does not exist.

If there exists in $V_n^{(r)}$ $(1 \leqslant r \leqslant n-1)$ a coordinate system in which the tensor $g_{\alpha\beta}$ has the canonical form

$$\|g_{\alpha\beta}\| = \begin{Vmatrix} g_{\lambda\mu} & 0 \\ 0 & 0 \end{Vmatrix}, \quad \det\|g_{\lambda\mu}\| \neq 0; \tag{6}$$

$$\lambda, \mu, \nu, \pi, \varrho, \sigma, \tau = 1, \ldots, (n-r)\,,$$

then $V_n^{(r)}$ is called reducible. If there is a coordinate system in which the metric has the canonical form and $\partial_A g_{\lambda\mu} = 0$, then $V_n^{(r)}$ is called r-time and absolutely reducible [28]. The coordinate system in which the metric has the form (6) is called the canonical coordinate system.

A particular example of $V_3^{(1)}$ is the null hypersurface $\overset{*}{V}_3$ in space-time of the general theory of relativity. The interior metric of such a surface is degenerate and we cannot solve the equations (1) in the usual way. The metric degeneracy of the null hypersurface is inseparably joined with a lack of the normal rigging of such a hypersurface. Normal vectors to the null hypersurface $\overset{*}{V}_3$ are simultaneously tangent to it. These vectors determine the congruence of null geodesics in space-time filling completely $\overset{*}{V}_3$. The geodesics of this congruence are bicharacteristics of Einstein's field equations, while the null hypersurface $\overset{*}{V}_3$ is the characteristic manifold of these equations [18], [19], [23].

Many authors have examined the internal geometry of this hypersurface, e.g. [3], [4], [5], especially the problem of the internal linear connection determined by its immersion in space-time and the equivalent problem [1] of the internal parallel transport on it. Lemmer [15] used the method of projection of the affinity from space-time by means of rigging of the hypersurface.

The affine connection obtained in this way is not metric and is not uniquely determined because there is no natural rigging [8]. The procedure of Daŭtcourt [5] is similar to ours, however the class of connections which he obtains is metric only on an absolutely reducible hypersurface $\overset{*}{V}_3$. The same class of connections on the null hypersurface $\overset{*}{V}_3$ was examined by Kemmerer [12]. Treder [26] proposes parallel transport on the null hypersurface which uses only Christoffel's symbols of the first kind which are uniquely determined.

We shall show in Section V that Treder's proposition is a usual parallel transport only on the absolutely reducible $\overset{*}{V}_3$ and that, in this case, it corresponds to infinitely many metric and symmetric linear connections.

The problem of the metric connection in the general case of $V_n^{(r)}$ has been examined by E. Bortolotti [2] (absolutely reducible $V_n^{(r)}$) and by O. Vogel [28]. See also the papers of A. Jakubowicz [10] and C. Jankiewicz [11].

II. METRIC CONNECTION ON A NULL HYPERSURFACE $\overset{*}{V}_n$

We shall consider the case of an n-dimensional null hypersurface $\overset{*}{V}_n$, immersed in a Riemannian manifold V_{n+1} with a sygnature $(++, \ldots, +-)$. Such a hypersurface is a particular example of $V_n^{(1)}$. O. Vogel has proved two important theorems concerning linear metric connection in $V_n^{(r)}$.

Theorem 1 [28]. $V_n^{(r)}(1 \leqslant r \leqslant n-1)$ admits of metric and symmetric linear connection if and only if it is r-time and absolutely reducible.

Theorem 2 [28]. Equations (1) always admit of solutions $\Gamma^{\alpha}_{\beta\gamma} \neq \Gamma^{\alpha}_{\gamma\beta}$, i.e. the non-symmetric metric connection on the $V_n^{(r)}(1 \leqslant r \leqslant n-1)$.

The analysis of equations (1) and their solutions can, therefore, be divided in a natural manner into two cases:

a) an absolutely reducible $\overset{*}{V}_n$,

b) a general $\overset{*}{V}_n$.

a) Absolutely Reducible $\overset{*}{V}_n$.

We shall prove Theorem 1. The proof is in fact the solution of the problem for this case. We shall show first that $\overset{*}{V}_n$, which admits of a symmetric and metric linear connection, is absolutely reducible.

Every $\overset{*}{V}_n$ is reducible, i.e. admits of the canonical coordinate system in which $g_{\alpha n} = g_{n\alpha} = 0$ and $k^{\alpha} = \delta^{\alpha}_n$, where k^{α} are the components of the null vector, satisfying equations

$$g_{\alpha\beta} k^{\alpha} = 0 . \tag{7}$$

Let us consider in this coordinate system the equations

$$\nabla_\alpha k^\beta = \partial_\alpha k^\beta + \Gamma^\beta_{\alpha\gamma} k^\gamma = l_\alpha k^\beta , \tag{8}$$

which follow from (7) and from the metric character of the connection [17].

In the canonical system

$$\nabla_\alpha k^\beta = l_\alpha \delta^\beta_n \tag{9}$$

and therefore

$$\Gamma^\lambda_{\alpha n} = \Gamma^\lambda_{n\alpha} = 0 . \tag{9'}$$

Using (9′) we have

$$\nabla_n g_{\lambda\mu} = \partial_n g_{\lambda\mu} - \Gamma^\nu_{n\lambda} g_{\nu\mu} - \Gamma^\nu_{n\mu} g_{\lambda\nu} = \partial_n g_{\lambda\mu} = 0 \tag{10}$$

which means that the $\overset{*}{V}_n$ is absolutely reducible.

We shall now prove Theorem 1 in the opposite direction. Solutions of the equations (1) can be obtained most easily in the canonical coordinate system. We have

$$\nabla_\lambda g_{\mu\nu} \equiv \partial_\lambda g_{\mu\nu} - \Gamma^\varrho_{\lambda\mu} g_{\varrho\nu} - \Gamma^\varrho_{\lambda\nu} g_{\mu\varrho} = 0 , \tag{11.1}$$

$$\nabla_n g_{\mu\nu} \equiv \partial_n g_{\mu\nu} - \Gamma^\varrho_{n\mu} g_{\varrho\nu} - \Gamma^\varrho_{n\nu} g_{\mu\varrho} = 0 , \tag{11.2}$$

$$\nabla_\alpha g_{\mu n} \equiv \partial_\alpha g_{\mu n} - \Gamma^\gamma_{\alpha\mu} g_{\gamma n} - \Gamma^\gamma_{\alpha n} g_{\mu\gamma} = 0 , \tag{11.3}$$

$$\nabla_\alpha g_{nn} \equiv \partial_\alpha g_{nn} - \Gamma^\beta_{\alpha n} g_{\beta n} - \Gamma^\beta_{\alpha n} g_{n\beta} = 0 . \tag{11.4}$$

Eq. (11.4) is fulfilled identically, because in the canonical coordinate system $g_{n\alpha} = 0$.

From (11.3), $g_{n\alpha} = 0$ and $\det\|g_{\mu\nu}\| \neq 0$ it follows that $\Gamma^\varrho_{\alpha n} = \Gamma^\varrho_{n\alpha} = 0$. Eq. (11.2) is fulfilled identically because $\Gamma^\varrho_{n\alpha} = 0$ and $\partial_n g_{\mu\nu} = 0$. Since $\det\|g_{\mu\nu}\| \neq 0$, one can solve the equations (11.1) in the usual way [24] and get $\Gamma^\lambda_{\mu\nu} = \{{}^{\lambda}_{\mu\nu}\}$.

It is seen that in the canonical coordinate system, the metric determines uniquely only the components $\Gamma^\lambda_{\beta\gamma} = \Gamma^\lambda_{\gamma\beta}$; the remaining $\frac{n(n+1)}{2}$ components $\Gamma^n_{\alpha\beta} = \Gamma^n_{\beta\alpha}$ may be chosen arbitrarily.

On applying this theorem to the case of $\overset{*}{V}_3$ we see that the metric of the absolutely reducible $\overset{*}{V}_3$ admits of a whole class of symmetric and metric linear connections. This class depends on six arbitrary functions, namely $\Gamma^3_{\alpha\beta}$ in the canonical coordinate system.

The null curves (the generators of $\overset{*}{V}_3$), tangent at every point to the null vector, are geodesics for every connection of this class.

We shall denote the class of metric and symmetric linear connections on the absolutely reducible null hypersurface $\overset{*}{V}_n$ by $[\Gamma^\alpha_{\beta\gamma}]_s$.

Having the components of the linear connection in the canonical coordinate system we define them in every other coordinate system by means of the transformation rule

$$\Gamma^{\alpha'}_{\beta'\gamma'} = \frac{\partial x^{\alpha'}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial x^{\beta'}} \frac{\partial x^{\gamma}}{\partial x^{\gamma'}} \Gamma^{\alpha}_{\beta\gamma} + \frac{\partial x^{\alpha'}}{\partial x^{\alpha}} \frac{\partial^2 x^{\alpha}}{\partial x^{\beta'} \partial x^{\gamma'}} . \tag{12}$$

b) The General $\overset{*}{V}_n$.

The proof of Theorem 2 is trivial if the theorem of Wong [30] is used. This theorem, in application to the metric tensor $g_{\alpha\beta}$, says that the metric connection, i.e. the solution of the system (1) exists if and only if in every point of $V_n^{(r)}$ there exists the basis [2]) $\underset{(\beta)}{e^{\alpha}}$ in which the components of the tensor $g_{\alpha\beta}$ are constant and not all equal to zero. The tensor $g_{\alpha\beta}$ admitting of such a field $\underset{(\beta)}{e^{\alpha}}$ is called homogeneous and the field $\underset{(\beta)}{e^{\alpha}}$ the regular field [30]. It is easy to see that on the $\overset{*}{V}_n$ there are infinitely many regular fields (see Section III). Therefore, the system (1) is always compatible and, by virtue of Theorem 1, the solutions are generally non-symmetric.

As in the case of absolute reducibility, one can solve the equations (1) most easily in the canonical coordinate system in which they assume the form (11). Eq. (11.4) are fulfilled identically. It results from (11.3) that $\Gamma^{\lambda}_{an} = 0$. Eq. (11.1) can be solved in the usual way [24]: $\Gamma^{\lambda}_{\mu\nu} = \{{}^{\lambda}_{\mu\nu}\}$. It is more difficult to solve the system (11.2). It is the system of $\frac{n(n-1)}{2}$ equations (linear in every point) on $(n-1)^2$ unknown components $\Gamma^{\lambda}_{n\mu}$.

In the case of a null hypersurface $\overset{*}{V}_3$ in space-time, which is particularly interesting for us, this system gives three equations for four unknowns Γ^1_{31}, Γ^1_{32}, Γ^2_{31}, Γ^2_{32}. Summarizing, if the components $\Gamma^3_{\alpha\beta}$ and Γ^2_{32} are given in the canonical coordinate system, then, in the general case of $\overset{*}{V}_3$, the metric tensor determines the remaining components of linear connection. We put here, for simplicity, $\Gamma^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\nu\mu}$ (in the canonical coordinate system). In general, the number of arbitrary components increases to twelve.

The number f of arbitrary components for $\overset{*}{V}_n$ equals

$$f = \frac{n^2(n-1)}{2} + n . \tag{13}$$

The special class of the non-symmetric, linear and metric connections on the $\overset{*}{V}_n$ for which, in the canonical coordinate system, $\Gamma^{\lambda}_{\mu\nu} = \{{}^{\lambda}_{\mu\nu}\}$ we shall denote by $[\Gamma^{\alpha}_{\beta\gamma}]_{ns}$.

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[2] The indices in round brackets always refer to a local frame.

From Section IV we restrict ourselves to this special class with $\Gamma^{\lambda}_{n\mu}$ fixed in the canonical coordinate system by the condition

$$\Gamma^{\lambda}_{n\mu} = [n\mu, \varrho] g^{\varrho\lambda}. \tag{14}$$

$g^{\varrho\lambda}$ is here the matrix inverse to $g_{\lambda\mu}$, which is the non-vanishing part of $g_{\alpha\beta}$ in the canonical coordinate system.

III. THE STRUCTURE GROUP OF A LINEAR METRIC CONNECTION ON THE NULL HYPERSURFACE $\overset{*}{V}_n$

It is known [1], [13], [14], [16], [27] that the structure group of the linear metric connection on the Riemannian manifold V_n is $0(n)$ — a semi-simple and compact Lie group (non-compact Lorentz group $L(n)$ for pseudo-Riemannian V_n) acting on the principal fibre bundle of orthonormal frames $0(V_n)$ and preserving the canonic form of a local metric $g_{(\alpha)(\beta)}(p)$. On the null hypersurface $\overset{*}{V}_n$ orthonormal frames sensu stricto do not exist and, consequently, the group $0(n)$ does not appear here as a structure group of the linear metric connection. However, the degenerate metric structure of $\overset{*}{V}_n$ distinguishes at each of its points p a set of frames $\{\underset{(\beta)}{e^{\alpha}}(p)\}$ which we shall call quasiorthonormal frames.

The quasiorthonormal frame $\underset{(\beta)}{e^{\alpha}}(p)$ consists of vectors: $\underset{(\mu)}{e^{\alpha}}(p)-(n-1)$ linearly independent space-like vectors at point p, i.e. vectors for which $g_{\alpha\beta}(p)\underset{(\mu)}{e^{\alpha}}(p)\underset{(\mu)}{e^{\beta}}(p) > 0$ and $\underset{(n)}{e^{\alpha}}(p) = k^{\alpha}(p)$ — the null vector of the metric $g_{\alpha\beta}(p)$, satisfying the orthogonality relations

$$\begin{aligned} g_{\alpha\beta}(p)\underset{(\mu)}{e^{\alpha}}(p)\underset{(\nu)}{e^{\beta}}(p) &= \delta_{(\mu)(\nu)} = \underset{(\mu)}{e^{\alpha}}(p)\underset{(\nu)}{e_{\alpha}}(p) = \underset{(\mu)}{(e)^{\alpha}}(p)\overset{(\nu)}{e_{\alpha}}(p) \\ &= \delta^{(\nu)}_{(\mu)};\ \overset{(\nu)}{e_{\alpha}}(p) \overset{\mathrm{df}}{=} \underset{(\nu)}{e_{\alpha}}(p) = g_{\alpha\beta}(p)\underset{(\nu)}{e^{\beta}}(p)\,, \\ g_{\alpha\beta}(p)\underset{(\mu)}{e^{\alpha}}(p)\underset{(n)}{e^{\beta}}(p) &= 0\,, \qquad g_{\alpha\beta}(p)\underset{(n)}{e^{\alpha}}(p)\underset{(n)}{e^{\beta}}(p) = 0\,. \end{aligned} \tag{15}$$

Every frame $\underset{(\beta)}{e^{\alpha}}(p)$ determines uniquely the co-frame $\overset{(\beta)}{e_{\alpha}}(p)$:

$$\overset{(\beta)}{e_{\alpha}}(p) = (\overset{(\nu)}{e_{\alpha}}(p), \overset{(n)}{\gamma_{\alpha}}(p))\,,$$

where

$$\overset{(n)}{\gamma_{\beta}} = \frac{\varepsilon_{\beta\alpha_1, \ldots, \alpha_{(n-1)}}\underset{(1)}{e^{\alpha_1}}, \ldots, \underset{(n-1)}{e^{\alpha_{(n-1)}}}}{\varepsilon_{\gamma_1, \ldots, \gamma_n}\underset{(n)}{k^{\gamma_1}}\underset{(1)}{e^{\gamma_2}}, \ldots, \underset{(n-1)}{e^{\gamma_n}}}\,, \tag{16}$$

which satisfies the affine (not connected with the metric) normalization conditions

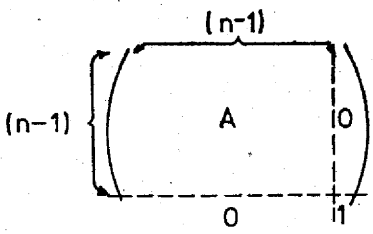
$$e^{\alpha}_{(n)}(p)\overset{(n)}{\gamma}_{\alpha}(p) = 1\,, \qquad e^{\alpha}_{(\mu)}(p)\overset{(n)}{\gamma}_{\alpha}(p) = 0\,. \tag{17}$$

$\varepsilon_{\alpha_1, \ldots, \alpha_n}$ denotes here the completely skew-symmetric permutation symbol, being the relative tensor of weight (-1) [8], [25]. The co-frame $\overset{(\gamma)}{e}_{\beta}(p)$ is the dual basis of the frame $e^{\alpha}_{(\beta)}(p)$. In quasiorthonormal frames the metric $g_{\alpha\beta}(p)$ possesses a constant, canonical form $g_{(\alpha)(\beta)}(p)$:

$$g_{(\alpha)(\beta)}(p) \overset{\mathrm{df}}{=} g_{\alpha\beta}(p)\, e^{\alpha}_{(\alpha)}(p)\, e^{\beta}_{(\beta)}(p) = \begin{pmatrix} \delta_{(\mu)(\nu)} & 0 \\ 0 & 0 \end{pmatrix}. \tag{18}$$

It can be seen that conditions (15) determine the frame $e^{\alpha}_{(\beta)}(p)$ with accuracy to the following transformation groups of Lie:

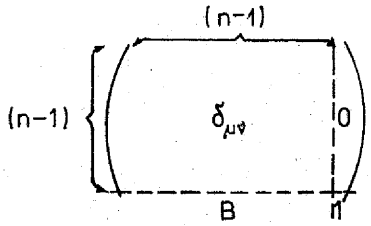
1. Rotations in such $(n-1)$ dimensional subspace of the tangent space $T_p(\overset{*}{V}_n)$, which is spanned by $e^{\alpha}_{(\mu)}(p) - a\dfrac{(n-1)(n-2)}{2}$ parameter subgroup G_1 of the linear group $Gl(n;\, R)$ consisting of all matrices of the shape

(19)


$$A\cdot A^T = A^T\cdot A = \delta_{\mu\nu}, \quad a_{n\mu} = a_{\mu n} = 0; \quad a_{nn} = 1.$$

This group is isomorphic to $0(n-1)$.

2. "Rotations" of such $(n-1)$-dimensional hyperplane of frame $e^{\alpha}_{(\beta)}(p)$ which is spanned by $e^{\alpha}_{(\mu)}(p)$ around the direction of the null vector $k^{\alpha}(p)$ preserving its space-like character — $a(n-1)$ parameter subgroup G_2 of the group $Gl(n, R)$. The group G_2 consists of all matrices of the shape

(20)


$$B = (a_{n1}, \ldots, a_{n(n-1)}) \quad a_{\mu n} = 0, \quad a_{nn} = 1$$

3. The scale transformation $\underset{(n)'}{k^a} = \lambda \underset{(n)}{k^a}$, $\lambda \neq 0$ and real — a one-parameter subgroup G_3 of the group $Gl(n;\ R)$. The transformation matrices of this subgroup have the form:

(21)
$$\begin{array}{c} \\ (n-1) \\ \\ \end{array}\overbrace{\left(\begin{array}{c|c} \delta_{\mu\nu} & 0 \\ \hline 0 & \lambda \end{array}\right)}^{(n-1)} \qquad \begin{array}{l} a_{n\mu} = a_{\mu n} = 0, \\ a_{nn} = \lambda \end{array}$$

The group G_3 is isomorphic to $Gl(1;\ R)$.

The generators of these transformations form a basis for the algebra of the $\frac{n(n-1)}{2} + l$ parameter Lie group, which will be denoted by G.

This group, containing G_1, G_2 and G_3 as its subgroups, is the group of all non-singular matrices possessing the form:

(22)
$$\begin{array}{c} \\ (n-1) \\ \\ \end{array}\overbrace{\left(\begin{array}{c|c} A & 0 \\ \hline B & \lambda \end{array}\right)}^{(n-1)} \qquad \begin{array}{l} A \cdot A^T = A^T \cdot A = \delta_{\mu\nu}, \\ B = (a_{n1}, \ldots, a_{n(n-1)}), \\ a_{nn} = \lambda \end{array}$$

It is a non-compact and not semi-simple subgroup of the linear group $Gl(n;\ R)$. The non-compactness of the group G is obvious. In order to show that it is not semi-simple it is most convenient to use the commutation relations for generators of the linear group $Gl(n;\ R)$ — the structure group of the principal fibre bundle of linear frames $L(\overset{*}{V}_n)$ [9], [29]:

(23)
$$[G_{\alpha\beta}, G_{\gamma\delta}] = \delta_{\gamma\beta} G_{\alpha\delta} - \delta_{\alpha\delta} G_{\gamma\beta}\,.$$

The generators of the group G arise from the generators of $Gl(n;\ R)$ which generate the subgroups G_1, G_2 and G_3, i.e. from $G_{n\mu}$, G_{nn} and $G_{\mu\nu}$. $G_{n\mu}$ are the generators of G_2, G_{nn} are the generators of the group G_3 and from the generators $G_{\mu\nu}$ we construct [29] the generators $C_{[\mu\nu]} \overset{\mathrm{df}}{=} G_{\nu\mu} - G_{\mu\nu}$ of the group G_1.

From (23) we have the following commutation relations for these generators:

(24)
$$\begin{gathered} [G_{n\mu}, G_{n\nu}] = 0,\ [G_{n\mu}, G_{nn}] = -G_{n\mu},\ [G_{nn}, G_{nn}] = 0 \\ [G_{n\mu}, G_{[\nu\varrho]}] = \delta_{\varrho\mu} G_{n\nu} - \delta_{\nu\mu} G_{n\varrho},\ [G_{nn}, G_{[\mu\nu]}] = 0 \\ [G_{[\mu\nu]}, G_{[\varrho\sigma]}] = \delta_{\mu\varrho} G_{[\nu\sigma]} + \delta_{\nu\sigma} G_{[\mu\varrho]} - \delta_{\nu\varrho} G_{[\mu\sigma]} - \delta_{\mu\sigma} G_{[\nu\varrho]}\,. \end{gathered}$$

It can be seen from (24) that the algebra of the group G_2 is an abelian invariant ideal of the algebra $\mathfrak{g}$ of the group G. Consequently, the group G_2 is a $(n-1)$-parameter abelian invariant subgroup of the group [3]) G [20]. Therefore, the group G is not semi-simple.

In the case of $\overset{*}{V}_3$ — null hypersurface in space-time the group G is a four-parameter group and acts on the quasiorthonormal frame $\underset{(b)}{e^a}(p)$, $(a, (b) = 1, 2, 3)$ as follows

$$(25) \qquad \underset{(1)'}{(t^a}, \underset{(2)'}{\overline{t}^a}, \underset{(3)'}{k^a}) = \underset{(1)}{(t^a}, \underset{(2)}{\overline{t}^a}, \underset{(3)}{k^a}) \begin{pmatrix} e^{-i\mu} & 0 & 0 \\ 0 & e^{i\mu} & 0 \\ (-)\bar{k}\lambda e^{-i\mu}, & (-)k\lambda e^{i\mu}, & \lambda \end{pmatrix}$$

where

$$t^a \overset{\mathrm{df}}{=} \frac{1}{\sqrt{2}} (\underset{(1)}{e^a} + i \underset{(2)}{e^a}) .$$

k^a, t^a, $\overline{t}^a$ form a null triad. The parameter k is complex, λ, μ are real parameters.

For $k = \lambda - 1 = 0$ we get the transformations of the group G_1, for $\mu = \lambda - 1 = 0$ — the transformations of the group G_2 and, finally, for $\mu = k = 0$ the transformations of the group G_3.

It is easy to see that the group G_2 is an invariant abelian subgroup of the group G, i.e. that the group G is not semi-simple.

R. Penrose [15] discovered a similar property for the subgroup of the Lorentz group which leaves one null direction (the generator direction) unchanged. This subgroup is isomorphic to G. The group G is the structure group of the principal fibre bundle of quasiorthonormal frames $Q(\overset{*}{V}_n)$.

Theorem. The group G is the structure group of the metric connection on V_n.

Proof. It is sufficient to show that the connection Γ in the bundle of linear frames $L(\overset{*}{V}_n)$ is metric if and only if it is determined by some connection $\overline{\Gamma}$ in the bundle $Q(\overset{*}{V}_n)$. Let $\overline{\Gamma}$ be an arbitrary connection in the bundle $Q(\overset{*}{V}_n)$. At a parallel transport in $Q(\overset{*}{V}_n)$ with respect to this, the quasiorthonormal frame $\underset{(\beta)}{e^a}(p)$ remains quasiorthonormal. The same is valid for a parallel transport in the bundle $L(\overset{*}{V}_n)$ with respect to the connection Γ determined by $\overline{\Gamma}$. The metric $g_{\alpha\beta}(x)$ of $\overset{*}{V}_n$ determines the fibre metric, i.e. the scalar product in the tangent bundle $T(\overset{*}{V}_n)$, associated with $L(\overset{*}{V}_n)$ [14]. According to definition [14], a connection in $L(\overset{*}{V}_n)$ is metric if the parallel transport (with respect to it) of the fibres of $T(\overset{*}{V}_n)$ preserves the fibre metric.

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[3]) Using the matrix form of the elements of G and of G_2, one can also prove this statement by direct calculation.

In the tangent bundle $T(\overset{*}{V}_n)$ one can define the parallel transport of the vector $\vec{v} \in T_p(\overset{*}{V}_n)$ along the curve $x^a(t)$ of $\overset{*}{V}_n$ and with respect to the connection Γ in $L(\overset{*}{V}_n)$ [1] in the following manner:

1. We transport parallel along $x^a(t)$ an arbitrary linear frame $\underset{(\beta)}{e^a}(p)$.
2. If in the point $p, \vec{v} = v^{(\alpha)}(p)\underset{(\alpha)}{e^a}(p)n_a(p)$ then $\vec{v}(t) = v^{(\alpha)}(p)\underset{(\alpha)}{e^a}(t)n_a(t)$ is the transported parallel vector $\vec{v}(p)$. n_a denotes here a natural frame.

Let Γ be a linear connection determined by an arbitrary connection $\overline{\Gamma}$ in $Q(\overset{*}{V}_n)$. Let us refer the transported parallel vectors $\vec{u}, \vec{v} \in T_p(\overset{*}{V}_n)$ with respect to connection Γ to a transported parallel quasiorthonormal frame $\underset{(\beta)}{e^a}(p)$. Then, at a parallel transport of these vectors along the curve $x^a(t)$: $x^a(t_0) = p^a$ of manifold $\overset{*}{V}_n$, we have

$$(26) \qquad g_{(\alpha)(\beta)}(t)u^{(\alpha)}(t)v^{(\beta)}(t) = g_{(\alpha)(\beta)}(t_0)u^{(\alpha)}(t_0)v^{(\beta)}(t_0) = \text{const}$$

$$= g_{(\alpha)(\beta)}(t)\underset{(\alpha)}{e_a}(t)u^a(t)\underset{(\beta)}{e_\beta}(t)v^\beta(t) = g_{a\beta}(t)u^a(t)v^\beta(t) = \text{const}.$$

From the facts:

1. $Gl(n; R)$ acts transitively on every fibre of the bundle $L(\overset{*}{V}_n)$.
2. the parallel transport in $L(\overset{*}{V}_n)$ commutes with the right action of $Gl(n; R)$ on it, it results that if we refer the parallel transported vectors $\vec{u}, \vec{v} \in T_p(\overset{*}{V}_n)$ to an arbitrary, parallel transported linear frame $\underset{(\beta)}{e^a}(p)$, then we also get $g_{a\beta}(t)u^a(t)v^\beta(t) = \text{const}$ at this parallel transport.

This means that every linear connection Γ determined by some connection $\overline{\Gamma}$ in $Q(\overset{*}{V}_n)$ is metric.

On the other hand, if we refer the parallel transported vectors $\vec{u}, \vec{v} \in T_p(\overset{*}{V}_n)$ to the parallel transported quasiorthonormal frame $\underset{(\beta)}{e^a}(p)$, then we have by their parallel transport with respect to the arbitrary metric connection Γ in the bundle $L(\overset{*}{V}_n)$

$$(27) \qquad g_{(\alpha)(\beta)}(t)u^{(\alpha)}(p)v^{(\beta)}(p) = g_{(\alpha)(\beta)}(p)u^{(\alpha)}(p)v^{(\beta)}(p) = \text{const}.$$

From (27) it results that $g_{(\alpha)(\beta)}(t) = g_{(\alpha)(\beta)}(p)$, i.e. that the quasiorthonormal frame $\underset{(\beta)}{e^a}(p)$ remains quasiorthonormal at this parallel transport. Consequently, this means that Γ is determined by some connection $\overline{\Gamma}$ in $Q(\overset{*}{V}_n)$.

Finally, we come to an interesting conclusion:
the group metric of the structure group G of a linear metric connection on the degenerate manifold $\overset{*}{V}_n$ is degenerate. It can be shown that this conclusion is true for a general $V_n^{(r)}$ [6].

IV. THE PARALLEL TRANSPORT WITHIN A NULL HYPERSURFACE $\overset{*}{V}_n$

It is most natural to define the parallel transport with respect to the whole class of admissible linear and metric connection, i.e. with respect to $[\Gamma^{\alpha}_{\beta\gamma}]_s$ in the case of absolute reducibility and with respect to $[\Gamma^{\alpha}_{\beta\gamma}]_{ns}$ in the general case. For such a parallel transport a class of solutions corresponds to a given initial value. It should be noted, however, that in the case of absolute reducibility and in the canonical coordinate system the metrically substantial components v^{λ} of a vector are transported in a unique fashion and do not depend on the metrically non-substantial component v^n.

For general $\overset{*}{V}_n$ one can get the same if one defines parallel transport of a vector $\vec{v}$ along differentiable curve $x^{\beta}(t)$ by means of the formula

$$\frac{dv^{\alpha}}{dt} = (-)\,[\Gamma^{\alpha}_{\beta\gamma}]_{ns} v^{\gamma} \frac{dx^{\beta}}{dt} \tag{28}$$

and fixes $\Gamma^{\lambda}_{n\mu}$ by the condition (14).

In what follows we are interested mainly in the metric side of the problem. We propose to consider the parallel transport of metrically equivalent classes of vectors.

Let $T_p(\overset{*}{V}_n)$ be the set of all vectors at the point $p \in \overset{*}{V}_n$, i.e. the space tangent to $\overset{*}{V}_n$ in p. We introduce in $T_p(\overset{*}{V}_n) \times T_p(\overset{*}{V}_n)$ an equivalence relation R^e defined as follows:

$$\underset{1}{\vec{v}}\; R^e \underset{2}{\vec{v}} \leftrightarrow g_{\alpha\beta}(p) \underset{1}{v^{\alpha}} \underset{1}{v^{\beta}} = g_{\alpha\beta}(p) \underset{2}{v^{\alpha}} \underset{2}{v^{\beta}},\; g_{\alpha\beta}(p) \underset{1}{v^{\alpha}} = g_{\alpha\beta}(p) \underset{2}{v^{\alpha}} \tag{29}$$

The relation R^e divides $T_p(\overset{*}{V}_n)$ into disjoint classes of vectors which are metrically equivalent. Two vectors $\vec{\mathbf{v}}(p)$ and $\vec{\mathbf{w}}(p)$ belong to the same class of equivalence if and only if

$$\vec{\mathbf{v}}(p)\; R^e\, \vec{\mathbf{w}}(p)\,.$$

We denote by $[\vec{\mathbf{v}}(p)]$ the class of equivalence to which the vector $\vec{\mathbf{v}}(p)$ belongs. This class is an infinite set. It consists of all vectors of the form

$$\vec{\mathbf{v}}(p) + a\vec{\mathbf{k}}(p)\,, \qquad a \in R \tag{30}$$

where $\vec{k}(p)$ is the null vector of the metric $g_{\alpha\beta}(p)$. By $T_p(\overset{*}{V}_n)/R^e$ we shall denote the quotient of $T_p(\overset{*}{V}_n)$ by R^e. Because the relation R^e is compatible with the structure of $T_p(\overset{*}{V}_n)$, the set $T_p(\overset{*}{V}_n)/R^e$ forms the quotient structure. This is also the vector space. We extend the relation R^e on all $T(\overset{*}{V}_n) \times T(\overset{*}{V}_n)$ in a natural manner:

$$(\vec{\mathbf{v}}, p;\; \vec{\mathbf{w}}, q) \in R^e \leftrightarrow p = q, \text{ i.e. } \vec{\mathbf{v}}, \vec{\mathbf{w}} \in T_p(\overset{*}{V}_n) \text{ and } \vec{\mathbf{v}}, \vec{\mathbf{w}} \text{ satisfy (29)}\,. \tag{31}$$

This extension we shall denote by R^{ex}.

It is obvious that the topological sum $\bigcup_{p \in \overset{*}{V}_n} T_p(\overset{*}{V}_n)/R^e$ is the quotient of $T(\overset{*}{V}_n)$ by R^{ex}. We shall denote it by $T(\overset{*}{V}_n)/R^{ex}$.

The relation R^{ex} is compatible with the operation of parallel transfer in the bundle $T(\overset{*}{V}_n)$ with respect to the classes of metric connections described in Section II.

This permits us to define, in a natural manner, the parallel transport in the quotient $T(\overset{*}{V}_n)/R^{ex}$, i.e. the parallel transport of the metrically equivalent classes of vectors along the curves of the manifold $\overset{*}{V}_n$:

$$PT[\vec{v}(p)] = [PT\vec{v}(p)] \,. \tag{32}$$

PT denotes here "parallel transport".

This quotient parallel transport is unique. In the V_n it is the usual parallel transport.

Let us restrict, in the tangent spaces $T_p(\overset{*}{V}_n)$, $p \in \overset{*}{V}_n$, to the special frames with $\vec{e}_{(n)} = \vec{k}$ which are determined by the canonical coordinate systems on the $\overset{*}{V}_n$. The set of all these frames at all points of $\overset{*}{V}_n$ is the principal fibre bundle over $\overset{*}{V}_n$ which we denote $L^s(\overset{*}{V}_n)$. The structure group G' of this bundle is determined by the canonical coordinate transformations on the $\overset{*}{V}_n$, i.e. by coordinate transformations of the shape

$$\begin{aligned} x^{\mu'} &= \varphi^{\mu'}(x^{\mu}) \,, \qquad \mu, \mu' = 1, \ldots, (n-1) \,, \\ x^{n'} &= f(x^{a}) \,, \qquad a = 1, \ldots, n \,. \end{aligned} \tag{33}$$

These coordinate transformations preserve the canonical coordinate system on the $\overset{*}{V}_n$. The tangent bundle with $T_p(\overset{*}{V}_n)$ refering only to the basis mentioned above, we shall denote by $T^s(\overset{*}{V}_n)$. This bundle is associated with the principal bundle $L^s(\overset{*}{V}_n)$.

It is easy to show that the quotient $T^s(\overset{*}{V}_n)/R^{ex}$ is the quotient bundle with the structure group G' acting on it as follows

$$G'[\vec{v}(p)] = [G'\vec{v}(p)] \tag{34}$$

V. DISCUSSION OF TREDER'S PROPOSITION FOR $\overset{*}{V}_n$

In this Section we shall discuss the parallel transport on $\overset{*}{V}_n$ proposed by Treder [26].

According to Treder, the vector $\vec{v}(p)$ is transported parallel along the curve $x^a(t)$ if its components fulfil the equations

$$g_{\alpha\beta} \frac{dv^{\beta}}{dt} = (-) g_{\alpha\beta} \left\{ {\beta \atop \gamma\delta} \right\} v^{\gamma} \frac{dx^{\delta}}{dt} \equiv (-) [\gamma\delta, \alpha] v^{\gamma} \frac{dx^{\delta}}{dt} \,. \tag{35}$$

On contracting these equations with the null vector $\vec{k}$ of the metric $g_{\alpha\beta}(x)$ we get

$$[\gamma\delta, \alpha]k^{\alpha}v^{\gamma}\dot{x}^{\delta} = 0\,, \tag{36}$$

where

$$\dot{x}^{\delta} = \frac{dx^{\delta}}{dt}\,.$$

The equation (36) is the condition of compatibility of the system (35). In the case of absolute reducibility [4])

$$[\gamma\delta, \alpha]k^{\alpha} \overset{df}{=} 0 \tag{37}$$

and, therefore, the equation (36) is satisfied identically. Consequently, the system (35) is compatible and its solution depends on one arbitrary function (as a result, one of the components $\vec{v}$, e.g. v^n can be chosen arbitrarily).

In general, the equation (36) is fulfilled automatically along the null curves (the generators of $\overset{*}{V}_n$) for which

$$[\gamma\delta, \alpha]k^{\alpha}\dot{x}^{\delta} = 0\,. \tag{38}$$

The transport along the generators also depends on one arbitrary function. In these two cases we use effectively only $(n-1)$ equations. This can be seen most easily in the canonical coordinate system in which we shall discuss Treder's transport in the general case. In this coordinate system the equations (35) take on the form

$$\begin{cases} g_{\lambda\mu}\dfrac{dv^{\lambda}}{dt} + [\mu, \nu\varrho]v^{\nu}\dot{x}^{\varrho} + \partial_n g_{\nu\mu}v^{\nu}\dot{x}^{n} + \partial_n g_{\varrho\mu}v^{n}\dot{x}^{\varrho} \overset{*}{=} 0\,, \\ v^{\sigma}\dot{x}^{\tau}\partial_n g_{\sigma\tau} \overset{*}{=} 0\,, \end{cases} \tag{39}$$

and the condition of compatibility the form

$$v^{\sigma}\dot{x}^{\tau}k^{n}\partial_n g_{\sigma\tau} \overset{*}{=} 0\,, \qquad k^{n} \neq 0\,. \tag{40}$$

We see that the compatibility condition is equivalent with the last equation of the system (39). Therefore, the system (39) is not in involution [5]) with the last equation as a constraint equation. Moreover, this system does not give, in general, what is understood by "parallel transport", because there does not exist a linear connection with respect to which this transport is the parallel transport. This is proved by the following facts:

1. One cannot transport the tangent space along a given differentiable curve (exception: the generators).

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[4]) The equations (37) are the criterion of the absolute reducibility of $\overset{*}{V}_n$ because their left side is proportional to the Lie derivative of $g_{\alpha\beta}$ with respect to the null vector $\vec{k}$.

[5]) The notion of „involution" is defined by A. Z. Petrov [18].

2. The system of equations (algebraic in every point)

$$\Gamma^{\alpha}_{\beta\gamma} g_{\alpha\delta} = [\beta\gamma, \delta] \tag{41}$$

which should permit the determination of the components of the linear connection is not compatible.

Only in the absolutely reducible case is this system compatible. From the facts indicated above there follow three possible ways of treatment of the whole Treder system with the condition of compatibility.

a) It is most natural to start from the condition of compatibility which does not depend on a curve and a vector, i.e. from (37). The transport, which we get in this case, is the usual parallel transport corresponding to the class $[\Gamma^{\alpha}_{\beta\gamma}]_s$ of symmetric and metric linear connections. It is easily seen if we solve the equation (41) in the canonical coordinate system.

These equations in this coordinate system take on the form

$$\Gamma^{\lambda}_{\alpha\beta} g_{\lambda\mu} = [\alpha\beta, \mu]\,. \tag{42}$$

On contracting them with $g^{\mu\nu}$ we get

$$\Gamma^{\nu}_{\alpha\beta} = g^{\mu\nu}[\alpha\beta, \mu]\,. \tag{43}$$

It follows immediately from (43) that

$$\Gamma^{\nu}_{\varrho\tau} = \left\{{\nu \atop \varrho\tau}\right\}, \Gamma^{\nu}_{\beta n} = \Gamma^{\nu}_{n\beta} = 0\,, \qquad \Gamma^{n}_{\alpha\beta} \text{ — arbitrary}\,.$$

Using the notation introduced in Section II, we can write

$$\Gamma^{\alpha}_{\beta\gamma} \equiv [\Gamma^{\alpha}_{\beta\gamma}]_s\,.$$

b) We can start from the condition of compatibility in the form (38), which does not depend on a vector. This condition is satisfied by every differentiable curve on an absolutely reducible $\overset{*}{V}_n$ and by the null curves on general $\overset{*}{V}_n$. Therefore Treder's parallel transport along the null curves possesses, in general, the usual properties and is determined up to one arbitrary function.

c) We can start from the condition of compatibility (40) and add to it the $(n-1)$ equations (39) which contain dv^{λ}/dt. The condition (40) allows the determination of one of the components v^{λ} and, together with the equations containing dv^{λ}/dt, permits transport uniquely along the suitable, differentiable curve $x^{\beta}(t)$ of the vector $\vec{v}$ which satisfies at $t = t_0$

$$v^{\sigma}(t_0)\,\dot{x}^{\tau}(t_0)\,\partial_n g_{\sigma\tau}(t_0) = 0$$

However this transport is not the usual parallel transport.

Summarizing, the Treder equations are the equations of the usual parallel transport only in the case of absolute reducibility and this parallel transport is determined up to one arbitrary function. In this case it permits the definition of

the parallel transport of the metrically equivalent classes of vector which is the same as described in Section IV.

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