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Analytic Continuation of Series of Homogeneous Polynomials of n Complex Variables

INTRODUCTION

Let f be an analytic function of n complex variables defined by a series of homogeneous polynomials

$$f(z) = \sum_{\nu=1}^{\infty} f_\nu(z) \qquad (\deg f_\nu = \nu) \tag{1}$$

that converges in a neighbourhood of zero in the space C^n. Let G_f denote the Mittag-Leffler star of the function f, i.e. the largest starlike (with respect to zero) open set such that f can be analytically continued to G_f. The question arises how the analytic continuation of the function f to G_f can be found. If $n = 1$ an answer is given by a known result (see e.g. [6], p. 493), which can be extended to the n-dimensional case as follows:

Theorem 1. There is a triangular array of complex numbers $c_0^\nu, c_1^\nu, \ldots, c_{k_\nu}^\nu$ $(\nu = 0, 1, \ldots)$ such that for every analytic function f given by (1) we have

$$f(z) = \sum_{\nu=0}^{\infty} [c_0^\nu f_0(z) + c_1^\nu f_1(z) + \ldots + c_{k_\nu}^\nu f_{k_\nu}(z)]$$

for $z \in G_f$, the convergence being uniform on compact subsets of G_f.

Proof (due to J. Siciak). Choose a point $z \in C^n \setminus \{0\}$ and write $G^z = \{\lambda \in C : \lambda z \in G_f\}$. Note that for each $z \in G_f$ $1 \in G^z$ and the function $f_z : \lambda \to f(\lambda z)$ is analytic in G^z. So, by the Cauchy formula:

$$f(z) = (2\pi i)^{-1} \int_{\lambda_z} f(\lambda z)(\lambda - 1)^{-1} d\lambda = (2\pi i)^{-1} \int_{\lambda_z} f(\lambda z)\left(1 - \frac{1}{\lambda}\right)^{-1} \lambda^{-1} d\lambda, \quad z \in G_f,$$

where γ_z is any positively oriented regular Jordan curve contained in G^z such that $1 \in \operatorname{int}\gamma_z$ (where $\operatorname{int}\gamma_z$ denotes the bounded component of $C\setminus\gamma_z$).

Take an arbitrary point $z_0 \in G_f\setminus\{0\}$ and a starlike (with respect to zero) positively oriented regular Jordan curve γ contained in G^{z_0} so that the points $0,1 \in \operatorname{int}\gamma$. Then $\overline{\operatorname{int}\gamma}\cdot z_0 = \{\lambda z_0 : \lambda \in \overline{\operatorname{int}\gamma}\}$ is contained in G_f. Since $\overline{\operatorname{int}\gamma}$ is compact and G_f is open one can find a number $\varrho > 0$ such that

$$\Delta = \{\zeta \in C^n : \zeta = \lambda z,\ \lambda \in \overline{\operatorname{int}\gamma},\ \|z-z_0\| \leqslant \varrho\} \subset G_f .$$

For any $z \in B(z_0, \varrho)$ we have

$$f(z) = (2\pi i)^{-1} \int_\gamma \left(1-\frac{1}{\lambda}\right)^{-1} \frac{f(\lambda z)}{\lambda} d\lambda .$$

Note that $\{\lambda^{-1} : \lambda \in \gamma\} \subset C\setminus[1, \infty)$. By the Runge approximation theorem, there exists a sequence $\{P_\nu\}_{\nu=0}^{\infty}$ of polynomials of one complex variable such that

$$\frac{1}{1-\tau} = \sum_{\nu=0}^{\infty} P_\nu(\tau), \quad \tau \in C\setminus[1, \infty), \tag{2}$$

the convergence being uniform on every compact subset of $C\setminus[1, \infty)$. Set

$$P_\nu(\tau) = c_0^\nu + c_1^\nu \tau + \ldots + c_{k_\nu}^\nu \tau^{k_\nu} .$$

Using (2) we obtain

$$f(z) = (2\pi i)^{-1} \int_\gamma \sum_{\nu=0}^{\infty} P_\nu\left(\frac{1}{\lambda}\right) \frac{f(\lambda z)}{\lambda} d\lambda, \quad z \in B(z_0, \varrho) . \tag{3}$$

Since series (2) converges uniformly on $\{\lambda^{-1} : \lambda \in \gamma\}$ and since the function $(\lambda, z) \to \frac{f(\lambda z)}{\lambda}$ is analytic in a neighborhood of $\gamma \times \overline{B(z_0, \varrho)}$, the right-hand side of (3) may be integrated term by term. Hence

$$f(z) = \sum_{\nu=0}^{\infty} \left[c_0^\nu (2\pi i)^{-1} \int_\gamma \frac{f(\lambda z)}{\lambda} d\lambda + c_1^\nu (2\pi i)^{-1} \int_\gamma \frac{f(\lambda z)}{\lambda^2} d\lambda + \ldots + c_{k_\nu}^\nu (2\pi i)^{-1} \int_\gamma \frac{f(\lambda z)}{\lambda^{k_\nu+1}} d\lambda \right]$$

$$= \sum_{\nu=0}^{\infty} [c_0^\nu f_0(z) + c_1^\nu f_1(z) + \ldots + c_{k_\nu}^\nu f_{k_\nu}(z)]$$

for $z \in B(z_0, \varrho)$ and the convergence is uniform in $B(z_0, \varrho)$. The proof is concluded.

Repeating the argument of the proof of Theorem 1 we obtain

Theorem 2 (for $n = 1$ see [5], th. 135). Let

$$\Phi_\delta(\lambda) = \sum_{\nu=0}^{\infty} a_\nu(\delta) \lambda^\nu, \quad \lambda \in C,\ \delta > 0 .$$

Assume that

(i) Φ_δ is an entire function for every positive δ;

(ii) $\Phi_\delta(\lambda)\to(1-\lambda)^{-1}$ when $\delta\to 0$, uniformly in any compact set containing no point of the line $[1, \infty)$;

(iii) f is an analytic function represented by series (1) in a neighborhood of $0 \in C^n$.

Set

$$F_\delta(z) = \sum_{\nu=0}^{\infty} a_\nu(\delta) f_\nu(z)\,, \qquad z \in C^n\,.$$

Then

$$F_\delta(z) \underset{\delta\to 0}{\to} f(z)\,, \qquad z \in G_f\,,$$

the convergence being uniform on compact subsets of G_f.

If $n = 1$, by taking $a_\nu(\delta) = 1/\Gamma(1+\delta\nu)$ we obtain relation (b) of the following

Theorem 3 ([4], th. 3.3.). Let f be an analytic function of one complex variable λ, represented in a neighborhood of zero by a series

$$f(\lambda) = \sum_{\nu=0}^{\infty} a_\nu \lambda^\nu\,.$$

Put

$$F_\delta(\lambda) = \sum_{\nu=0}^{\infty} \frac{a_\nu}{\Gamma(1+\delta\nu)} \lambda^\nu\,, \qquad \lambda \in C \qquad (\delta > 0).$$

Then we have:

(a) For an arbitrary compact subset D of G_f there exists $\delta_0 > 0$ such that

$$f(\lambda) = \int_0^\infty e^{-t} F_\delta(t^\delta \lambda)\, dt\,, \qquad \text{when} \qquad \lambda \in D,\ \delta \leqslant \delta_0\,;$$

(b) $f(\lambda) = \lim\limits_{\delta\to 0} F_\delta(\lambda)$ for $\lambda \in G_f$, the convergence being uniform on compact subsets of G_f.

Remark 1. If $n = 2$ and $a_\nu(\delta) = 1/\Gamma(1+\delta\nu)$ Theorem 2 was proved by Bertil Almer in 1922 (see [2]).

The aim of this paper is to generalize (a) of Theorem 3 for the case of n complex variables. Our result will include some results of L. A. Aizenberg and W. M. Trutniev (see [1], [9]) concerning Borel's method of summability of n-tuple power series. We shall give a characterization (analogous to a characterization given in [1] and [9]) of domains of summability of series (1).

1. THE MAIN RESULT

Given a function f represented by series (1), we write

$$F_k(z) = \sum_{\nu=0}^{\infty} f_\nu(z)/\Gamma\left(1+\frac{\nu}{k}\right), \qquad z \in C^n \tag{4}$$

and call the *k-th function associated with* f $(k \in N)$.

Note that for any $k \in N$, F_k is an entire function. Moreover, there exist positive constants M_k, a_k such that

$$|F_k(z)| \leqslant M_k \exp(a_k \|z\|^k), \; z \in C^n . \tag{5}$$

So, for any $k \in N$ and any $z \in C^n$ the order ϱ of the function $F_{k,z} : C \ni \lambda \mapsto F_k(\lambda z) \in C$ does not exceed k and if $\varrho = k$ the type of $F_{k,z}$ is finite.

The function

$$H_k(z) = \limsup_{\zeta \to z} \limsup_{t \to \infty} t^{-k} \ln |F_k(t\zeta)|$$

is called a *regularized radial indicator* of the function $F_k (k \in N)$ (see e.g. [7]). For any $k \in N$ the function H_k is plurisubharmonic in C^n (i.e. H_k is upper-semi-continuous and for every $z, w \in C^n$ the function $\lambda \to H_k(z+\lambda w)$ is subharmonic in C). Moreover H_k is positively homogeneous of order k, i.e. $H_k(tz) = t^k H_k(z)$, $t > 0$. These and other properties of the regularized radial indicator may be found in [7].

Set

$$A_k = \{\lambda \in C : Re\,\lambda^{-k} \geqslant 1, \; |\mathrm{Arg}\,\lambda| \leqslant \pi/2k\}$$

and observe that the sequence $\{A_k\}_{k \in N}$ is decreasing and $\bigcap_{k=1}^{\infty} A_k = [0, 1]$.

We define

$M_k \equiv M_k(f) = \{z \in C^n : A_k \cdot z \subset G_f\}$, where $A_k \cdot z = \{\lambda z : \lambda \in A_k\}$;

$B_k \equiv B_k(f) = \mathrm{int}\{z \in C^n : \int_0^\infty e^{-t} F_k(t^{1/k} z)\, dt$ converges$\}$;

$B'_k \equiv B'_k(f) = \{z \in C^n : \int_0^\infty e^{-t} F_k(t^{1/k} z)\, dt$ converges absolutely and uniformly in a neighborhood of $z\}$;

$C_k \equiv C_k(f) = \{z \in C^n : H_k(z) < 1\}$.

With these denotations we may now state the main result of this note:

Theorem 4. $B'_k = B_k = M_k = C_k$, $k \in N$.

Remark 2. Let U be the domain of convergence of series (1). Observe that

$$f(z) = \int_0^\infty e^{-t} F_k(t^{1/k} z)\, dt, \qquad z \in U, \; k \in N . \tag{6}$$

Indeed, we have

$$\int_0^x e^{-t} \sum_{\nu=0}^{p} \left[f_\nu(t^{1/k}z)/\Gamma\left(1+\frac{\nu}{k}\right)\right] dt = \sum_{\nu=0}^{p} \left[f_\nu(z)/\Gamma\left(1+\frac{\nu}{k}\right)\right] \int_0^x e^{-t} t^{\nu/k} dt$$

for all $z \in C^n$, $x \in (0, \infty)$ and $p, k \in N$. Let us fix $z \in U$ and $k \in N$. Since $\Gamma\left(1+\frac{\nu}{k}\right) \geqslant \int_0^x e^{-t} t^{\nu/k} dt$, the series

$$\sum_{\nu=0}^{\infty} \left[f_\nu(z)/\Gamma\left(1+\frac{\nu}{k}\right) \int_0^x e^{-t} t^{\nu/k} dt$$

converges uniformly with respect to $x \in (0, \infty)$. So, by the standard theorem on iterated limits, we have

$$\int_0^\infty e^{-t} \sum_{\nu=0}^{\infty} \left[f_\nu(t^{1/k}z)/\Gamma\left(1+\frac{\nu}{k}\right)\right] dt = \sum_{\nu=0}^{\infty} \left[f_\nu(z)/\Gamma\left(1+\frac{\nu}{k}\right)\right] \int_0^\infty e^{-t} t^{\nu/k} dt\,,$$

as claimed.

It follows from Theorem 4 that formula (6) gives the analytic continuation of series (1) to each domain M_k. We note that for every compact set $D \subset G_f$ there exists a $k_0 \in N$ such that $D \subset M_k$ for all $k \geqslant k_0$.

Definition 1. The domain $B_k \equiv B_k(f)$ is called the *domain of k-summability* of series (1).

As already mentioned, a characterization of domains of k-summability (if $k = 1$) of n-tuple power series has been given in [1] and [9]. Theorem 4 gives the analogous characterization for series of homogeneous polynomials with an arbitrary $k \in N$.

2. Proof of Theorem 4. Fix a number $k \in N$. To prove Theorem 4 it is enough to show that

$$B'_k \subset B_k \subset M_k \subset C_k \subset B'_k\,.$$

1° The inclusion $B'_k \subset B_k$ is obvious.

2° To prove that $B_k \subset M_k$ we shall need the following Propositions 1 and 2.

Proposition 1. Let

$$f(\lambda) = \sum_{\nu=0}^{\infty} a_\nu \lambda^\nu$$

be an analytic function in $B = B(0, r) \subset C$, $0 < r \leqslant 1$.

Given a positive integer k, suppose that the integral $\int_0^\infty e^{-t} F_k(t^{1/k})\, dt$ converges, where F_k is defined by (4) (for $n = 1$). Then there exists an analytic function $\tilde{f}$ in $A_k^0 = \operatorname{int} A_k$ such that $\tilde{f}|_{B\cap A_k^0} = f|_{B\cap A_k^0}$.

Proposition 2. Let G be an open subset of the space C^n such that $P = \{z \in C^n : |z_j| < r, j = 1, 2, \ldots, n\} \subset G$. Assume that for every $z \in P$ the set $D_z = \{\lambda \in C : \lambda z \in G\}$ is connected. Let $f : G \to C$ be analytic in the polydisc P and for every $z \in P$ the function $f_z : D_z \ni \lambda \to f(\lambda z) \in C$ be analytic in D_z. Then f is analytic in G.

We owe the statement and the proof of Proposition 2 to J. Siciak, who believes Proposition 2 to be known, but we could not find any relevant paper.

Proof of Proposition 1. By (6) we have

$$f(\lambda) = \int_0^\infty e^{-t} F_k(t^{1/k}\lambda)\, dt\,, \qquad \lambda \in B\,.$$

Let $\lambda \in (0, r)$. If we make the substitution $t = \tau\lambda^{-k}$, we obtain

$$f(\lambda) = \lambda^{-k} \int_0^\infty \exp(-\tau\lambda^{-k}) F_k(\tau^{1/k})\, d\tau\,, \qquad \lambda \in (0, r)\,.$$

Put $\lambda = \varphi(s)$, where $\varphi : (r^{-k}-1, \infty) \ni s \to (1+s)^{-1/k} \in (0, r)$. Then

$$f(\varphi(s)) = (s+1) \int_0^\infty e^{-s\tau} e^{-\tau} F_k(\tau^{1/k})\, d\tau\,, \qquad s \in (r^{-k}-1, \infty)\,. \tag{7}$$

But, by hypothesis, the integral on the right-hand side of (7) converges also for $s = 0$; hence it converges uniformly on compact subsets of the half-plane $\{Re\, s > 0\}$ (see e.g. [8], p. 218). So, the function

$$g(s) = (s+1) \int_0^\infty e^{-s\tau} e^{-\tau} F_k(\tau^{1/k})\, d\tau$$

is analytic in $\{Re\, s > 0\}$.

Now put $\tilde{f}(\lambda) = g(\lambda^{-k}-1)$ for $\lambda \in A_k^0$. By (7) we have $\tilde{f}(\lambda) = g(\varphi^{-1}(\lambda)) = f(\varphi(\varphi^{-1}(\lambda))) = f(\lambda)$, $\lambda \in (0, r)$. Since $(0, r) \subset A_k^0 \cap B$ and $A_k^0 \cap B$ is connected the proof is concluded.

Proof of Proposition 2. Observe that, for any $z \in P$, $\overline{B(0, 1)} \subset D_z$. It is easy to see that $D = \{(z, \lambda) \in C^{n+1} : z \in P, \lambda \in D_z\}$ is open. Set $F(z, \lambda) = f(\lambda z)$ for $(z, \lambda) \in D$. The function F is analytic in the domain $P \times B(0, 1)$ and for every $z \in P$ the function $F_z : D_z \ni \lambda \to F(z, \lambda) \in C$ is analytic in D_z. Thus, by the generalized Hartogs theorem ([3], p. 141) the function F is analytic in D.

Now let us fix an arbitrary $a \in G$. We can write $a = \lambda' a'$ with $\lambda' > 0$ and $a' \in P$. Obviously $(a', \lambda') \in D$. Thus F is analytic in a neighborhood of (a', λ'). The mapping T: $C^{n+1} \to C^{n+1}$ defined by $T(\xi, \tau) = \left(\frac{\xi}{\tau}, \tau\right)$ is analytic in a neighborhood of (a, λ'). Since $T(a, \lambda') = (a', \lambda')$, $F \circ T$ is analytic in a neighborhood of (a, λ'). But $F(T(\xi, \tau) = f(\xi)$. So f is analytic in a neighborhood of a. The proof is concluded.

We return to the proof of the inclusion $B_k \subset M_k$.

Fix a point $z_0 \in B_k$. Let V be an open neighborhood of z_0 such that $V \subset B_k$. Then, for any $z \in V$ the integral $\int_0^\infty e^{-t}F_k(t^{1/k}z)\,dt$ converges. By Proposition 1, for each $z \in V$, the function $f_z : \lambda \to f(\lambda z)$ is analytic in the domain A_k^0. Observe that $G \overset{\mathrm{df}}{=} G_f \cup \bigcup_{z \in V} A_k^0 \cdot z$ is a starlike domain. Therefore, since by Proposition 2 f is analytic in G, we have $G = G_f$. Thus, $A_k \cdot z_0 \subset G_f$, whence $z_0 \in M_k$.

3° The proof of the inclusion $M_k \subset C_k$ is based on the generalized Polya theorem (see [4], th. 6.6.).

First we give some definitions.

Given $\theta \in (-\pi, \pi]$, $k \in N$, and $c \geqslant 0$ consider the curve

$$L_k(\theta, c) = \{\zeta \in C : Re(\zeta e^{-i\theta})^k = c,\ |\mathrm{Arg}(\zeta e^{-i\theta})| \leqslant \pi/2k\}\,.$$

If $k = 1$ we admit also $c < 0$.

The curve $L_k(\theta, c)$ cuts the complex plane in two disjoint domains: $D_k^*(\theta, c)$ and $D_k(\theta, c)$ containing the intervals $\{0 < |\zeta| < c^{1/k}, \mathrm{Arg}\,\zeta = \theta\}$ and $\{c^{1/k} < |\zeta| < \infty, \mathrm{Arg}\,\zeta = \theta\}$, respectively.

The closed set $\overline{D_k^*(\theta, c)}$ is called an *elementary k-convex set*. We say that a closed set $M \subset C$ is *k-convex* if it is the intersection of a family of elementary k-convex sets. By a *k-convex hull* of a set $M \subset C$ we mean the intersection of all elementary k-convex sets containing M.

Let $W \subset C$ be a bounded k-convex set (hence W is compact). Write $W(k, \theta) = W \cap D_k(\theta, 0)$.

We define the function $\varkappa_k$ as follows:

$$\varkappa_1(\theta) = \max_{\zeta \in W} Re(\zeta e^{-i\theta})$$

$$\varkappa_k(\theta) = \begin{cases} \max\limits_{\zeta \in W(k,\theta)} Re(\zeta e^{-i\theta})^k, & \text{when } W(k, \theta) \neq \emptyset \\ 0 & \text{, when } W(k, \theta) = \emptyset \end{cases}, \text{ for } k > 1$$

and call $\varkappa_k$ a *k-supporting function* of the set W.

Assume that the sum of the series

$$\Phi(\lambda) = \sum_{\nu=0}^{\infty} b_\nu \lambda^\nu / \Gamma\left(1 + \frac{\nu}{k}\right), \qquad \lambda \in C \tag{8}$$

is an entire function of order k and type σ, $(k, \sigma \in (0, \infty))$. Write

$$\varphi_k(\lambda) = \sum_{\nu=0}^{\infty} b_\nu \lambda^{-\nu-1}\,, \quad |\lambda| > \sigma^{1/k}\,. \tag{9}$$

The function φ_k is called *$\mathfrak{B}_k$-transformation* of Φ (or the generalized Borel transformation of Φ).

Let W_k denote the least starlike set such that φ_k can be analytically continued to $C\setminus W_k$ and let $\widetilde{W}_k$ denote the k-convex hull of W_k. Let h_k be the indicator function of Φ, i.e.

$$h_k(\theta) = \limsup_{t\to\infty} t^{-k}\ln|\Phi(te^{i\theta})|,\ \theta\in(-\pi,\pi].$$

Now let $\widetilde{\varkappa}_k$ denote the k-supporting function of $\widetilde{W}_k$.

With these denotations we have the following known (see [4], th. 6.6).

Theorem 5. (the generalized Polya theorem). The equality

$$h_k(\theta) = \widetilde{\varkappa}_k(-\theta)$$

is valid in the following cases:

(α) $\theta\in(-\pi,\pi]$, $k=1$

(β) $\theta\in(-\pi,\pi]\cap\{\theta : h_k(\theta)\geqslant 0\}$, $k>1$.

We now proceed to the proof of the inclusion $M_k\subset C_k$.

Fix a point $z_0\in M_k$. Since $A_k\cdot z_0\subset G_f$, $A_k\cdot z_0$ is compact and G_f is open, we can find a neighborhood U of the point z_0 and a number $\varepsilon>0$ such that

$$A_k^\varepsilon\cdot z\subset G\ ,\ z\in U \tag{10}$$

where

$$A_k^\varepsilon = \{\lambda\in C : Re\,\lambda^{-k}\geqslant 1-\varepsilon,\ |\mathrm{Arg}\,\lambda|\leqslant \pi/2k\}.$$

Given any $z\in U$, consider the entire function of variable λ, represented by the series

$$F_{k,z}(\lambda) = \sum_{\nu=0}^{\infty} f_\nu(z)\lambda^\nu/\Gamma\left(1+\frac{\nu}{k}\right),\ \lambda\in C.$$

Since for each $z\in U$ the order ϱ_z of the function $F_{k,z}$ does not exceed k and if $\varrho_z=k$ the type σ_z of $F_{k,z}$ is finite, we have the following three possibilities:

(i) $\varrho_z<k$,

(ii) $\varrho_z=k$, $\sigma_z=0$,

(iii) $\varrho_z=k$, $\sigma_z\in(0,\infty)$.

In both cases (i) and (ii) we have

$$\limsup_{t\to\infty} t^{-k}\ln|F_k(tz)| = \limsup_{t\to\infty} t^{-k}\ln|F_{k,z}(t)|\leqslant 0. \tag{11}$$

In the third case we can apply Theorem 5 to the function $\Phi=F_{k,z}$. The $\mathcal{B}_k$-transformation of the function $F_{k,z}$ is of the form $\varphi_k(\lambda)=\lambda^{-1}f_z(\lambda^{-1})$. It follows from (10) that the function f_z is analytic in a neighborhood of A_k^ε, when $z\in U$. Hence the function φ_k is analytic in $D_k(0,1-\varepsilon)=\{\lambda\in C : \lambda^{-1}\in \mathrm{int}\,A_k^\varepsilon\}$. Therefore $\widetilde{W}_k\subset\overline{D_k^*(0,1-\varepsilon)}$ and so we have:

$$\widetilde{\varkappa}_k(0)\leqslant \max\{Re\,\zeta^k : \zeta\in\overline{D_k^*(0,1-\varepsilon)}\cap D_k(0,0)\}\leqslant 1-\varepsilon.$$

Consequently, by theorem 5, we obtain

$$\limsup_{t\to\infty} t^{-k}\ln|F_k(tz)| = \limsup_{t\to\infty} t^{-k}\ln|F_{k,z}(t)| = h_k(0) \leqslant 1-\varepsilon\,. \tag{12}$$

Finally, by (11) and (12),

$$\limsup_{t\to\infty} t^{-k}\ln|F_k(tz)| \leqslant 1-\varepsilon, \quad z\in U\,.$$

So

$$H_k(z_0) = \limsup_{z\to z_0}\limsup_{t\to\infty} t^{-k}\ln|F_k(tz)| \leqslant 1-\varepsilon < 1\,.$$

4° It remains to prove that $C_k \subset B'_k$.

Let $H_k(z_0) < 1$ and let $0 < \varepsilon < 1-H_k(z_0)$. Take a neighbourhood U of the point z_0 such that

$$H_k(z) < H_k(z_0)+\varepsilon/2, \quad z\in U\,.$$

Then

$$\limsup_{t\to\infty} t^{-k}\ln|F_k(tz)| < H_k(z_0)+\varepsilon/2, \quad z\in U\,.$$

Taking U to be compact and using the Hartogs lemma for subharmonic functions (see [7]) we obtain:

$$t^{-k}\ln|F_k(tz)| \leqslant H_k(z_0)+\varepsilon = \delta < 1, \quad t\geqslant t_0 = t_0(U,\varepsilon), \quad z\in U\,.$$

So

$$|F_k(tz)| \leqslant e^{\delta t^k}, t\geqslant t_0, \quad z\in U\,.$$

Hence

$$|F_k(t^{1/k}z)| \leqslant \exp(\delta t), t\geqslant T_0 = t_0^k, \quad z\in U\,.$$

Thus, the integral $\int_0^\infty e^{-t}F_k(t^{1/k}z)\,dt$ is absolutely and uniformly convergent in U. So $z_0\in B'_k$. The proof is completed.

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