ZESZYTY NAUKOWE UNIWERSYTETU JAGIELLOŃSKIEGO
CCCCXLI PRACE MATEMATYCZNE, ZESZYT 18 1977

Feliks Barański, Jan Musiałek

On Mixed Boundary-Value Problems for a Spherical Octant

1. In the paper [1] we solved a mixed boundary value problem of the Neumann and Dirichlet type for the quarter circle. In the present paper we shall construct the solutions of four boundary value problems for the equation

$$\Delta u(X) = 0, \; X = (x, y, z) \tag{1}$$

in the domain $K = \{(x, y, z)\colon x^2+y^2+z^2<R^2, x>0, y>0, z>0\}$ with boundary Neumann and Dirichlet conditions on the different parts of the boundary.

2. Let us denote: $S = \{(x, y, z)\colon x^2+y^2+z^2 = R^2\}$,

$S_1 = \{(x, y, z)\colon x^2+y^2+z^2 = R^2, x>0, y>0, z>0\}$,

$S_2 = \{(x, y, z)\colon 0<x<R, y = 0, 0<z<(R^2-x^2)^{\frac{1}{2}}\}$,

$S_3 = \{(x, y, z)\colon x = 0, 0<y<(R^2-z^2)^{\frac{1}{2}}, 0<z<R\}$,

$S_4 = \{(x, y, z)\colon 0<x<(R^2-y^2)^{\frac{1}{2}}, 0<y<R, z = 0\}$.

We shall consider four cases. We wish to find functions $u_i(X)$, $i = 1, 2, 3, 4$ harmonic in K satisfying the boundary conditions

$$u_1(X) = f_1(X) \quad \text{for} \quad X \in S_1 \tag{2}$$

$$D_y u_1(X) = f_2(x, z) \quad \text{for} \quad X \in S_2 \tag{3}$$

$$D_x u_1(X) = f_3(y, z) \quad \text{for} \quad X \in S_3 \tag{4}$$

$$D_z u_1(X) = f_4(x, y) \quad \text{for} \quad X \in S_4 \tag{5}$$

and

$$u_2(X) = q_1(X) \quad \text{for} \quad X \in S_1 \tag{6}$$

$$u_2(X) = q_2(x, z) \quad \text{for} \quad X \in S_2 \tag{7}$$

$$u_2(X) = q_3(y, z) \quad \text{for} \quad X \in S_3 \tag{8}$$

(9) $$u_2(X) = q_4(x, y) \quad \text{for} \quad X \in S_4$$

and

(10) $$u_3(X) = h_1(X) \quad \text{for} \quad X \in S_1$$

(11) $$u_3(X) = h_2(x, z) \quad \text{for} \quad X \in S_2$$

(12) $$D_x u_3(X) = h_3(y, z) \quad \text{for} \quad X \in S_3$$

(13) $$D_z u_3(X) = h_4(x, y) \quad \text{for} \quad X \in S_4$$

and

(14) $$u_4(X) = p_1(X) \quad \text{for} \quad X \in S_1$$

(15) $$u_4(X) = p_2(x, z) \quad \text{for} \quad X \in S_2$$

(16) $$u_4(X) = p_3(y, z) \quad \text{for} \quad X \in S_3$$

(17) $$D_z u_4(X) = p_4(x, y) \quad \text{for} \quad X \in S_4 .$$

The problem of finding a solution of (1) satisfying (2), (3), (4), (5) or (6), (7), (8), (9) or (10), (11), (12), (13) or (14), (15), (16), (17) will be called briefly (D, N, N, N), (D, D, D, D), (D, D, N, N), and (D, D, D, N) respectively.

These problems will be solved by means of Green functions.

3. Given any points $X(x, y, z) = X_1 \in K$, $Y(s, t, w) \in \overline{K}$, where $\overline{K}$ denote the closure of the set K, let $\overline{X}(x, \bar{y}, \bar{z})$ denotes the conjugate point to X with respect to the sphere S. Let $X_2(-x, y, z)$, $X_3(-x, -y, z)$, $X_4(x, -y, z)$, $X_5(x, y, -z)$, $X_6(-x, y, -z)$, $X_7(-x, -y, -z)$ and $X_8(x, -y, -z)$ denote the symmetric images of the points X_1, X_2, X_3, X_4 X_5, X_6, X_7 with respect to the corresponding coordinate planes. Let $\overline{X}_i = (e_i^1 \bar{x}, e_i^2 \bar{y}, e_i^3 \bar{z})$ denote conjugate points to $X_i = (e_i^1 x, e_i^2 y, e_i^3 z)$ with respect to S. Let

$$r_i^2 = (s+e_i^1 x)^2 + (t+e_i^2 y)^2 + (w+e_i^3 z)^2 ,$$

$$\bar{r}_i^2 = (s+e_i^1 \bar{x})^2 + (t+e_i^2 \bar{y})^2 + (w+e_i^3 \bar{z})^2$$

where

$$e_i^1 = \begin{cases} 1 & \text{for} \quad i = 2, 3, 6, 7 \\ -1 & \text{for} \quad i = 1, 4, 5, 8 \end{cases} \qquad e_i^2 = \begin{cases} 1 & \text{for} \quad i = 3, 4, 7, 8 \\ -1 & \text{for} \quad i = 1, 2, 5, 6 \end{cases}$$

$$e_i^3 = \begin{cases} 1 & \text{for} \quad i = 5, 6, 7, 8 \\ -1 & \text{for} \quad i = 1, 2, 3, 4 \end{cases}$$

Let $\varrho^2 = x^2+y^2+z^2$.

4. First we shall solve the (D, N, N, N) problem.

Let us consider the plane (p): $As+Bt+Cw = 0$ and the points P_1, P_2 symmetric with respect to (p) and $P_1 \neq P_2$. Let n denote the normal to (p) at the point $Q \in (p)$, $Q = Q(s, t, w)$. Let $d_1^2 = (s-x_1)^2+(t-y_1)^2+(w-z_1)^2$, $d_2^2 = (s-x_2)^2+(t-y_2)^2+$ $+(w-z_2)^2$, (x_i, y_i, z_i), $i = 1, 2$, being coordinates of the points P_i.

Lemma 1. Let the function $v(d)$, $d>0$ be of class C^1, then

$$D_{n_Q}(v(d_1)+v(d_2)) = 0\,.$$

We omit the simple proof.

Let

$$G(X, Y) = \sum_{i=1}^{8} G_i(X, Y)\,, \tag{18}$$

where

$$G_i(X, Y) = r_i^{-1} - RQ^{-1}(\bar{r}_i)^{-1}\,.$$

Theorem 1. The function $G(X, Y)$ defined by formula (18) is the Green function with the pole X for the (D, N, N, N) problem.

Proof. $G(X, Y)$ is a harmonic function with respect to the point $Y \neq X$, since $G_i(X, Y)$ are harmonic functions of the argument Y. For $Y \in S_1$, $G_i(X, Y) = 0$, $i = 1, \ldots, 8$, ([2], p. 250). From Lemma 1 it follows that $D_n G(X, Y) = 0$ for $Y \in S_2 \cup S_3 \cup S_4$. In virtue of (18) we get

$$G(X, Y)|_{s=0} = 2(r_{1,2}^{-1}+r_{3,4}^{-1}+r_{5,6}^{-1}+r_{7,8}^{-1}) - 2RQ^{-1}(\bar{r}_{1,2}^{-1}+\bar{r}_{3,4}^{-1}+\bar{r}_{5,6}^{-1}+\bar{r}_{6,7}^{-1})\,,$$

where

$r_i = r_{i+1} = r_{i,\,i+1}$, $\bar{r}_i = \bar{r}_{i+1} = \bar{r}_{i,\,i+1}$, for $s = 0$, $i = 1, 3, 5, 7$.

$$G(X, Y)|_{t=0} = 2(r_{1,4}^{-1}+r_{2,3}^{-1}+r_{5,8}^{-1}+r_{6,7}^{-1}) - 2RQ^{-1}(\bar{r}_{1,4}^{-1}+\bar{r}_{2,3}^{-1}+\bar{r}_{5,8}^{-1}+\bar{r}_{6,7}^{-1})\,,$$

where

$r_i = r_{i+3} = r_{i,\,i+3}$, $r_j = r_{j+1} = r_{j,\,j+1}$, $\bar{r}_j = \bar{r}_{j+1} = \bar{r}_{j,\,j+1}$, $\bar{r}_i = \bar{r}_{i+3} = \bar{r}_{i,\,i+3}$ for $t = 0$, $i = 1, 5$, and $j = 2, 6$.

$$G(X, Y)|_{w=0} = 2(r_{1,5}^{-1}+r_{2,6}^{-1}+r_{3,7}^{-1}+r_{4,8}^{-1}) - 2RQ^{-1}(\bar{r}_{1,5}^{-1}+\bar{r}_{2,6}^{-1}+\bar{r}_{3,7}^{-1}+\bar{r}_{4,8}^{-1})$$

where

$r_i = r_{i+4} = r_{i,\,i+4}$, $\bar{r}_i = \bar{r}_{i+4} = \bar{r}_{i,\,i+4}$ for $w = 0$, $i = 1, 2, 3, 4$.

5. We shall prove that under certain assumptions concerning f_i, $i = 1, \ldots, 4$, that the function defined by formula

$$u_1(X) = A \iint_{S_1} f_1(Y) D_n G(X, Y)|_{Y \in S_1}\, dS_1 + A_1 \iint_{S_2} f_2(s, w) G(X, Y)|_{t=0}\, ds dw + \\ + A_1 \iint_{S_3} f_3(t, w) G(X, Y)|_{s=0}\, dt dw + A_1 \iint_{S_4} f_4(s, t) G(X, Y)|_{w=0}\, ds dt\,, \tag{19}$$

where $A = (4\pi R)^{-1}$, $A_1 = (2\pi)^{-1}$, n being inward normal to S_1, or the function

$$u_1(X) = \sum_{i=1}^{4} g_i(X)\,, \tag{19a}$$

where

$$g_1(X) = A \iint_{S_1} f_1(Y)(R^2-\varrho^2) \sum_{i=1}^{8} r_i^{-3}\, dS_1,$$

$$g_2(X) = A_1 \iint_{S_2} f_2(s, w)\big(r_{1,4}^{-1}+r_{2,3}^{-1}+r_{5,8}^{-1}+r_{6,7}^{-1}-R\varrho^{-1}(\bar{r}_{1,4}^{-1}+\bar{r}_{2,3}^{-1}+\bar{r}_{5,8}^{-1}+\bar{r}_{6,7}^{-1})\big)\, ds\, dw,$$

$$g_3(X) = A_1 \iint_{S_3} f_3(t, w)\big(r_{1,2}^{-1}+r_{3,4}^{-1}+r_{5,6}^{-1}+r_{7,7}^{-1}-R\varrho^{-1}(\bar{r}_{1,2}^{-1}+\bar{r}_{3,4}^{-1}+\bar{r}_{5,6}^{-1}+\bar{r}_{7,8}^{-1})\big)\, dt\, dw,$$

$$g_4(X) = A_1 \iint_{S_4} f_4(s, t)\big(r_{1,5}^{-1}+r_{2,6}^{-1}+r_{3,7}^{-1}+r_{4,8}^{-1}-R\varrho^{-1}(\bar{r}_{1,5}^{-1}+\bar{r}_{2,6}^{-1}+\bar{r}_{3,7}^{-1}+\bar{r}_{4,8}^{-1})\big)\, ds\, dt$$

is a solution of the (D, N, N, N) problem.

Lemma 2. Let the functions f_i, $i = 1, \ldots, 4$ be continuous and bounded in the sets $\bar{S}_i$, then the function $u_1(X)$ defined by formula (19a) is harmonic in K.

Proof. Since $X \neq Y$ and $Y \in \bar{S}_1 \cup \bar{S}_2 \cup \bar{S}_3 \cup \bar{S}_4$ the integrals

$$(20)\qquad \begin{cases} J_1^{pqr}(X) = A \iint_{S_1} f_1(Y)\, D_{x^p y^q z^r} D_n G(X, Y)|_{Y\in S_1}\, dS_1, \\ J_2^{pqr}(X) = A_1 \iint_{S_2} f_2(s, w)\, D_{x^p y^q z^r} G(X, Y)|_{t=0}\, ds\, dw, \\ J_3^{pqr}(X) = A_1 \iint_{S_3} f_3(t, w)\, D_{x^p y^q z^r} G(X, Y)|_{s=0}\, dt\, dw, \\ J_4^{pqr}(X) = A_1 \iint_{S_4} f_4(s, t)\, D_{x^p y^q z^r} G(X, Y)|_{w=0}\, ds\, dt, \end{cases}$$

where $p, q, r = 0, 1, 2$; $p+q+r \leqslant 2$, are locally uniformly convergent in every point $X \in K$, and consequently there exist the derivatives $D_{x^p y^q z^r} g_j(X)$, $j = 1, \ldots, 4$.

From (20) and the symmetry of the Green function we get

$$\begin{aligned} \Delta_X u_1(X) = {} & A \iint_{S_1} f_1(Y)\, D_n \Delta_Y G(X, Y)|_{Y\in S_1}\, dS_1 + \\ & + A_1 \iint_{S_2} f_2(s, w)\, \Delta_Y G(X, Y)|_{t=0}\, ds\, dw + \\ & + A_1 \iint_{S_3} f_3(t, w)\, \Delta_Y G(X, Y)|_{s=0}\, dt\, dw + \\ & + A_1 \iint_{S_4} f_4(s, t)\, \Delta_Y G(X, Y)|_{w=0}\, ds\, dt\,. \end{aligned}$$

6. Assuming that the functions f_i, $i = 1, \ldots, 4$ satisfy the conditions of Lemma 2 we shall verify the boundary conditions (2), (3), (4) and (5).

Lemma 3. Let the function f_1 be continuous and bounded on $\bar{S}_1$, then $\lim u_1(X) = f_1(X_0)$, as $X \to X_0(x_0, y_0, z_0) \in S_1$ and $X \in K$.

Proof. Let

$$\bar{f}_1(Y) = \begin{cases} f_1(Y) & \text{for} \quad Y \in S_1 \\ 0 & \text{for} \quad Y \in S - \bar{S}_1 \end{cases}$$

and let

$$g_1(X) = W_1(X) + W_2(X),$$

where

$$W_1(X) = A \iint_S \bar{f}_1(Y)(R^2-\varrho^2)r_1^{-3}\,dS$$

and

$$W_2(X) = A \iint_{S_1} f_1(Y)(R^2-\varrho^2)\sum_{i=2}^{8} r_i^{-3}\,dS_1 .$$

From ([2], p. 253) it follows that $\lim W_1(X) = f_1(X_0)$, as $X \to X_0$. If $r_i > \delta > 0$, then

$$|W_2(X)| \leqslant (R^2-\varrho^2)M \iint_{S_1} \delta^{-3}\,dS_1 \to 0 \quad \text{as} \quad \varrho \to R ,$$

where $M = \sup_{S_1} |f_1(Y)|$.

The integrals $g_i(X)$, $i = 2, 3, 4$ are uniformly convergent at the point X_0 and consequently continuous and we get

$$\lim g_i(X) = g_i(X_0) = 0 \quad \text{as} \quad X \to X_0,\ \varrho \to R$$

and $\bar{r}_{i,j} \to r_{i,j}$, $i, j = 1, 2, \ldots, 8$.

Now we shall prove only the boundary condition (4); the proof of the conditions (3) and (5) being analogous.

Let (p_1) be a plane and n the normal versor to (p_1); let $P(x, y, z)$, $Y(s, t, w)$, $Y_1(s_1, t_1, w_1)$, $Y_2(s_2, t_2, w_2)$ be arbitrary points such that $P \in (p_1)$, $Y \in (p_1)$, $Y \neq P$, $Y_1 \neq P$, $Y_2 \neq P$ and Y_1, Y_2 lying in the opposite half spaces respectively to the plane (p_1).

Let

$$r_i^2 = (s_i-x)^2+(t_i-y)^2+(w_i-z)^2, \quad (i = 1, 2)$$

Lemma 4. Let $v(r) \in C^1$ for $r > 0$, and let $Y_1 \to Y$, $Y_2 \to Y$, then

$$\lim D_n\big(v(r_1)+v(r_2)\big) = 0 .$$

We omit the simple proof.

From Lemma 4 follows the following

Lemma 5. Let the functions f_i, $i = 1, 2, 4$, be continuous and bounded in $\bar{S}_4$, then

$$\lim D_x g_i(X) = 0 \quad \text{as} \quad X \to X_0(0, y_0, z_0),\ i = 1, 2, 4 .$$

Let us now prove that

$$\lim D_x g_3(X) = f_3(y_0, z_0) \quad \text{as} \quad X \to X_0(0, y_0, z_0) .$$

In order to prove this proposition we shall need some lemmas.

Let $V(X, t, w) = [x^2+(y-t)^2+(z-w)^2]^{-3/2}$.

Lemma 6. If $x > 0$, then

$$A_1 x \iint_{E_2} V(X, t, w)\,dt\,dw = 1$$

where $E_2 = \{(t, w)\colon -\infty < t < \infty,\ -\infty < w < \infty\}$

We omit the simple proof.

Lemma 7. Let the function $f_3(t, w)$ be continuous and bounded for $(t, w) \in \bar{S}_3$. Then the function

$$L(X) = A_1 x \iint_{S_3} f_3(t, w) V(X, t, w) dt dw$$

is convergent to $f_3(y_0, z_0)$ when $X \to X_0(0^+, y_0, z_0)$, $(y_0, z_0) \in S_3$.

Proof. Let

$$\bar{f}_3(t, w) = \begin{cases} f_3(t, w) & \text{for} \quad (t, w) \in \bar{S}_3 \\ 0 & \text{for} \quad (t, w) \in E_2 - \bar{S}_3 \end{cases}$$

Hence

$$L(X) = A_1 x \iint_{E_2} \bar{f}_3(t, w) V(X, t, w) dt dw .$$

By Lemma 6 we can represent $L(X)$ in the form

$$L(X) = f_3(y_0, z_0) + T(X) ,$$

where

$$T(X) = A_1 x \iint_{E_2} (\bar{f}_3(t, w) - f_3(y_0, z_0)) V(t, t, w) dt dw .$$

It is sufficient to verify that

$$\lim T(X) = 0 \quad \text{as} \quad X \to X_0(0^+, y_0, z_0) .$$

Let

$$T(X) = T_1(X) + T_2(X) ,$$

where

$$T_1(X) = A_1 x \iint_{K_\delta} (\bar{f}_3(t, w) - f_3(y_0, z_0)) V(X, t, w) dt dw$$

and

$$T_2(X) = A_1 x \iint_{E_2 - K_\delta} (\bar{f}_3(t, w) - f_3(y_0, z_0)) V(X, t, w) dt dw ,$$

where K_δ denotes the circle of the radius δ and the center at the point (y_0, z_0).

Let ε be an arbitrary positive number and let $M = \sup |f_3|$. From the continuity of the function f_3 at the point (y_0, z_0) it follows that there exists a number $\delta > 0$ such that

$$|f_3(t, w) - f_3(y_0, z_0)| < \varepsilon$$

for the points $(t, w) \in K_\delta$.

Hence

$$|T_1(X)| \leqslant \varepsilon A_1 x \iint_{E_2} V(X, t, w) dt dw = \varepsilon$$

$$|T_2(X)| \leqslant 2M A_1 x \iint_{E_2 - K_\delta} V(X, t, w) dt dw = t_2(X) .$$

Introducing in the integral $t_2(X)$ the transformation $t = y+R\cos\varphi$, $w=z+R\sin\varphi$, where $\delta<R<\infty$, $0\leqslant\varphi\leqslant 2\pi$ we get

$$|T_2(X)|\leqslant 4\pi Mx\int_{\delta}^{\infty}(R^2+x^2)^{-\frac{3}{2}}RdR\leqslant 4\pi Mx\delta^{-1}\to 0 \text{ as } X\to(0^+, y_0, z_0)$$

From Lemmas 4 and 7 we get.

Lemma 8. Under the assumptions of lemmas 4 and 7 we have

$$\lim D_x g_3(X) = f_3(y_0, z_0) \text{ as } X\to(0^+, y_0, z_0)\,.$$

By Lemmas 2, 3, 5 and 8 we obtain.

Theorem 2. Let the functions f_i be continuous and bounded in $\bar{S}_i$, $i = 1, \ldots, 4$, respectively, then the function $u_1(X)$ defined by formula (19a) is the solution of the (D, N, N, N) problem.

6. By an argument similar to that used in the proof of Theorem 2 we show the following results.

Theorem 3. Let the functions q_i be bounded and continuous on the sets $\bar{S}_i$, $i = 1, 2, 3, 4$, respectively, then the function defined by formula

$$u_2(X) = \sum_{i=1}^{11} v_i(X)\,,$$

where

$$v_1(X) = A\iint_{S_1} q_1(Y)(R^2-\varrho^2)r_1^{-3}\,dS_1\,,$$

$$v_2(X) = A\iint_{S_1} q_1(Y)(R^2-\varrho^2)(-r_2^{-3}+r_3^{-3}-r_4^{-3}-r_5^{-3}+r_6^{-3}-r_7^{-3}+r_8^{-3})dS_1\,,$$

$$v_3(X) = A_1 y\iint_{S_2} q_2(s, w)r_{1,4}^{-3}\,ds\,dw\,,$$

$$v_4(X) = A_1 y\iint_{S_2} q_2(s, w)(r_{6,7}^{-3}-r_{2,3}^{-3}-r_{5,8}^{-3})ds\,dw\,,$$

$$v_5(X) = A_1 R\varrho^{-1}\bar{y}\iint_{S_2} q_2(s, w)(\bar{r}_{1,4}^{-3}-\bar{r}_{2,3}^{-3}+\bar{r}_{6,7}^{-3}-\bar{r}_{5,8}^{-3})ds\,dw\,,$$

$$v_6(X) = A_1 x\iint_{S_3} q_3(t, w)r_{1,2}^{-3}dt\,dw\,,$$

$$v_7(X) = A_1 x\iint_{S_3} q_3(t, w)(r_{7,8}^{-3}-r_{3,4}^{-3}-r_{5,6}^{-3})dt\,dw\,,$$

$$v_8(X) = A_1 R\bar{x}\varrho^{-1}\iint_{S_3} q_3(t, w)(\bar{r}_{1,2}^{-3}-\bar{r}_{3,4}^{-3}-\bar{r}_{5,6}^{-3}+\bar{r}_{7,8}^{-3})dt\,dw\,,$$

$$v_9(X) = A_1 z\iint_{S_4} q_4(s, t)r_{1,5}^{-3}ds\,dt\,,$$

$$v_{10}(X) = A_1 z\iint_{S_4} q_4(s, t)(r_{3,7}^{-3}-r_{2,6}^{-3}-r_{4,8}^{-3})ds\,dt\,,$$

$$v_{11}(X) = A_1 R\bar{z}\varrho^{-1}\iint_{S_4} q_4(s, t)(\bar{r}_{1,5}^{-3}-\bar{r}_{2,6}^{-3}+\bar{r}_{3,7}^{-3}-\bar{r}_{4,8}^{-3})ds\,dt\,.$$

is the solution of the problem (D, D, D, D).

Theorem 4. Let the functions h_i be continuous and bounded on the sets $\bar{S}_i$, $i = 1, \ldots, 4$, respectively, then the function

$$u_3(X) = \sum_{i=1}^{11} H_i(X),$$

where

$$H_1(X) = A \iint_{S_1} h_1(Y)(R^2-\varrho^2) r_1^{-3} dS_1,$$

$$H_2(X) = A \iint_{S_1} h_1(Y)(R^2-\varrho^2)(r_2^{-3}-r_3^{-3}-r_4^{-3}+r_5^{-3}+r_6^{-3}-r_7^{-3}-r_8^{-3}) dS_1,$$

$$H_3(X) = A_1 y \iint_{S_2} h_2(s, w)(r_{1,4}^{-3}) ds\, dw,$$

$$H_4(X) = A_1 y \iint_{S_2} h_2(s, w)(r_{2,3}^{-3}+r_{5,8}^{-3}+r_{6,7}^{-3}) ds\, dw,$$

$$H_5(X) = -A_1 R\bar{y}\varrho^{-1} \iint_{S_2} h_2(s, w)(r_{1,4}^{-3}+r_{2,3}^{-3}+r_{5,8}^{-3}+r_{6,7}^{-3}) ds\, dw,$$

$$H_6(X) = A_1 \iint_{S_3} h_3(t, w) r_{1,2}^{-1} dt\, dw,$$

$$H_7(X) = A_1 \iint_{S_3} h_3(t, w)(r_{5,6}^{-1}-r_{3,4}^{-1}-r_{7,8}^{-1}) dt\, dw,$$

$$H_8(X) = -A_1 R\varrho^{-1} \iint_{S_3} h_3(t, w)(\bar{r}_{1,2}^{-1}-\bar{r}_{3,4}^{-1}+\bar{r}_{5,6}^{-1}-\bar{r}_{7,8}^{-1}) dt\, dw,$$

$$H_9(X) = A_1 \iint_{S_4} h_4(s, t) r_{1,5}^{-1} ds\, dt,$$

$$H_{10}(X) = A_1 \iint_{S_4} h_4(s, t)(r_{2,6}^{-1}-r_{3,7}^{-1}-r_{4,8}^{-1}) ds\, dt,$$

$$H_{11}(X) = -A_1 R\varrho^{-1} \iint_{S_4} h_4(s, t)(\bar{r}_{1,5}^{-1}-\bar{r}_{4,8}^{-1}+\bar{r}_{2,6}^{-1}-\bar{r}_{3,7}^{-1}) ds\, dt$$

is the solution of the problem (D, D, N, N).

Theorem 5. Let the functions p_i be continuous and bounded in sets $\bar{S}_i$, $i = 1, \ldots, 4$ respectively, then the function

$$u_4(X) = \sum_{i=1}^{11} P_i(X),$$

where

$$P_1(X) = A \iint_{S_1} p_1(Y)(R^2-\varrho^2) r_1^{-3} dS_1,$$

$$P_2(X) = A \iint_{S_1} p_1(Y)(R^2-\varrho^2)(r_1^{-3}-r_2^{-3}+r_3^{-3}-r_4^{-3}+r_5^{-3}-r_6^{-3}+r_7^{-3}-r_8^{-3}) dS_1,$$

$$P_3(X) = A_1 \iint_{S_2} p_2(s, w) r_{1,4}^{-3} ds\, dw,$$

$$P_4(X) = A_1 \iint_{S_2} p_2(s, w)(r_{5,8}^{-3}-r_{2,3}^{-3}-r_{6,7}^{-3}) ds\, dw,$$

$$P_5(X) = -A_1 R\bar{y}\varrho^{-1} \iint_{S_3} p_2(s, w)(\bar{r}_{1,4}^{-3} - \bar{r}_{2,3}^{-3} + \bar{r}_{5,8}^{-3} - \bar{r}_{6,7}^{-3})\,ds\,dw\,,$$

$$P_6(X) = -A_1 x \iint_{S_3} p_3(t, w) r_{1,2}^{-3}\,dt\,dw\,,$$

$$P_7(X) = A_1 x \iint_{S_3} p_3(t, w)(r_{3,4}^{-3} + r_{5,6}^{-3} + r_{7,8}^{-3})\,dt\,dw\,,$$

$$P_8(X) = -A_1 Rx\varrho^{-1} \iint_{S_3} p_3(t, w)(\bar{r}_{3,4}^{-3} - \bar{r}_{1,2}^{-3} - \bar{r}_{5,8}^{-3} + \bar{r}_{7,8}^{-3})\,dt\,dw\,,$$

$$P_9(X) = A_1 \iint_{S_4} p_4(s, t)(r_{1,5}^{-1})\,ds\,dt\,,$$

$$P_{10}(X) = A_1 \iint_{S_4} p_4(s, t)(r_{3,7}^{-1} - r_{2,6}^{-1} - r_{4,8}^{-1})\,ds\,dt\,,$$

$$P_{11}(X) = -A_1 R\varrho^{-1} \iint_{S_4} p_4(s, t)(\bar{r}_{1,5}^{-1} - \bar{r}_{4,8}^{-1} + \bar{r}_{3,7}^{-1} - \bar{r}_{2,6}^{-1})\,ds\,dt$$

is the solution of the problem (D, D, D, N).

REFERENCES

[1] S. Krypel i J. Musiałek, *O mieszanym problemie brzegowym dla ćwiartki koła.* (in press: Zeszyty Nauk. AGH).

[2] M. Krzyżański, *Partial differential equations of second order*, vol. I, Warszawa 1972.