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On Some Comparison Theorems for a Functional Inequality Connected with Cosine

§ 1. In the present paper we shall deal with the functional inequality

$$\psi(x+y) \leqslant \psi(x)\psi(y) - \sqrt{1-\psi^2(x)}\sqrt{1-\psi^2(y)}, \tag{1}$$

where ψ is an unknown real function fulfilling the condition

$$\psi(0) = 1. \tag{2}$$

As has been proved in [3] continuous solutions ψ of (1) satisfying (2) are either constant functions or there is a positive a such that ψ is positive in $(0, a)$.

Thus in the sequel we shall study the following class of functions:

Definition. Let $a>0$ be a fixed number. A function $\psi\colon [0, a]\to[0, 1]$ is said to belong to the class A if it is a solution of (1), continuous in $[0, a]$ and such that

$$\psi(a) = 0 \quad \text{and} \quad \psi(x)>0 \quad \text{for} \quad x \in (0, a). \tag{3}$$

§ 2. We shall need some results from [3], which we quote here as lemmas.

Lemma 1. If $\psi \in A$, then

$$\psi(x) = \cos f(x), \quad x \in [0, a] \tag{4}$$

where f is a superadditive function, i.e.

$$f(x+y) \geqslant f(x)+f(y) \tag{5}$$

fulfilling the conditions

$$f\colon [0, a] \to [0, \pi/2]. \tag{6}$$

$$f(0) = 0 \quad \text{and} \quad f(a) = \pi/2 \tag{7}$$

Lemma 2. Let $\psi \in A$ and let φ be the continuous solution of the equation

$$\varphi(x+y) = \varphi(x)\varphi(y) - \sqrt{1-\varphi^2(x)}\sqrt{1-\varphi^2(y)} \tag{8}$$

such that

(9) $$\varphi(0) = 1 \quad \text{and} \quad \varphi(a) = 0 .$$

If there exists $x_0 \in (0, a]$ such that

$$\psi(x_0) = \varphi(x_0)$$

then

(10) $$\psi\left(\frac{x_0}{2^n}\right) \geqslant \varphi\left(\frac{x_0}{2^n}\right).$$

Lemma 3. If $\psi \in A$, φ is a continuous solution of equation (8) fulfilling conditions (9) and

(11) $$\psi\left(\frac{a}{2^n}\right) = \varphi\left(\frac{a}{2^n}\right) \quad \text{for} \quad n=1, 2, \ldots$$

then

$$\psi(x) = \varphi(x) \quad \text{for} \quad x \in [0, a] .$$

Moreover the general continuous solution of equation (8) satisfying conditions (9) is given by the formula

(12) $$\psi(x) = \cos cx, \quad \text{where} \quad c = \pi/2a$$

(see [2]).

§ 3. In the present section we prove two comparison theorems for solutions of inequality (1).

Theorem 1. If $\psi \in A$ and if there exists the limit

(13) $$c = \lim_{r=0} \frac{\arccos \psi(r)}{r}$$

and

(14) $$0 < c \leqslant \pi/a$$

then

(15) $$\psi(x) \leqslant \cos cx \quad \text{for} \quad x \in [0, a]$$

and

(16) $$c \in \left(0, \frac{\pi}{2a}\right]$$

Proof. It follows from Lemma 1 that the function $f(x) = \arccos \psi(x)$ is superadditive. We see from (13) that

$$\lim_{r \to 0} \frac{f(r)}{r} = c$$

There also exists the limit

$$\lim_{r \to 0} \frac{f(rx)}{r} = cx$$

Since $\frac{1}{n} \to 0$ and f fulfils (5) and (6), then

$$f(x) \geqslant \frac{f(x/n)}{1/n} \to cx$$

Since $c \in \left(0, \frac{\pi}{a}\right]$ and $x \in [0, a]$, then $cx \in [0, \pi]$ and

$$\cos f(x) \leqslant \cos cx \quad \text{for} \quad x \in [0, a].$$

Therefore we obtain (15), by virtue of (4).

If $c \in \left(\frac{\pi}{2a}, \frac{\pi}{a}\right]$ then from the inequality (15) it follows for $x = a$ that

$$\psi(a) \leqslant \cos ca < \cos \frac{\pi}{2} = 0$$

which contradicts (2). This ends the proof.

Theorem 2. Let the assumptions of theorem 1 be fulfilled. If moreover

(17) $$c = \pi/2a$$

then

(18) $$\psi(x) = \cos cx = \varphi(x) \quad \text{for} \quad x \in [0, a].$$

Proof. Inequality (15) is fulfilled by virtue of theorem 1. Condition (17) implies (9), whence we obtain

(19) $$\psi\left(\frac{a}{2^n}\right) \geqslant \varphi\left(\frac{a}{2^n}\right) \quad n = 1, 2, \ldots,$$

by virtue of Lemma 2, because $x_0 = a$. Inequalities (19) and (15) imply condition (11), and by virtue of Lemma 3 we have (18).

§ 4. Here we shall prove another comparison theorem for which we need the following result from [1].

Lemma 4. If g is a superadditive function fulfilling the conditions (5)—(7), then there exists an extension g^* of the function g on the interval $(0, \infty)$ which is a superadditive function and we have

(20) $$g^*(x) = g(x) \quad \text{for} \quad x \in [0, a].$$

If $\psi \in A$, then it follows from Lemma 1 that $f(x) = \arccos \psi(x)$ is a superadditive function and that the conditions (5)—(7) are fulfilled. Hence by virtue of Lemma 4 there exists a superadditive function f^* which is an extension of the function f into $(0, \infty)$.

Theorem 3. Let $\psi \in A$. If there exists a positive limit

(21) $$0 < \lim_{r \to \infty} \frac{f^*(r)}{r} = d \leqslant \frac{\pi}{a}$$

then

(22) $$\psi(x) \geqslant \cos dx \quad \text{for} \quad x \in [0, a]$$

and

(23) $$d \in \left[\frac{\pi}{2a}, \frac{\pi}{a}\right].$$

Proof. Let $\psi \in A$. From (21) we have

$$\lim_{r\to\infty} \frac{f^*(rx)}{r} = dx.$$

Since $n \to \infty$ and f^* fulfils inequality (5) then

$$f^*(x) \leqslant \frac{f^*(nx)}{n} \to dx.$$

We obtain by virtue of (20) and (21)

$$\psi(x) = \cos f(x) \geqslant \cos dx$$

for $x \in [0, a]$.

If $d \in \left(0, \frac{\pi}{2a}\right)$ then putting in (22) we have

$$\psi(a) \geqslant \cos da > \cos\frac{\pi}{2} = 0$$

which contradicts (2). This ends the proof.

REFERENCES

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