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Attempted generalization of the idea of a tensor product

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0. Introduction. In this paper I shall try to generalize the definition of a tensor product for any concrete category and to formulate the theorems known in the category of modules for any category. I wish to express my gratitude to Dr. E. Tutaj, who helped me in the section on topology, Dr. B. Grell, thanks to whom the paper has been accomplished, Mr K. M. Werber, thanks to whom the idea of these concepts was conceived.

Terminology

$f\colon A \hookrightarrow B$	— f is an injection A in B
$f\colon A \twoheadrightarrow B$	— f is a surjection A on B
$\exists!$	— there exists one and only one
$\underset{i\in I}{\times} A_i$	— a Cartesian product
$A = (\underline{A}, \operatorname{str} A)$	— an object of concrete category
$\underline{A}$	— the underlying set of A
$\operatorname{str} A$	— the structure of A
$\mathcal{C}(A, B)$	— the set of morphisms $A \to B$ in category $\mathcal{C}$ for $f \in \mathcal{C}(A, B)$ we note also $f\colon A \underset{\mathcal{C}}{\to} B$ or $f\colon A \to B$
$\mathcal{A}_\nu[N]$	— the free ν-algebra spread on N (the ν-algebra of terms on N) $N = \{0, 1, 2, \ldots\}$
$\mathcal{A}_{\nu\tau}$	— the category of ν-algebras, defined by the relation where $\nu\colon S \to N$, $\tau \in (\mathcal{A}_\nu[N])^2$
$S_i = \{k\colon \nu(k) = i\}$	
$\operatorname{top} A$	— the family of open subsets of A, i.e. $\operatorname{str} A = \operatorname{top} A$ for $A \in \operatorname{obj} \Pi$, where Π is the category of topological spaces.

1. Fundamental concepts. Let $\mathcal{C}$ be any concrete category, I be any set. Let $A\colon I \to \operatorname{obj}\mathcal{C}$ $i \in I$ $x \in \underset{j\in I\setminus\{i\}}{\times} \underline{A_j}$. We define $\operatorname{in}^x = \operatorname{in}_i^x\colon \underline{A_i} \to \underset{j\in I}{\times} \underline{A_j}$, by the formula $\operatorname{in}^x(y) = x \cup \{(i, y)\}$.

Definition: We call a map $h\colon \underset{i\in I}{\times} \underline{A_i} \to \underline{B}$ $(B \in \operatorname{obj}\mathcal{C})$ the I-morphism (polymorphism) iff $\forall i \in I$ $\forall x \in \underset{j\in I\setminus\{i\}}{\times} \underline{A_j}$ $h^x = h \circ \operatorname{in}^x \in \mathcal{C}(A_i, B)$.

Definition: A universal I-morphism $\underset{i\in I}{\times} \underline{A_i}$ in an object of category $\mathcal{C}$ is called a tensor product of the family $\{A_i\}_{i\in I}$ of the objects of the category $\mathcal{C}$, i.e. $\otimes$ is a tensor product of a family $\{A_i\}_{i\in I}$
iff

1° $\otimes$: $\underset{i\in I}{\times} \underline{A_i} \to \underline{B}$ $(B \overset{\mathrm{df}}{=} \underset{i\in I}{\otimes} A_i = \otimes A)$
2° $\otimes$ is an I-morphism
3° $\forall h$: $\underset{i\in I}{\times} \underline{A_i} \to \underline{C}$ if h is an I-morphism, then $\exists!\, \varphi \in \mathcal{C}(B, C)$: $\varphi \circ \otimes = h$

$$\begin{tikzcd}
\underset{i\in I}{\times} \underline{A_i} \arrow[dr, "\otimes"'] \arrow[dr, phantom, shift left=3ex, "h"] & \\
\underline{B} \arrow[r, "\varphi"'] & \underline{C}
\end{tikzcd}$$

THEOREM: *A tensor product is unequivocally determined exact to an isomorphism.*
Proof: Suppose two different tensor products $\otimes$ and $\overline{\otimes}$ exist, $\otimes$: $\underset{i\in I}{\times} \underline{A_i} \to \underline{B}$ and $\overline{\otimes}$: $\underset{i\in I}{\times} \underline{A_i} \to \underline{C}$.

$$\begin{tikzcd}
\underset{i\in I}{\times} \underline{A_i} \arrow[dr, "\otimes"'] \arrow[dr, phantom, shift left=3ex, "\overline{\otimes}"] & \\
\underline{B} \arrow[r, shift right, "\overline{\varphi}"'] & \underline{C} \arrow[l, shift right, "\varphi"']
\end{tikzcd}$$

$\otimes$ is an I-morphism, hence $\exists!\ \varphi \in \mathcal{C}(C, B)$: $\varphi \circ \overline{\otimes} = \otimes$
$\overline{\otimes}$ is an I-morphism, hence $\exists!\ \bar{\varphi} \in \mathcal{C}(B, C)$: $\bar{\varphi} \circ \otimes = \overline{\otimes}$
$\bar{\varphi} \circ \varphi \in \mathcal{C}(C, C)$ and $(\bar{\varphi} \circ \varphi) \circ \overline{\otimes} = \otimes$ and id_C satisfy the same properties so $\bar{\varphi} \circ \varphi = \mathrm{id}_C$ and analogously $\varphi \circ \bar{\varphi} = \mathrm{id}_B$, hence $\varphi \in \mathrm{iso}\,\mathcal{C}$

2. Problem of associativity and commutativity. Let be φ: $J \hookrightarrow I$
THEOREM: *If* $\exists \underset{i\in I}{\otimes} A_i$, $\exists \underset{j\in J}{\otimes} A_{\varphi(j)}$, *then* $\exists!\ \varphi^*$: $\underset{i\in I}{\otimes} A_i \to \underset{j\in J}{\otimes} A_{\varphi(j)}$: $\varphi^*(\underset{I}{\otimes}(x)) = \underset{j\in J}{\otimes} x_{\varphi(j)}$.
Proof: $\exists \bar{\varphi}$: $\underset{i\in I}{\times} \underline{A_i} \to \underset{j\in J}{\times} \underline{A_{\varphi(j)}}$ defined by formula $\bar{\varphi}(x) = x \circ \varphi$ i.e. $[\bar{\varphi}(x)]_j = x_{\varphi(j)}$

$$\begin{tikzcd}
\underset{i\in I}{\times} A_j \arrow[r, "\bar{\varphi}"'] \arrow[d, "\otimes"] & \underset{j\in J}{\times} A_{\varphi(j)} \arrow[d, "\otimes"] \\
\underset{i\in I}{\otimes} A_i \arrow[r, dashed, "\varphi^*"'] & \underset{j\in J}{\otimes} A_{\varphi(j)}
\end{tikzcd}$$

For proving the existence and uniqueness of φ^* we must prove, that $\otimes \circ \bar{\varphi}$ is an I-morphism.

Let $x \in \underset{j \in I \setminus \{i\}}{\times} \underline{A}_j$ $\mathrm{in}^x\colon \underline{A}_i \to \underset{j \in I}{\times} \underline{A}_i$ $\mathrm{in}^x(y) = x \cup \{i; y)\}$,

$\mathrm{in}^x(y) \circ \varphi = x \circ \varphi \cup \{(\varphi^{-1}(i); y)\} = \mathrm{in}^{x \circ \varphi}(y)$

$\otimes \circ \bar{\varphi} \circ \mathrm{in}^x(y) = \otimes \circ \mathrm{in}^x(y) \circ \varphi = \otimes \circ \mathrm{in}^{x \circ \varphi}(y)$.

Corollary: *If $\varphi\colon J \hookrightarrow\!\!\!\!\rightarrow I$ is a bijection, then* $\underset{i \in I}{\otimes} A_i \cong \underset{j \in J}{\otimes} A_{\varphi(j)}$.

THEOREM: *If $I = \bigcup_{j \in I} I_j$-separable union and the category $\mathcal{C}$ is with tensor products, then*

$$\exists!\, \varphi\colon \underset{i \in I}{\otimes} A_i \to \underset{j \in J}{\otimes} \underset{i \in I_j}{\otimes} A_i\colon \varphi(\underset{i \in I}{\otimes} x_i) = \underset{j \in J}{\otimes} \underset{i \in I_j}{\otimes} x_i$$

Proof:

$$\begin{tikzcd}
\underset{i \in I}{\times} \underline{A}_i \arrow[r] \arrow[d, "\otimes"'] & \underset{j \in J}{\times} (\underset{i \in I_j}{\times} \underline{A}_i) \arrow[d, "\times \otimes_j"] \arrow[dd, bend left, "p"] \\
 & \underset{j \in J}{\times} (\underset{i \in I_j}{\otimes} A_i) \arrow[d, "\otimes"] \\
\underset{i \in I}{\otimes} A_i \arrow[r, dashed, "\varphi"] & \underset{j \in J}{\otimes} (\underset{i \in I_j}{\otimes} A_i)
\end{tikzcd}$$

where $\otimes_j\colon \underset{i \in I_j}{\times} A_i \to \underset{i \in I_j}{\otimes} A_i$

$p(x) = \underset{j \in J}{\otimes} \underset{i \in I_j}{\otimes_j} x_i$

Let now $x \in \underset{i \in I \setminus \{i_0\}}{\times} \underline{A}_i$ $x(j) = x | I_j$ $i_0 \in I_{j_0}$

$\mathrm{in}^x(y) = x \cup \{(i_0; y)\} = \bigcup_{j \in J \setminus \{j_0\}} x(j) \cup x | I_{j_0} \cup \{(i_0; y)\}$

$= \bigcup_{j \in J \setminus (j_0)} x(j) \cup \mathrm{in}^{x(j_0)}(y) = ((\mathrm{in}^{(x(j))_{j \in J \setminus \{j_0\}}}) \cdot \mathrm{in}^{x(j_0)})(y)$

then $p \circ \mathrm{in}^x = \underbrace{\otimes \circ (\mathrm{in}^{(\otimes_j x(j))_{j \in J \setminus \{j_0\}}})}_{\text{morphism}} \circ \underbrace{\otimes_j \circ \mathrm{in}^{x(j_0)}}_{\text{morphism}}$

Then p is an I-morphism, hence $\exists!\varphi$ such that $\varphi(\otimes x) = p(x)$.

Definition: A tensor product is said to be associative, iff φ (from the preceding theorem) is an isomorphism.

3. Tensor products in $\mathcal{A}_{\nu\tau}$

THEOREM: *The category $\mathcal{A}_{\nu\tau}$ is with tensor products.*

Proof: Let $A\colon I \ni i \to (\underline{A}_i; a^i) = A_i \in \mathrm{obj}\,\mathcal{A}_{\nu\tau}$. I spread a free $\nu\tau$-algebra $B = (\underline{B}, \beta) = \mathcal{A}_{\nu\tau}[\underset{i \in I}{\times} \underline{A}_i]$ on $\underset{i \in I}{\times} \underline{A}_i$.

I determine the relation in B:

1° $\quad k \in S_0 \; x \in \underset{i \in I}{\times} \underline{A}_i \; \exists i \in I\colon x_i = \alpha_k^i$, then $x \approx \beta_k$

2° $\quad k \in S_n \; n > 0 \; x^1 \ldots x^n \in \underset{i \in I}{\times} \underline{A}_i; \; \exists i \in I\colon x_j^1 = \ldots = x_j^n$ for $j \neq i$

$$y_j = \begin{cases} x_j^1 & \text{for} \quad j \neq i \\ \alpha_k^i(x_i^1 \ldots x_i^n) & \text{for} \quad j = i \end{cases}$$

then $y \approx \beta_k(x^1 \ldots x^n)$.

I denote the least congruence containing $\approx$ by $\sim$ and introduce

$$\underset{i \in I}{\otimes} A_i = B/\sim = (\underset{i \in I}{\otimes} A_i; \gamma) \quad \text{and} \quad \otimes = s \circ \iota$$

where $\iota\colon \underset{i \in I}{\times} \underline{A}_i \hookrightarrow \underline{B}$ is the canonical injection and $s\colon \underline{B} \twoheadrightarrow \underline{B}/\sim$ is the canonical surjection.

Now I must prove that $\otimes$ is an I-morphism.

Let $k \in S_0 \; x \in \underset{i \in I}{\times} \underline{A}_i \; \exists i\colon x_i = \alpha_k^i$

$$\otimes x = [x] = [\beta_k] = \gamma_k$$

Now let $k \in S_n \; n > 0 \; x^1 \ldots x^n \in \underset{i \in I}{\times} \underline{A}_i \quad x_j^1 = \ldots = x_j^n$ for $j \neq i$

$$y_j = \begin{cases} x_j^1 & \text{for} \quad j \neq i \\ \alpha_k^i(x_i^1 \ldots x_i^n) & \text{for} \quad j = i \end{cases}$$

$$\otimes y = [y] = [\beta_k(x^1 \ldots x^n)] = \gamma_k([x^1] \ldots [x^n]) = \gamma_k(\otimes x^1 \ldots \otimes x^n) .$$

Now I shall prove the universality

$$\begin{tikzcd}
 & \underset{i \in I}{\times} \underline{A}_i \arrow[dl, "\iota"'] \arrow[dr, "\varphi"] \arrow[dll, bend right, "\otimes"'] & \\
B \arrow[d, "s"] \arrow[drr, dashed, "\bar{\varphi}"] & & \\
\underset{i \in I}{\otimes} A_i \arrow[rr, dashed] & & D
\end{tikzcd}$$

Let $\varphi\colon \underset{i \in I}{\times} \underline{A}_i \to \underline{D}$ be an I-morphism, where $D = (\underline{D}; \delta)$.

We have (from the definition of B):

$$\exists! \, \bar{\varphi} \in \mathcal{A}_{pr}(B; D)\colon \varphi = \bar{\varphi} \circ \iota$$

We have still to show $\bar{\varphi}$ is transported on $\underset{i \in I}{\otimes} A_i$.

For this purpose we must show that if $x \sim y$ then $\varphi x = \varphi y$. But it is satisfied by the relation $\approx$ which generates the relation $\sim$ and $\bar{\varphi}$ is a morphism; hence $\bar{\varphi}$ is transported on $\underset{i \in I}{\otimes} A_i$.

4. Tensor products of topological spaces

THEOREM: *The category Π of topological spaces and its continuous maps is with tensor products.*

Proof: Let $A\colon I \to \mathrm{obj}\,\Pi$ be a family of topological spaces. I determine $\underset{i\in I}{\otimes} A_i = \underset{i\in I}{\times} \underline{A_i}$ and

$$V \in \mathrm{top} \underset{i\in I}{\otimes} A_i \Leftrightarrow \forall j \in I\ \forall x \in \underset{i\in I\setminus\{j\}}{\times} A_i \quad V_x = (\mathrm{in}^x)^{-1}(V) \in \mathrm{top}\, A_j$$

I determine $\otimes = \mathrm{id}$. I-continuity is evident from the definition itself. Let now $\varphi\colon \underset{i\in I}{\times} \underline{A_i} \to \underline{B}$ be I-continuous i.e.

$\forall j \in I\ \forall x \in \underset{i\in I\setminus\{j\}}{\times} \underline{A_i}$ $\varphi \circ \mathrm{in}^x$ is continuous. Let $j \in I$ $x \in \underset{i\in I\setminus\{j\}}{\times} \underline{A_i}$ $U \in \mathrm{top}\, B$

$$[\varphi^{-1}(U)]_x = (\mathrm{in}^x)^{-1}\varphi^{-1}(U) = (\varphi \circ \mathrm{in}^x)^{-1}(U) \in \mathrm{top}\, A_j\,.$$

Therefore φ is continuous in $\underset{i\in I}{\otimes} A_i$, but is the only possible mapping commutating the diagram, and therefore $\otimes$ is a tensor product.

Remark: The topology of the tensor product is stronger than the Tichonov topology.

THEOREM: *Tensor products in a category of topological spaces and its continuous maps are associative.*

Proof: $\underset{i\in I}{\otimes} A_i = \underset{i\in I}{\times} \underline{A_i} = \underset{j\in J}{\times} \underset{i\in I_j}{\times} \underline{A_i} = \underset{j\in J}{\times} \underset{i\in I_j}{\otimes} A_i = \underset{j\in I}{\otimes} \underset{i\in I_j}{\otimes} A_i$. Therefore I have proved $\varphi\colon \underset{i\in I}{\otimes} A_i \to \underset{j\in I}{\otimes} \underset{i\in I_j}{\otimes} A_i$ is a one-to-one function. The map is continuous, because it is a morphism. Finally I shall prove the open access:

Let $V \in \mathrm{top} \underset{i\in I}{\otimes} A_i$ $x \in \underset{j\in J\setminus\{j_0\}}{\times} \underset{i\in I_j}{\times} \underline{A_i}$ $\bar{x} \in \underset{i\in I\setminus I_{j_0}}{\otimes} A_i$ where $\bar{x}_i = (x_j)_i$ for $i \in I_j$ and let $z \in \underset{i\in I_{j_0}\setminus\{i_0\}}{\otimes} A_i$

$$(\mathrm{in}^z)^{-1}(\mathrm{in}^x)^{-1}(V) = (\mathrm{in}^x \circ \mathrm{in}^z)^{-1}(V) = (\mathrm{in}^{\bar{x}\cup z})^{-1}(V) \in \mathrm{top}\, A_{i_0}.$$