ZESZYTY NAUKOWE UNIWERSYTETU JAGIELLOŃSKIEGO
DLXXX PRACE MATEMATYCZNE, ZESZYT 22 1981

On invariant measures for piecewise convex transformation on the half line

by Piotr KASPROWSKI

Abstract. A class of piecewise convex transformations of the half line $(0, \infty)$ into itself is shown to have an absolutely continuous invariant measure.

In recent years many authors have discussed the problem of the existence of invariant measures. But in most cases the spaces under considerations were compact [2, 3, 4, 5]. The question arises if one can extend these results on non-compact spaces. The purpose of the present paper is to show an example of such extension related to a theorem of A. Lasota [3].

Let $\{a_i\}_{i=0}^{n,\infty}$ be an at most countable sequence of real numbers such that

$$a_0 = 0, \quad \inf(a_{i+1}-a_i) \geqslant \varepsilon > 0$$

and let $\{\varphi_i\}_{i=0}^{n,\infty}$ be a sequence of functions defined on the intervals $[a_i, a_{i+1})$ respectively, i.e.

$$\varphi_i\colon [a_i, a_{i+1}) \to [0, \infty).$$

We admit that each function φ_i is convex $\big(\varphi_i(\alpha x+(1-\alpha)y) \leqslant \alpha\varphi_i(x)+(1-\alpha)\varphi_i(y)\big)$, and that $\varphi_i(a_i) = 0$. For any function φ_i we define its generalized inverse by setting

$$\psi_i(x) = \begin{cases} \varphi_i^{-1}(x) & \text{if } 0 \leqslant x \leqslant \varphi_i(a_{i+1}-0), \\ a_{i+1} & \text{if } \varphi_i(a_{i+1}-0) < x. \end{cases}$$

Given the sequence $\{\varphi_i\}$ we define the transformation $\tau\colon [0, \infty) \to [0, \infty)$ by formulas

$$\tau(x) = \varphi_i(x) \qquad x \in [a_i, a_{i+1}).$$

THEOREM. *If the function $\sigma(x) = \sup_i \psi_i'(x)$ is integrable and $\sigma(0) < 1$, then the transformation τ admits an absolutely continuous probabilistic invariant measure.*

Proof. For transformation τ, the Frobenius–Perron operator P_τ has the form [6]

$$(*) \qquad P_\tau f(x) = \frac{d}{dx} \int_{\tau^{-1}[0,x)} f(s)\,ds = \sum_{i=0}^{n,\infty} f(\psi_i(x))\psi_i'(x).$$

Consider the set

$$S = \left\{ f \in L^1([0,\infty)) \colon 0 \leqslant f(x) \leqslant \frac{\eta(x)}{\varepsilon(1-\eta(0))},\ f\text{-decreasing},\ \|f\| \leqslant 1 \right\}$$

where

$$\eta(x) = \sigma(x) + \frac{\varepsilon}{1+x^2} \quad \text{and} \quad 0 < \varepsilon < 1 - \sigma(0).$$

We claim that

$$P_\tau(S) \subset S$$

or more precisely that the following conditions hold

1) $\|f\| = \|P_\tau f\|$ for $f \in L^1([0,\infty))$, $f \geqslant 0$,

2) if f is decreasing, then $P_\tau f$ is decreasing,

3) $f \in S$ implies $P_\tau f \leqslant \eta \dfrac{1}{\varepsilon(1-\eta(0))}$.

Conditions 1, 2 follow immediately from formula (*). In order to prove 3 assume that $f \in S$. We have

$$\sum_{i=0}^{n,\infty} f(\psi_i(x)) \leqslant \sum_{i=0}^{n,\infty} f(\psi_i(0)) = \sum_{i=0}^{n,\infty} f(a_i) = \frac{1}{\varepsilon} \sum_{i=1}^{n,\infty} f(a_i)\varepsilon + f(0)$$

$$\leqslant \frac{1}{\varepsilon} \sum_{i=1}^{n,\infty} f(a_i)(a_i - a_{i-1}) + f(0) \leqslant \frac{1}{\varepsilon} \int_0^\infty f(x)\,dx + f(0) \leqslant \frac{1}{\varepsilon} + \frac{\eta(0)}{(1-\eta(0))} = \frac{1}{\varepsilon(1-\eta(0))}.$$

From the above inequality we obtain

$$P_\tau f(x) = \sum_{i=0}^{n,\infty} f(\psi_i(x))\psi_i'(x) \leqslant \eta(x) \sum_{i=0}^{n,\infty} f(\psi_i(x)) \leqslant \eta(x) \frac{1}{\varepsilon(1-\eta(0))},$$

which finishes the proof of the claim.

Since S is a convex set we have

$$\frac{1}{n} \sum_{k=0}^{n-1} P_\tau^k f \in S \quad \text{for } n = 1, 2, \ldots, \quad \text{and } f \in S.$$

The set S is weakly compact in $L^1([0,\infty))$ (because $\dfrac{\eta(x)}{\varepsilon(1-\eta(0))}$ is integrable) and consequently for each $f \in S$

$$\left\{ \frac{1}{n} \sum_{k=0}^{n-1} P_\tau^k f \right\}_{n=1}^{\infty}$$

is weakly relatively compact.

It is easy to see that linear combinations of functions from S are dense in $L^1([0, \infty))$.

Therefore from the Kakutani–Yosida ergodic theorem it follows that for each function $f \in L^1([0, \infty))$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} P_\tau^k f = f_0 \text{ strongly in } L^1([0, \infty))$$

and $P_\tau f_0 = f_0$.

If $||f||>0$, then $||f_0||>0$ and the function $\frac{f_0}{||f_0||}$ is the density of a probabilistic measure invariant under τ.

This completes the proof of the Theorem.

Example. Define

$$\tau(x) = 2\operatorname{tg}\left(x-\frac{\pi i}{2}\right) \quad \text{for } x \in \left[\frac{\pi i}{2}, \frac{\pi(i+1)}{2}\right),$$

then

$$\psi_i(x) = \varphi_i^{-1}(x) = \operatorname{arctg}\frac{1}{2}x+\frac{\pi i}{2}, \quad \left(\varphi_i = \tau \Big/ \left[\frac{\pi i}{2}, \frac{\pi(i+1)}{2}\right)\right)$$

and

$$\sigma(x) = \sup_i \psi_i'(x) = \frac{2}{4+x^2}$$

thus all conditions of our theorem are satisfied and the transformation τ admits an absolutely continuous probabilistic invariant measure.

References

[1] N. Dunford and J. T. Schwartz, *Linear operators. I. General Theory*, Pure Appl. Math. 7, Interscience, New York 1958.

[2] K. Krzyżewski and W. Szlenk, *On invariant measures for expanding differentiable mappings*, Studia Math. 33 (1969), 83–92.

[3] A. Lasota, *On the existence of invariant measures for Markov processes*, Ann. Polon. Math. 28 (1973), 207–211.

[4] — and G. Pianigiani, *Invariant measures on topological spaces*, Bolletino U.M.I. (5) 14–B (1977), 592–603.

[5] — and J. A. Yorke, *On the existence of invariant measures for piecewise monotonic transformations*, Trans. Amer. Math. Soc. 186 (1973), 481–488.

[6] S. M. Ulam, *A collection of mathematical problems*, Interscience Tracts in Pure Appl. Math. 8 Interscience, New York 1960.

UNIWERSYTET ŚLĄSKI
INSTYTUT MATEMATYKI
KATOWICE (POLAND)