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Local triviality of a bundle of geometric objects

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Summary. Let M, E be differentiable manifolds and $\pi\colon E \to M$ be a differentiable surjection. We suppose that for some pseudogroup Γ of local transformations of M there is defined an operation of lifting of transformations of Γ to transformations of E. In this note we prove that if Γ is transitive at each point of M and Γ is transitive then $\pi\colon E \to M$ is a locally trivial bundle.

Differentiability always means that of class C^∞.

Let M be a manifold and Γ be a pseudogroup of local diffeomorphisms of M. If φ is an element of Γ we denote by D_φ the domain of φ. D_φ and $\varphi(D_\varphi)$ are open subsets of M. Let E be a manifold and $\pi\colon E \to M$ be a surjective mapping of class C^∞. A *Γ-geometry* on $\pi\colon E \to M$ is a mapping such that to each transformation φ of Γ there corresponds a transformation

$$\tilde{\varphi}\colon \pi^{-1}(D_\varphi) \to \pi^{-1}(\varphi(D_\varphi))$$

in such a way that the following conditions are satisfied:

(A) for every φ the diagram

$$\begin{tikzcd}
\pi^{-1}(D_\varphi) \arrow[r, "\tilde{\varphi}"] \arrow[d, "\pi"'] & \pi^{-1}(\varphi(D_\varphi)) \arrow[d, "\pi"] \\
D_\varphi \arrow[r, "\varphi"] & \varphi(D_\varphi)
\end{tikzcd}$$

commutes,

(B) for two elements φ and ψ of Γ, $\widetilde{\varphi \circ \psi} = \tilde{\varphi} \circ \tilde{\psi}$,

(C) $\widetilde{id}_M = id_E$,

(D) if φ is an element of Γ and U is an open subset of D_φ then

$$\widetilde{\varphi|U} = \tilde{\varphi}|\pi^{-1}(U)\,,$$

(E) if $\Phi\colon K \to M$ is a differentiable mapping (where K is an open subset of $R \times M$) such that for all $t \in R$

$$\Phi_t\colon K_t \to M, \qquad \Phi_t(x) = \Phi(t, x)$$

belongs to Γ, where $K_t = \{x \in M\colon (t, x) \in K\}$, then the mapping

$$(t, e) \to \tilde{\Phi}_t(e)$$

is of class C^∞ on its domain (which is an open subset of $R \times E$).

This definition of Γ-geometry has been given by A. Zajtz.

If $\Gamma = \Gamma_0$ is the pseudogroup of all local diffeomorphisms of M then $\pi\colon E \to M$ with some Γ_0-geometry is called a *bundle of geometric objects* (see [3]) or a *natural bundle*.

A pseudogroup Γ is called *transitive* if for two any points x and y of M there is a transformation φ of Γ such that $\varphi(x) = y$.

We denote by $L(\Gamma)$ the set of all local vector fields on M (that is, vector fields defined on open subsets of M) such that if φ_t is a local 1-parameter group of transformations of X then φ_t is an element of Γ for all t.

A pseudogroup Γ is called *transitive at a point* x of M if the tangent space T_xM is spanned by the set

$$\{X_x\colon X \in L(\Gamma), X \text{ is defined at } x\}.$$

We shall need the following lemma (see [1] and [2], p. 183).

LEMMA. *Let Γ be transitive at a point x_0 and $X_1, \ldots, X_n$ be elements of $L(\Gamma)$ such that $X_1(x_0), \ldots, X_n(x_0)$ is a basis of $T_{x_0}M$. If we denote by $\varphi_t^{(i)}$ a local 1-parameter group of transformations of X_i in some neighbourhood of x_0 then there is $\varepsilon > 0$ such that*

$$\Phi\colon (-\varepsilon, +\varepsilon)^n \ni (t_1, \ldots, t_n) \to (\varphi_{t_1}^{(1)} \circ \ldots \circ \varphi_{t_n}^{(n)})(x_0) \in M$$

is a diffeomorphism of $(-\varepsilon, +\varepsilon)^n$ onto some open neighbourhood of x_0.

From this lemma it follows (see [1] and [2]).

COROLLARY. *If Γ is a transitive pseudogroup at each point of a connected manifold M then Γ is transitive.*

We shall now prove the following theorem.

THEOREM. *If Γ is transitive at each point of M and transitive on M and some Γ-geometry is defined on $\pi\colon E \to M$ then $\pi\colon E \to M$ is a locally trivial fibre bundle.*

Proof. Since $\pi\colon E \to M$ is surjective, by Sard's theorem there is a non-critic value x_0 of π. We fix such a point x_0 of M. Now $\pi^{-1}(x_0)$ is a submanifold of E and $\dim \pi^{-1}(x_0) = \dim E - \dim M$. Let $X_1, \ldots, X_n$ be elements of $L(\Gamma)$ such that $X_1(x_0), \ldots, X_n(x_0)$ is a basis of $T_{x_0}M$, and let $\varphi_t^{(i)}$ be a local 1-parameter group of transformations of X_i in a neighbourhood of x_0. By the Lemma, there are $\varepsilon > 0$ and an open neighbourhood U of x_0 such that

$$\Phi\colon (-\varepsilon, +\varepsilon)^n \ni (t_1, \ldots, t_n) \to (\varphi_{t_1}^{(1)} \circ \ldots \circ \varphi)_{t_n}^{(n)}(x_0) \in U$$

is a diffeomorphism. We can suppose that $\varphi_t^{(i)}$ is defined on U for all $t \in (-\varepsilon, +\varepsilon)$ and $i = 1, \ldots, n$. Let

$$U \ni x \to (t_1(x), \ldots, t_n(x)) \in (-\varepsilon, +\varepsilon)^n$$

be the inverse diffeomorphism to Φ. We define

$$\Psi\colon\ U\times\pi^{-1}(x_0)\to\pi^{-1}(U),\qquad \psi(x,e)=\overline{(\varphi^{(1)}_{t_1(x)}}\circ\ldots\circ\overline{\varphi^{(n)}_{t_n(x)})}(e)\,.$$

From (E) it follows that Ψ is a differentiable mapping and by (A), (B) and the definition of the functions $t_1,\ldots,t_n$, we have

$$\begin{aligned}(\pi\circ\Psi)(x,e)&=(\varphi^{(1)}_{t_1(x)}\circ\ldots\circ\varphi^{(n)}_{t_n(x)})(\pi(e))\\&=\Phi(t_1(x),\ldots,t_n(x))\\&=x\,,\end{aligned}$$

that is, the diagram

$$\begin{tikzcd}U\times\pi^{-1}(x_0)\arrow[rr, "\Psi"]\arrow[dr, "p_1"'] & & \pi^{-1}(U)\arrow[dl, "\pi"] \\ & U & \end{tikzcd}$$

commutes, where $p_1\colon U\times\pi^{-1}(x_0)\to U$ is the projection on the first factor.

In order to show that Ψ is a diffeomorphism we construct a differentiable mapping

$$\overline{\Psi}\colon\ \pi^{-1}(U)\to U\times\pi^{-1}(x_0)$$

$$\overline{\Psi}(e)=\big(\pi(e),\overline{(\varphi^{(1)}_{t_1(\pi(e))}}\circ\ldots\circ\overline{\varphi^{(n)}_{t_n(\pi(e))})}^{-1}(e)\big)\,.$$

To show that $\overline{\Psi}$ is well-defined we must verify the formula

$$L(e)=[\pi\circ\overline{(\varphi^{(1)}_{t_1(\pi(e))}}\circ\ldots\circ\overline{\varphi^{(n)}_{t_n(\pi(e))})}^{-1}](e)=x_0$$

for all $e\in\pi^{-1}(U)$. In fact, the conditions (A) and (B) imply that

$$\pi\circ\overline{(\varphi^{(1)}_{t_1}}\circ\ldots\circ\overline{\varphi^{(n)}_{t_n})}^{-1}=(\varphi^{(1)}_{t_1}\circ\ldots\circ\varphi^{(n)}_{t_n})^{-1}\circ\pi\,,$$

and hence

$$L(e)=(\varphi^{(1)}_{t_1(\pi(e))}\circ\ldots\circ\varphi^{(n)}_{t_n(\pi(e))})^{-1}(\pi(e))=x_0$$

because

$$(\varphi^{(1)}_{t_1(\pi(e))}\circ\ldots\circ\varphi^{(n)}_{t_n(\pi(e))})(x_0)=\Phi\big(t_1(\pi(e)),\ldots,t_n(\pi(e))\big)=\pi(e)\,.$$

Now it is easy to see that

$$\Psi\circ\overline{\Psi}=id,\qquad\overline{\Psi}\circ\Psi=id\,,$$

hence Ψ is a diffeomorphism. Thus, we have proved that E is trivial on U.

Now, let y be any point of M. There is an element φ of Γ such that $\varphi(x_0)=y$. By (B) and (C), $\tilde{\varphi}\colon\pi^{-1}(D_\varphi)\to\pi^{-1}(\varphi(D_\varphi))$ is a diffeomorphism. $U_y=\varphi(U\cap D_\varphi)$ is an open neighbourhood of y and

$$\varkappa=\tilde{\varphi}\circ\psi\circ(\varphi^{-1}\times id_F)\colon\ U_y\times F\to\pi^{-1}(U_y)$$

is a diffeomorphism, where $F = \pi^{-1}(x_0)$, and the diagram

$$\begin{tikzcd} U_y \times F \arrow[rr, "\partial\varphi"] \arrow[dr, "p_1"'] & & \pi^{-1}(U_y) \arrow[dl, "\pi"] \\ & U_y & \end{tikzcd}$$

commutes. Thus $\pi\colon E \to M$ is a locally trivial fibre bundle.

As an immediate consequence we have:

COROLLARY. *If Γ is transitive at each point of a connected manifold M and some Γ-geometry is defined on $\pi\colon E \to M$ then $\pi\colon E \to M$ is a locally trivial fibre bundle.*

COROLLARY. *Each bundle of geometric objects is locally trivial.*

References

[1] P. Lecomte, *Infinitesimal automorphisms and complete integrability of distributions*, Bull. Soc. Royal de Liège, 46 (1977), 44–50.

[2] H. I. Susmann, *Orbits of families of vector fields and integrability of distributions*, Trans. Amer. Math. Soc. 180 (1973), 173–188.

[3] S. E. Salvioli, *On the theory of geometric objects*, J. Diff. Geom. 7 (1972), 257–278.

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