ZESZYTY NAUKOWE UNIWERSYTETU JAGIELLOŃSKIEGO
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Polynomial Automorphisms of C^n

by Kamil Rusek and Tadeusz Winiarski

Abstract. The present paper consists of two parts. In the first part we give a sharp estimation of the degree of the inverse to a polynomial automorphism of C^n. The second part contains the theorem on limits of sequences of polynomial automorphisms with some of its applications.

Introduction. In Part One we give a sharp estimation of the degree of the inverse to a polynomial authomorphism of C^n. (Theorem 1.5). This result is based on the affine version of Bezout's theorem. Since we can not find in bibliography such a version of Bezout's theorem we present its proof based on Rouche's theorem and on the well known projective version of Bezout's theorem (cf. [6], p. 191).

In Part Two a structure of some sets of polynomial automorphisms is examined. In particular, a formal analogue of known Cartan's theorem on sequences of biholomorphisms of a bounded domain (cf. [3], p. 78) is proved (Theorem 2.3). The constructibility and algebraicity of the sets of polynomial automorphisms of bounded degree and fixed jacobian are also given (Theorems 2.5 and 2.6).

We introduce some notations and definitions which will be frequently used in the sequel.

Let M, N be complex vector spaces. Let $F\colon M \to N$ be a polynomial mapping and let $F = \sum_{0}^{\infty} F_k$ be the decomposition of F into the homogeneous components. As usual, the number $\deg F = \max\{k\colon F_k \neq 0\}$ is called the degree of F. Let us write, for $k \in \mathbf{N}$,

$$\mathscr{P}^k(M, N) = \{F\colon M \to N,\ F \text{ is a polynomial and } \deg F \leqslant k\},$$

$$\mathscr{P}^k(M) = \mathscr{P}^k(M, M),\ \boldsymbol{C}^k[M] = \mathscr{P}^k(M, \boldsymbol{C}),$$

$$\mathscr{P}(M, N) = \bigcup_{1}^{\infty} \mathscr{P}^k(M, N),\ \mathscr{P}(M) = \mathscr{P}(M, M),$$

$$\boldsymbol{C}[M] = \mathscr{P}(M, \boldsymbol{C}).$$

Definition. We call a mapping $F\colon M \to M$ a *polynomial automorphism* of the space M, if F is one-to-one polynomial transformation of M onto M and the mapping F^{-1} is polynomial too.

We write

$$\mathscr{P}^k_A(M) = \{F \in \mathscr{P}^k(M)\colon F \text{ is a polynomial automorphism of } M\},$$

$$\mathscr{P}_A(M) = \bigcup_1^\infty \mathscr{P}^k_A(M).$$

Assume that M is a finite-dimensional space.

The mapping $J\colon \mathscr{P}(M) \ni F \to \det F' \in \boldsymbol{C}[M]$ is a polynomial mapping. Thus for $\lambda \in \boldsymbol{C}$, $k \in \mathbf{N}$, the set

$$\mathscr{P}^k_\lambda(M) = \{F \in \mathscr{P}^k(M)\colon J(F) = \lambda\}$$

is algebraic in the vector space $\mathscr{P}^k(M)$.

We shall need also some notations from the projective geometry.

If M is a finite dimensional complex vector space, we denote by $\boldsymbol{P}(M)$ the respective projective space, i.e. the space of vector lines through the origin in M. Define $\varphi\colon M \ni z \to \boldsymbol{P}((1, z)) = \boldsymbol{C}(1, z) \in \boldsymbol{P}(\boldsymbol{C} \times M)$. For an arbitrary set $S \subset M$ the set $\varphi(S)$ will be denoted by $\boldsymbol{P}(S)$.

Part One. In the beginning we present an affine version of Bezout's theorem.

PROPOSITION 1.1. *Let* $\dim M = n \geqslant 2$ *and* $F = (f_1, \ldots, f_n)\colon M \to \boldsymbol{C}^n$ *be a polynomial mapping such that* $F^{-1}(0) = \{a_1, \ldots, a_k\}$. *Then*

$$\nu(F) = \sum_{i=1}^{k} m_{a_i} F \leqslant \deg f_1 \ldots \deg f_n,$$

where $m_{a_i} F$ *denotes the multiplicity of* F *at the point* a_i (see e.g. Stoll [7] for the definition).

The proof is preceded by two lemmas which permit us to use the projective version of Bezout's theorem

LEMMA 1.2. *Let* V_1, V_2 *be homogeneous algebraic sets in* M *of pure dimensions* k, l, *respectively* $(1 \leqslant k \leqslant l < n)$. *Then there exists a linear isomorphism* σ *of* M *such that*

$$\dim(V_1 \cap \sigma(V_2)) < k.$$

Proof. Let $V_1 = V_1^1 \cup \ldots \cup V_1^s$ be the decomposition of V_1 into its irreducible components (which are also homogeneous). Let us fix a point a_j in $\boldsymbol{P}(V_1^j)$ for $j = 1, \ldots, s$, and a hyperplane X in M such that $X \cap a_j = 0$, $j = 1, \ldots, s$. Let $Y \subset M$ be a vector line, complementary to X such that $Y \cap V_2 = \{0\}$. Fix in turn a point $y_0 \in Y \setminus \{0\}$ and two constants $r, R > 0$ such that

$$(a_1 \cup \ldots \cup a_s) \cap (y_0 + X) \subset y_0 + B_X(R),$$

$$y_0 + B_X(r) \subset (y_0 + X) \setminus V_2$$

($B_X(t)$ denotes the ball of center 0 and radius t in the space X). We define, for $\lambda > \frac{R}{r}$, the mapping

$$\sigma\colon X+Y \ni x+y \to \lambda x+y \in M .$$

Then σ is an isomorphism and $a_j \notin \boldsymbol{P}(\sigma(V_2))$. Thus $\dim V_1 \cap \sigma(V_2) < k$.

Let $\mathscr{G} = [\mathrm{End}(M)]^n$, $\mathscr{G}_0 = [\mathrm{Isom}(M)]^n$, where $\mathrm{End}(M)$ (resp.: $\mathrm{Isom}(M)$) denotes the set of all linear endomorphisms (resp.: isomorphisms) of M. If $\sigma = (\sigma_1, \ldots, \sigma_n) \in \mathscr{G}$ and $G = (g_1, \ldots, g_n)\colon M \to \boldsymbol{C}^n$ we put $G_\sigma = (g_1 \circ \sigma_1, \ldots, g_n \circ \sigma_n)$. Under these notations we have

LEMMA 1.3. *Let $H = (h_1, \ldots, h_n)\colon M \to \boldsymbol{C}^n$, where h_i is a homogeneous polynomial and $\deg h_i \geq 1$, $i = 1, \ldots, n$. Then the set*

$$X_H = \{\sigma \in \mathscr{G}\colon H_\sigma^{-1}(0) \neq \{0\}\}$$

is algebraic, properly included in the space $\mathscr{G}$.

Proof. The set X_H is the projection on $\mathscr{G}$ of the constructible set

$$\{(\sigma, z) \in \mathscr{G} \times M\colon H_\sigma(z) = 0,\ z \neq 0\},$$

and, on the other hand, of the closed set

$$\{(\sigma, z) \in \mathscr{G} \times \{z \in M\colon |z| = 1\}\colon H_\sigma(z) = 0\}.$$

Thus, the set X_H is algebraic in $\mathscr{G}$ (being constructible and closed). Applying Lemma 1.2 we can easily deduce that $(\mathscr{G} \setminus X_H) \cap \mathscr{G}_0 \neq \emptyset$. This completes the proof.

Proof of Proposition 1.1. Let $F^* = (f_1^*, \ldots, f_n^*)$ (if $g \in \boldsymbol{C}[M]$, $g = g_0 + \ldots + g_k$ is the natural decomposition of g into the homogeneous components, we put $g^* = g_k$). Then $\deg f_i^* \geq 1$ for $i = 1, \ldots, n$, and, by Lemma 1.3, the set X_{F^*} is algebraic, properly included in $\mathscr{G}$. If $I^n = (I, \ldots, I) \notin X_{F^*}$ (I denotes the identity), our statement follows immediately from the projective version of Bezout's theorem.

Let us suppose that $I^n \in X_{F^*} \cap \mathscr{G}_0$. It is easy to find an affine complex line L in $\mathscr{G}$ and an open neighborhood $U_1 \subset \mathscr{G}_0$ of the point I^n such that $(L \cap X_{F^*}) \cap U_1 = \{I^n\}$ and the set $\Omega = L \cap U_1$ is connected.

Let us fix $R > 0$ such that $F^{-1}(0) \subset B(R)$ and define the holomorphic mapping

$$\Psi\colon \Omega \times B(2R) \ni (\sigma, z) \to F_\sigma(z) \in \boldsymbol{C}^n$$

Since $\Psi(I^n, z) \neq 0$ for every $z \in B(2R) \setminus B(R)$, we can find a connected neighborhood U of I^n, $U \subset \Omega$ such that $F_\sigma(z) \neq 0$ for $\sigma \in U$, $z \in \overline{B(2R)} \setminus B(R)$. We know that $\# F_\sigma^{-1}(0) < \infty$ for every $\sigma \in U$ (since $\dim F_\sigma^{-1}(0) = 0$ for $\sigma \in U$). Therefore, the Rouché theorem (see Stoll [7]) implies that the function $U \ni \sigma \to \nu(F_\sigma|_{B(R)}) \in \boldsymbol{N}$ is constant. Let us fix an element σ in $U \setminus \{I^n\}$. Applying the projective version of Bezout's theorem we obtain the desired estimation, namely

$$\nu(F) = \nu(F_{I^n}) \leq \nu(F_\sigma) = \deg f_1 \ldots \deg f_n .$$

The following theorem summarizes the properties of invertible polynomial transformations.

THEOREM 1.4. *Let F: $M \to M$, $\dim M = n$, be one-to-one polynomial mapping. Then*
(i) *F is a polynomial automorphism of M;*
(ii) $\deg F^{-1} \leqslant (\deg F)^{n-1}$.

Proof. (i) Following the classical theorem of Clements, the set $F(M)$ is open and the inverse mapping G: $F(M) \to M$ is holomorphic. Since the set $S = M \setminus F(M)$ is closed and constructible, it is algebraic. Suppose that $\operatorname{codim} S = 1$ and let S_0 be an irreducible component of S of codimension one. Then there exists a nonconstant polynomial $g \in \boldsymbol{C}[M]$ such that $S_0 = g^{-1}(0)$. Since the polynomial $g \circ F \in \boldsymbol{C}[M]$ has no zeros in M, it is constant. This contradicts the assumption that $\operatorname{codim} S_0 = 1$. Therefore $\operatorname{codim} S \geqslant 2$. By the suitable version of the extension theorem there exists a holomorphic mapping $\tilde{G}$: $M \to M$ such that $\tilde{G}|_{F(M)} = G$. By the identity principle $F \circ \tilde{G} = \mathrm{id}_M$. Hence $F(M) = F(\tilde{G}(M)) = M$, i.e. the mapping F is one-to-one transformation of M onto M. The fact that F^{-1} is a polynomial mapping is a special case of Serre's theorem ([5], Prop. 9). A more elementary proof can be given (see e.g. [4], [8]) but not by the brutal computation. Thus F is a polynomial automorphism of M.

(ii). Let us fix a vector line $L_1 \subset M$ and its complementary subspace M_1 such that $F^{-1} = h + H$, where h: $M \to L_1$ is a polynomial satisfying the condition $\deg F^{-1} = \deg h$. Let us observe that the set $V = h^{-1}(0) = F(M_1)$ is irreducible and the polynomial h is irreducible too.

An affine line $L \subset M$ can be found with $\# V \cap L = \deg h = s$. Indeed, we can find, in a standard way, two vector subspaces X, Y of M such that $M = X + Y$, $\dim X = n-1$, $\dim Y = 1$ and

$$h(x+y) = \alpha\big(y^s + h_1(x)y^{s-1} + \ldots + h_s(x)\big) \quad \text{for } x \in X,\ y \in Y,$$

where $\alpha \in \boldsymbol{C} \setminus \{0\}$, $\deg h_j \leqslant j$ for $j = 1, 2, \ldots, s$.

It is easy to see that there exists $a \in X$ such that $\# V \cap (a + Y) = s$. Then the line $L = a + Y$ has the desired property.

Now, using the properties of polynomial automorphism F^{-1} and Proposition 1.1, we get

$$\deg F^{-1} = \deg h = \# V \cap L = \# F^{-1}(V \cap L) = \# M_1 \cap F^{-1}(L) \leqslant (\deg F)^{n-1}.$$

Here is an immediate consequence of Theorem 1.4:

COROLLARY 1.5. *Let F: $\boldsymbol{C}^n \to \boldsymbol{C}^n$ be one-to-one polynomial mapping with $F(0) = 0$. Let us define, for $r > 0$,*

$$\varrho(F; r) = \inf\{|F(x)|: |x| = r\}$$

and let $\alpha = (1/\deg F)^{n-1}$.

Then there exist $r_0 > 0$, $A > 0$ such that

$$\varrho(F; r) \geqslant A r^{\alpha} \quad \textit{for } r > r_0.$$

Remark 1.6. In the case $M = C^2$, the inequality (ii) implies the equality $\deg F = \deg F^{-1}$ for $F \in \mathscr{P}_A(C^2)$. On the other hand, we obtain this equality immediately from known Jung's theorem (see e.g. [1], [2]). This difficult theorem states that the group $\mathscr{P}_A(C^2)$ is generated by the set $\operatorname{Isom}(C^2) \cup \mathscr{P}_A(C^2)$, where

$$\mathscr{P}_A(C^2) = \{F \in \mathscr{P}_A(C^2)\colon F(x, y) = (x, y+\lambda x^r),\ r>1,\ \lambda \in C\setminus\{0\}\}.$$

It is easy to see that the theorem of Jung may be formulated in the following equivalent form: if $F = (f_1, f_2) \in \mathscr{P}_A(C^2)$ then $\deg f_1 | \deg f_2$ or $\deg f_2 | \deg f_1$.

Remark 1.7. When $n \geqslant 3$, the inequality (ii) is, in general, sharp. As the class of examples we propose the mappings

$$F_\alpha\colon C^n \ni x \to (x_1, x_2+x_1^{\alpha_1}, \ldots, x_n+x_{n-1}^{\alpha_{n-1}}) \in C^n$$

for $\alpha = (\alpha_1, \ldots, \alpha_{n-1}) \in \mathbf{N}^{n-1}$.

Let us observe that $\deg F_\alpha = \max\{\alpha_1, \ldots, \alpha_{n-1}\}$ and $\deg F_\alpha^{-1} = \alpha_1 \ldots \alpha_{n-1}$. These examples show also that our estimation is the best possible.

Part Two. We need two elementary lemmas which hold true for every finite dimensional space M (real or complex).

LEMMA 2.1. *Let $\{F_\nu\}_{\nu \in N}$ be a sequence of homeomorphisms of M onto M, locally uniformly convergent in M to a local homeomorphism $F\colon M \to M$. Then the mapping F is one-to-one.*

Proof. Suppose that there exist $x_1, x_2 \in M$, $x_1 \neq x_2$ such that $F(x_1) = F(x_2) = y$. Fix $\delta_0>0$ such that for every $0<\delta<\delta_0$ there exists an open bounded neighborhood $U_j(\delta)$ of the point x_j ($j = 1, 2$) with the properties: $\overline{U_1(\delta)} \cap \overline{U_2(\delta)} = \emptyset$ and $F|_{U_j(\delta)}\colon U_j(\delta) \to B(y, \delta)$ is a homeomorphism for $j = 1, 2$ ($B(y, \delta)$ denotes the open ball with the center y and the radius δ). Find $\varepsilon>0$ such that $0<2\varepsilon<\delta_0$. Since $F_\nu \to F$ uniformly in $\overline{U_j(\delta_0)}$ and the mappings F_ν are homeomorphic, there exists $\nu_0 \in \mathbf{N}$ such that for every $\nu>\nu_0$, $j = 1, 2$, we have

$$F_\nu(\partial U_j(\varepsilon)) \subset B\left(y, \frac{3}{2}\varepsilon\right) \setminus B\left(y, \frac{\varepsilon}{2}\right) \quad \text{and} \quad F_\nu(x_j) \in B\left(y, \frac{\varepsilon}{2}\right).$$

Hence $B\left(y, \frac{\varepsilon}{2}\right) \subset F_\nu(U_j(\varepsilon))$, i.e. $F_\nu^{-1}\left(B\left(y, \frac{\varepsilon}{2}\right)\right) \subset U_j(\varepsilon)$, $j = 1, 2$. This contradicts the assumption that $U_1(\varepsilon) \cap U_2(\varepsilon) = \emptyset$.

LEMMA 2.2. *Let $\{F_\nu\}_{\nu \in N}$ be a sequence of homeomorphisms of M onto M, locally uniformly convergent in M to a homeomorphism F of M onto M. Then*

(i) *the sequence $\{F_\nu^{-1}\}_{\nu \in N}$ is locally uniformly bounded;*

(ii) *$F_\nu^{-1}(y) \to F^{-1}(y)$ for every $y \in M$.*

Proof. (i). Let us fix $y_0 \in M$ and $\varepsilon > 0$. Let $x_0 = F^{-1}(y_0)$, $A(y, \varepsilon) = F^{-1}(B(y, \varepsilon))$. Since the set $A = \partial A(y, 2\varepsilon) \cup \{x_0\}$ is compact, there exists $\nu_0 \in N$ such that for every $\nu > \nu_0$ and every $x \in A$, $|F_\nu(x) - F(x)| < \frac{\varepsilon}{8}$. This means that

$$F_\nu(\partial A(y, 2\varepsilon)) \subset B(y, 3\varepsilon) \setminus B(y, \varepsilon) \text{ and } F_\nu(x_0) \subset B(y_0, \varepsilon) \text{ for } \nu > \nu_0.$$

Hence, for $\nu > \nu_0$, $B(y_0, \varepsilon) \subset F_\nu(A(y, 2\varepsilon))$, i.e. $F_\nu^{-1}(B(y, \varepsilon) \subset A(y, 2\varepsilon)$.

(ii) Let $x_\nu = F_\nu^{-1}(y_0)$, $\nu = 1, 2, \ldots$ It follows from (i) that the sequence $\{x_\nu\}$ is bounded and by assumption $F_\nu \rightarrow F$ uniformly on $\overline{\{x_\nu\}}$. Hence

$$\lim_{\nu \to \infty} (F_\nu(x_\nu) - F(x_\nu)) = \lim_{\nu \to \infty} (y_0 - F(x_\nu)) = 0 .$$

Thus $x_\nu \to x_0 = F^{-1}(y_0)$ and the proof is complete.

Now we obtain easily

THEOREM 2.3. *Let $\{F_\nu\}_{\nu \in N}$ be a sequence of polynomial automorphisms of M, locally uniformly convergent in M to a polynomial mapping F. Suppose that $J(F)$ is not identically zero in M.*
Then

(i) *F is a polynomial automorphism of M;*
(ii) *$F_\nu^{-1} \to F^{-1}$ locally uniformly in M.*

Proof. By the Weierstrass theorem $J(F) \in \mathbb{C} \setminus \{0\}$, i.e. F is a local homeomorphism and moreover, one-to-one mapping (by Lemma 2.1). Following Theorem 1.4, F is a polynomial automorphism of M. Applying Lemma 2.2 and the Vitali theorem, we obtain the statement (ii).

COROLLARY 2.4. *The set $\mathscr{P}_A(M)$ endowed with the compact-open topology and the standard group-operations is a topological group.*

THEOREM 2.5. *For every $k \in \mathbf{N}$ the set $\mathscr{P}_A^k(M)$ is constructible in the vector space $\mathscr{P}^k(M)$.*

Proof. Define the linear mapping

$$\Phi \colon \mathscr{P}^k(M) \ni F \to (M \times M \ni (x, y) \to F(x) - F(y) \in M) \in \mathscr{P}^k(M \times M, M) .$$

If we set

$$\Omega = \{H \in \mathscr{P}^k(M \times M, M) \colon H^{-1}(0) \cap (M^2 \setminus \Delta) = \emptyset\} ,$$

where $\Delta = \{(x, x) \colon x \in M\}$, then

$$\Omega' = \mathscr{P}^k(M \times M, M) \setminus \Omega = \{H \in \mathscr{P}^k(M \times M, M) \colon \exists (x, y) \in M^2 \setminus \Delta \text{ such that } H(x, y) = 0\} .$$

Define, in turn, the polynomial mapping

$$\varphi \colon \mathscr{P}^k(M \times M, M) \times M \times M \ni (H, x, y) \to H(x, y) \in M .$$

Let $p\colon \mathscr{P}^k(M\times M, M)\times M\times M \to \mathscr{P}^k(M\times M, M)$ denotes the natural projection and let

$$\Lambda = \varphi^{-1}(0)\cap\big(\mathscr{P}^k(M\times M, M)\times(M^2\setminus\Delta)\big).$$

It is easy to see that $p(\Lambda) = \Omega'$. Therefore, by Chevalley's theorem, the set Ω is constructible in the space $\mathscr{P}^k(M\times M, M)$. Thus the set $\Phi^{-1}(\Omega)$ is constructible in the space $\mathscr{P}^k(M)$. Observe that

$$\Phi^{-1}(\Omega) = \{F\in\mathscr{P}^k(M)\colon F \text{ is one-to-one}\}.$$

Applying Theorem 1.4 we obtain the equality $\Phi^{-1}(\Omega) = \mathscr{P}^k_A(M)$. This concludes the proof.

Observe that every set $\mathscr{P}^k_{A,\lambda}(M) = \mathscr{P}^k_A(M)\cap\mathscr{P}^k_\lambda(M)$ ($k\in\mathbf{N}$, $\lambda\in\mathbf{C}\setminus\{0\}$) is closed (by Theorem 2.3). Therefore, Theorem 2.5 implies immediately

THEOREM 2.6. *For every $k\in\mathbf{N}$ and $\lambda\in\mathbf{C}\setminus\{0\}$, the set $\mathscr{P}^k_{A,\lambda}(M)$ is algebraic in the space $\mathscr{P}^k(M)$.*

The following question arises:

Problem 2.7. *Is the set $\mathscr{P}^k_{A,\lambda}(M)$ irreducible?*

Remark 2.8. Theorem 2.6 may be an argument for the positive solution of so-called "jacobian problem" for the field C (see [1]). It can be formulated as a question: does the equality $\mathscr{P}^k_1(M) = \mathscr{P}^k_{A,1}(M)$ hold true for every $k\in\mathbf{N}$?

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