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A note on degenerate pseudo-Riemannian manifolds

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1. Introduction. In this paper we shall determine the first structure tensor of pseudo-Riemannian structure with degenerate metric and prove the necessary and sufficient condition for the existence of torsion-free $O(f)$-connection.

The deformation tensor for torsion-free $O(f)$-connection is given. Let M be a real Hausdorff paracompact smooth manifold, $\dim(M) = n$ and denote by g a positive semi-definite covariant symmetric tensor field of degree 2 on M.

DEFINITION 1.1. The Riemannian manifold (M, g) is called an *r-time degenerate* if $\operatorname{rank}(g) = n-r$, where $1 \leqslant r \leqslant n-1$.

DEFINITION 1.2. If there is an atlas of local maps in which the tensor field g has the form

$$g = \left[\begin{array}{c|c} \boldsymbol{g} & 0 \\ \hline 0 & 0 \end{array}\right] \quad \text{and } \det(\boldsymbol{g}) \neq 0, \quad \text{where } \boldsymbol{g} = (g_{ab}) \ a, b = 1, \ldots, n-r$$

then (M, g) is called *reducible*.

DEFINITION 1.3. If there is an atlas of local maps in which (M, g) is reducible and $\partial_A \boldsymbol{g} = 0$ for $A = n-r+1, \ldots, n$ then (M, g) is called *absolutely reducible*.

THEOREM 1.1. (O. Vogel). *Let (M, g) be an r-time degenerate Riemannian manifold. (M, g) admits a metric torsionless linear connection if and only if it is absolutely reducible.*

We shall prove an equivalent version of Vogel's theorem in the G-structure language and discuss the variety of the connection in questions. For proof of Theorem 1.1. see [1].

2. The degenerate pseudo-Riemannian structure and flatness problem. Let f be an r-time degenerate symmetric bilinear form on $V \times V$, where we denote by V the real vector space, $\dim(V) = n$, the signature of f is defined by $(+, \ldots, +, -, \ldots, -, \underbrace{0, \ldots, 0}_{r\text{-times}})$.

The degenerate pseudo-orthogonal group of f is

$$0(f) := \{a \in GL(V) | \forall x, y \in V\ f(ax, ay) = f(x, y)\}.$$

The Lie algebra of $O(f)$ is

$$o(f) := \{A \in gl(V) | \forall x, y \in V\ f(Ax, y) = -f(x, Ay)\}.$$

Let $L(M, \boldsymbol{p}, GL(n, R))$ be the principal bundle of linear frames over M. We denote the reduced bundle of $L(M, \boldsymbol{p}, GL(n, R))$ to the Lie subgroup $O(f)$ by $O(M, p, O(f))$ where $p := \boldsymbol{p}|O$ is the canonical projection.

DEFINITION 2.1. The $O(M, p, O(f))$-structure is called a *degenerate pseudo-Riemannian structure.*

Let g be a Lie subalgebra of $\mathrm{gl}(V)$.

DEFINITION 2.2. The *first prolongation* $g^{(1)}$ of g is defined by

$$g^{(1)} := g \otimes V^* \cap V \otimes S^2(V^*),$$

where $S^2(V^*)$ denotes the spaces of the symmetric tensor of degree 2 over V. In general

$$g^{(k)} := g \otimes \underbrace{V^* \otimes V^* \otimes \ldots \otimes V^*}_{k\text{-times}} \cap V \otimes S^{k+1}(V^*).$$

We say that g is of *finite type* if $g^{(k)} = 0$ for some k. If $g^{(k)} \neq 0$ for all k then g is said to be of *infinite type.*

LEMMA 2.1. *Let* $\ker(f) := \{x \in V | \forall y \in V\ f(x, y) = 0\}$ *and* $\boldsymbol{V}$ *be a complement to* $\ker(f)$ *in* V. *Then*

(1) $o(f) = o(\boldsymbol{f}) \oplus \ker(f) \otimes V^*$, *where* $\boldsymbol{f} := f|\boldsymbol{V}$,

(2) $o(f)^{(1)} = \ker(f) \otimes S^2(V^*)$ *and the Lie algebra* $o(f)$ *is of infinite type.*

Proof. ad(1). Corresponding to the direct sum decomposition $V = \boldsymbol{V} \oplus \ker(f)$, there is decomposition of $V \otimes V^*$:

$$V \otimes V^* = \boldsymbol{V} \otimes \boldsymbol{V}^* \oplus \boldsymbol{V} \otimes \ker(f)^* \oplus \ker(f) \otimes \boldsymbol{V}^* \oplus \ker(f) \otimes \ker(f)^*$$

but for all $A \in o(f)$, $x \in V$, $k \in \ker(f)$ since

$$f(Ak, x) = -f(k, Ax) = 0 \text{ it follows that } A(\ker(f)) \subset \ker(f).$$

Hence in the corresponding decomposition of any $A \in o(f)$ the $\boldsymbol{V} \otimes \ker(f)^*$ component is zero. Let $\boldsymbol{A}$ denote the restriction of A to $\boldsymbol{V}$ and $\boldsymbol{A}_1$ its $\boldsymbol{V} \otimes \boldsymbol{V}^*$ component, then for all $x = \boldsymbol{x} + k$, $y = \boldsymbol{y} + l \in \boldsymbol{V} \oplus \ker(f) = V$, we have

$$\begin{aligned} f(Ax, y) + f(x, Ay) &= f(\boldsymbol{A}\boldsymbol{x}, \boldsymbol{y}) + f(\boldsymbol{x}, \boldsymbol{A}\boldsymbol{y}) = 0 \\ &= \boldsymbol{f}(\boldsymbol{A}_1\boldsymbol{x}, \boldsymbol{y}) + \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{A}_1\boldsymbol{y}) \end{aligned}$$

hence $\boldsymbol{A}_1 \in o(\boldsymbol{f})$ and $o(f) \subset o(\boldsymbol{f}) \oplus \ker(f) \otimes V^*$.

On the other hand, let $a \in \ker(f) \otimes V^*$ and $A \in o(f)$ then

$$(a+A)x = ax + Ax$$

and

$$f((a+A)x, y) + f(x, (a+A)y) = f(ax, y) + f(x, ay) + f(Ax, y) + f(x, Ay) = 0$$

but $ax, ay \in \ker(f)$ it is immediate that $o(f) \oplus \ker(f) \otimes V^* \subset o(f)$

and (2).

For all $t \in o(f)^{(1)}$, $x, y \in V$

$$f(t_x y, z) = f(t_y x, z) = -f(x, t_y z) = -f(x, t_z y) = f(t_z x, y) = f(t_x z, y)$$
$$= -f(z, t_x y) = -f(t_x y, z)$$

thus $f(t_x y, z) = 0$ for arbitrary $x, y \in V$ and $z \in V$. Hence $t_x y \in \ker(f)$. Also

$$o(f)^{(1)} \subset \ker(f) \otimes S^2(V^*).$$

Because we have $o(f) \oplus \ker(f) \otimes V^* = o(f)$ hence

$$o(f)^{(1)} = (o(f) \oplus \ker(f) \otimes V^*) \otimes V^* \cap V \otimes S^2(V^*)$$

and $\ker(f) \otimes S^2(V^*) \subset o(f)^{(1)}$. Thus $o(f)^{(1)} = \ker(f) \otimes S^2(V^*)$.

It is evident that $\ker(f) \otimes S^2(V^*)$ is of infinite type, hence it appears that $o(f)$ is of infinite type.

Given a graded linear Lie algebra $\bigoplus_p g_p$, we define the cohomology group as follows (see, [2]):

$$C^{p,q} := g_{p-1} \otimes \Lambda^q(V^*).$$

We define the coboundary operator

$$\partial^{p,q}\colon C^{p,q} \to C^{p-1,q+1}$$

by

$$(\partial)^{p,q_t}(x_1, \ldots, x_{q+1}) := \sum_{i=1}^{q+1} (-1)^{i+1} [x_i, t(x_1, \ldots, x_i, \ldots, x_{q+1})]$$

then $\partial^{p,q} \circ \partial^{p+1,q-1} = 0$.

We shall denote *Spencer's cohomology groups* by

$$H^{p,q} := \frac{\ker \partial^{p,q}}{\operatorname{im} \partial^{p+1,q-1}}$$

We denote by ∂ the coboundary operator $\partial^{1,1}$.

Let θ be the *canonical form* of $O(M, p, O(f))$-structure, let V_l be the subspace of vectors tangent to the fibre throught l. Then V_l is isomorphic with the Lie algebra $o(f)$. A complement to V_l in $T_l O$ is called a *horizontal space at l*. Let H_l be a horizontal space at frame l. We define

$$C_H(l)(\theta(X), \theta(Y)) := d\theta(X, Y),$$

where

$$C_H\colon l \in 0 \to C^{0,2}.$$

The equivalence class $C(l)$ of $C_H(l)$ modulo im is independent of horizontal distribution H and defines a function

$$C\colon l\ 0 \to H^{0,2}$$

called the *first structure function.*

The first structure function commutes with an $O(f)$-automorphism q in the sense that $C \circ q = C$. It vanishes for the standard flat $O(f)$-structure. Thus a necessary condition for local flatness is $C = 0$. Such a $O(f)$-structure is called a first order flat.

THEOREM 2.1. (see, [3]). *Flatness of the first order is a necessary and sufficient condition for the existence of a torsionfree connection on the G-structure.*

3. Main result. Let $i(M)$ be a Lie algebra of *infinitesimal $O(f)$-automorphisms.* We define the r-dimensional distribution

$$\ker(g)\colon x \in M \to \ker(g)_x \subset T_x M,$$

where

$$g(X, Y)_x := f(l^{-1}X, l^{-1}Y) \quad \text{for all } X, Y \in T_x M,\ x = p(x)$$

and

$$\ker(g)_x := \{X \in T_x M \mid \forall\, Y \in T_x M\ \ g(X, Y)_x = 0\}$$

THEOREM 3.1. *$O(M, p, O(f))$-structures admit a torsionfree connection if and only if for all $x \in M$ $\ker(g)_x \subset i(M)$*

4. Proof of Theorem 3.1.

LEMMA 4.1. *Let V be a real n-dimensional vector space. If $W \leqslant V$ (subspace), $T \in V \otimes \otimes \Lambda^2(V^*)$ such that $T(W, W) \subset W$ then exist $S \in V \otimes V^* \otimes V^*$ such that for all $x \in V$*

a) $S_x W \subset W$,

b) $\partial S = T$.

Proof. Since every element T leaves W invariant this induces a linear transformation of $\frac{V}{W}$. We set

$$g_W := \{t \in V \otimes V^* \mid tW \subset W\}$$

and

$$\tau := \{T \in V \otimes \Lambda^2(V^*) \mid T(W, W) \subset W\}.$$

For

$$S \in g_W \otimes V^*, \quad x, y \in W \quad (\partial S)(x, y) = (S_x y - S_y x) \in W$$

and

$$\tau \supset \partial(g_W \otimes V^*).$$

Since

$$\begin{aligned}\dim(\operatorname{im}\partial) &= \dim(g_W \otimes V^*) - \dim(\ker\partial)\\ &= \dim(g_W \otimes V^*) - \dim g_W^{(1)}\end{aligned}$$

we obtain the following, if $k = \dim(W)$ then

$$\dim(g_W) = n^2 - n(n-k), \qquad \dim(\mathrm{gl}(V)) = \tfrac{1}{2}n^2(n+1)$$

and

$$\begin{aligned}\dim(\partial(g_W \otimes V^*)) &= n[n^2 - n(n-k)] + \tfrac{1}{2}k(k+1)(n-k) - \tfrac{1}{2}n^2(n+1)\\ &= \dim\tau\end{aligned}$$

it follows that

$$\tau = \partial(g_W \otimes V^*)$$

LEMMA 4.2. (see, [2]). *Let h be a non-degenerate symmetric bilinear form on $V \times V$, then we have $\partial(o(h) \otimes V^*) = V \otimes \Lambda^2(V^*)$.*

Proof. Assuming that we have element $A \in \ker\partial$ it follows that for all $x, y \in V$ $A_x y = A_y x$ and $o(h) \ni A_x$ thus

$$h(A_x y, z) = -h(y, A_x z) = -h(y, A_z y) = h(A_z y, x) = h(A_y z, x) = -h(z, A_y x).$$

Thus $h(A_x y, z) = 0$. Since z is arbitrary and h is non-degenerate, for all $x, y \in V$ $A_x y = 0$. Hence $A_x = 0$ for all $x \in V$. This implies $A = 0$. Furthermore

$$\dim(o(h) \otimes V^*)) = \dim(V \otimes \Lambda^2(V^*))$$

but $\ker\partial = 0$, hence is an isomorphism.

LEMMA 4.3. *Let f be an r-time degenerate symmetric bilinear form on $V \times V$. If $T \in V \otimes \otimes \Lambda^2(V^*)$ then the following conditions are mutually equivalent:*
(1) *exist t $o(f) \times V$ such that $T = \partial t$*
(2) *for all $k \in \ker(f)$, $x, y \in V$ $f(T(k, x), y) = -f(x, T(k, y))$.*

Proof. (1) $\Rightarrow$ (2)
$T = \partial t$ and $t \in o(f) \otimes V^*$ hence for all $x, y \in V$, $k \in \ker(f)$

$$f(T(k, x), y) + f(x, T(k, y)) = f(t_k x, y) + f(x, t_k y) - f(t_x k, y) - f(x, t_y k) = 0$$

since $t_k \in o(f)$ and $t_x(\ker(f)) \subset \ker(f)$, $t_y(\ker(f)) \subset \ker(f)$.

(2) $\Rightarrow$ (1). Observe first that if $l \in \ker(f)$, $f(T(k, l), x) = 0$ for all $x \in V$. Thus $T(\ker(f), \ker(f)) \subset \ker(f)$, apply Lemma 4.1 it then follows that exist $S \in V \otimes V^* \otimes V^*$ such that for all $x \in V$

$S_x \ker(f) \subset \ker(f)$ and $T = \partial S$. Hence

$$S = A+B+C \in \ker(f)\otimes V^*\otimes V^*\oplus V\otimes V^*\otimes V^*$$
$$= \ker(f)\otimes V^*\otimes V^*\oplus V\otimes V^*\otimes \ker(f)^*\oplus V\otimes V^*\otimes V^*$$

and

$$\partial S = \partial A(x, y)+B_p y-B_q x+\partial C(x, y),$$

where $x = x+p$, $y = y+q$ V. Next observe that

$$f(T(k, x), y)+f(x, T(k, y))$$
$$= f(B_k x, y)+f(x, B_k y) = f(B_k x, y)+f(x, B_k y) = 0$$

it follows that $B_k \in o(f)$.

Since $\partial C \in V\otimes \Lambda^2(V^*)$ and f is non-degenerate, using Lemma 4.2. exist $C' \in o(f)\otimes V^*$ such that $\partial C' = \partial C$.

Then for $S' = A+B+C'$ we have (apply Lemma 2.1.)

$$S' \in \ker(f)\otimes V^*\otimes V^*\oplus o(f)\otimes \ker(f)^*\oplus o(f)\otimes V^*$$
$$= \ker(f)\otimes V^*\otimes V^*\oplus o(f)\otimes V^* = o(f)\otimes V^*$$

hence $\partial S' = \partial S$ and $T \in \partial(o(f)\otimes V^*)$.

Lie algebras of vector fields on the neighbourhoods $U_l \subset O(M)$ and $U_{p(l)} \subset M$ will be denoted by $\mathscr{X}(U_l)$ and $\mathscr{X}(U_{p(l)})$ where p: $O(M) \to M$ canonical projection.

LEMMA 4.4. *Let O be a canonical form on $O(M, p, O(f))$. If $X, Y \in \mathscr{X}(U_{p(l)})$, $Z \in \mathscr{X}(U_{p(l)}) \cap \ker(g)$, $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \in \mathscr{X}(U_l)$ such that $dp(\boldsymbol{X}) = Y$, $dp(\boldsymbol{Y}) = Y$, $dp(\boldsymbol{Z}) = Z$ then*

$$2\{f(d\theta(\boldsymbol{Z}, \boldsymbol{X})_l, \theta(\boldsymbol{Y})_l)+f(\theta(\boldsymbol{X})_l, d\theta(\boldsymbol{Z}, \boldsymbol{Y})_l)\} = (L_Z g)(X, Y)_{p(l)}.$$

Proof. Define the function F: $U_1 \to R$ by $F(l) := f(\theta(\boldsymbol{X})_l, \theta(\boldsymbol{Y})_l)$ and the function G: $U_{p(l)} \to R$ by $G(p(l)) := g(X, Y)_{p(l)}$.

These functions have the properties:

a) $(ZG)_{p(l)} = ((dp\boldsymbol{Z})G)_{p(l)} = (\boldsymbol{Z}(Gp))_l = (\boldsymbol{Z}F)_l$,

b) $(\boldsymbol{Z}F)_l = (\boldsymbol{Z}(f(\theta(\boldsymbol{X}), \theta(\boldsymbol{Y})))_l = f((\boldsymbol{Z}\theta)(\boldsymbol{X})_l, \theta(\boldsymbol{Y})_l)+f(\theta(\boldsymbol{X})_l, (\boldsymbol{Z}\theta)(\boldsymbol{Y}))_l$,

c) $(ZG)_{p(l)} = (Z)(g(X, Y))_{p(l)} = (L_Z g)(X, Y)_{p(l)}+g([Z, X], Y)_{p(l)}+g(X, [Z, Y])_{p(l)}$.

(for the Lie derivative see e.g. [4]).

Hence

$$g([Z, X], Y)_{p(l)} f(l^{-1}\circ dp\boldsymbol{Z}, \boldsymbol{X}_l, l^{-1}\circ dp\boldsymbol{Y}_l) = f(\theta([\boldsymbol{Z}, \boldsymbol{X}])_l, \theta(\boldsymbol{Y})_l).$$

Similarly

$$g(X, [Z, Y])_{p(l)} = f(\theta(\boldsymbol{X})_l, \theta([\boldsymbol{Z}, \boldsymbol{Y}])_l).$$

Thus

$$(L_Z g)(X, Y)_{p(l)} = Z(g(X, Y))_{p(l)} - g([Z, X], Y)_{p(l)} - g(X, [Z, Y])_{p(l)} = \dots$$

from the properties b) and c) we obtain

$$\dots = f((Z\theta)X)_l, \theta(Y)_l) + f(\theta(X)_l, (Z\theta)(Y)_l) -$$
$$- f(\theta(X)_l, \theta([Z, Y]_l)) - f(\theta([Z, X])_l, \theta(Y)_l) = \dots$$

Since $2d\theta(X, Y) = X\theta(Y) - Y\theta(X) - \theta([X, Y])$ see [4], hence

$$\dots = f(2d\theta(Z, X)_l, \theta(Y)_l) + f(X\theta(Z), O(Y)_l) +$$
$$+ f(\theta(X)_l, 2d\theta(Z, Y)_l) + f(\theta(X)_l, Y\theta(Z)_l).$$

By the definition of θ we have:

$$\theta(Z)_l = l^{-1}(dpZ) = l^{-1}(Z)$$

thus for $Z \in \ker(g) \cap \mathscr{X}(U_{p(l)})$ this implies $\theta(Z) \in \ker(f)$. Hence for $Z \in \ker(g) \cap \mathscr{X}(U_{p(l)})$ we have:

$$f(\theta(Z), \theta(Y)) = 0$$

and by property c)

$$(XF)_l = (Xf(\theta(Z)_l, \theta(Y)_l) = f(X\theta(Z)_l, \theta(Y)_l) + f(\theta(Z)_l, (X\theta(Y))_l)$$
$$= f(X\theta(Z)_l, \theta(Y)_l) = 0.$$

Similary

$$(YF)_l = f(\theta(X)_l, (Y\theta(Z))_l) = .$$

Consequently for all $X, Y \in \mathscr{X}(U_{p(l)})$, $Z \in \mathscr{X}(U_{p(l)}) \cap \ker(g)$ we obtain

$$(L_Z g)(X, Y) = 2\{f(d\theta(Z, X)_l, \theta(Y)_l + f(\theta(X)_l, dO(Z, Y)_l).$$

Let H_l be a horizontal space at l. For $X, Y \in H_l$ the first structure function is $C_H(l)(\theta(X)_l, \theta(Y)_l) := d\theta(X, Y)_l$.

From Lemma 4.4. we obtain:

$$(L_Z g)(X, Y)_{p(l)} = 2\{f(C_H(l)(\theta(Z)_l, \theta(X)_l), \theta(Y)_l) + f(\theta(X)_l, C_H(l)(\theta(Z)_l, \theta(Y)_l))$$

and by Lemma 4.3.

$$(L_Z g)(X, Y)_{p(l)} = 0 \quad \text{iff } C_H(l) \in \partial(o(f) \otimes V^*) \quad \text{iff } C(l) = 0.$$

Finally we have by Lemma 2.1. a proof of Theorem 3.1.

Remark (see e.g. [4]). A vector field X on M is *an infinitesimal automorphism* of $O(M, p, O(f))$-structure iff $L_X g = 0$.

5. Equivalence of the theorems 1.1 and 3.1.

Proof. Let K_x be a complement to $\ker(g)_x$ in T_xM. (M, g) is absolutely reducible if and only if there is an atlas of local maps in which: for all $x \in M$

$$g_x|K_x^* \otimes K_x^* = \boldsymbol{g}_x \qquad g_x|K_x \otimes \ker(g)_x = 0$$
$$g_x|\ker(g)_x^* \otimes \ker(g)_x^* = 0 \qquad g_x|\ker(g)_x^* \otimes K_x^* .$$

But $\dim(\ker(g)_x) = r$, $\dim(K_x) = n-r$, $\dim(M) = n$, hence for all $X \in \ker(g)_x$ we have $X = (0, \ldots, X^{n-r+1}, \ldots, X^n)$.

If $(U_x\colon x^1, \ldots, x^n)$ be a local map such that definitions 1.2 and 1.3. are satisfied then

$$(L_X g)_{hk} = X^i \partial_i g_{hk} + g_{hi} \partial_k X^i + g_{ik} \partial_h X^i . \tag{5.1}$$

Since (M, g) is absolutely reducible hence $L_X g = 0$.

On the other hand the $O(M, p, O(f))$-structure has a torsionfree connection and by (5.1) we have absolutely reducible (M, g).

6. The variety of $O(f)$-connections.

THEOREM 6.1. *The arbitrary $O(f)$-torsionfree connection is uniquely determined up to the arbitrary* $\ker(f)$*-valued tensor field on M.*

Proof. Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be an open covering of M with a family of cross-sections $\{s_\lambda\}_{\lambda \in \Lambda}$. For each λ, $s_\lambda\colon U_\lambda \to O(M)$. Let Ω be the family of $O(f)$-torsionfree connection forms. For each $U_\lambda \cap U_\mu \neq \emptyset$ define $o(f)$-valued 1-form on U_λ by $\underset{\lambda}{\omega} := s_\lambda^* \omega$ where $\omega \in \Omega$. If $\omega = (\omega_j^i)$ then by Lemma 2.1. we have:

$$\underset{\lambda}{\omega} = \left[\begin{array}{c|c} \underset{\lambda}{\omega_b^a} & 0 \\ \hline \underset{\lambda}{\omega_b^A} & \underset{\lambda}{\omega_B^A} \end{array}\right],$$

where $a, b = 1, \ldots, r$; $i, j = 1, \ldots, n$; $A, B = n-r+1, \ldots, n$.

The part $(\underset{\lambda}{\omega_b^a})$ of ω is uniquely determined by restriction $\boldsymbol{g}$ where $\boldsymbol{g}_x := g_x|K_x^* \otimes K_x^*$ for all $x \in M$ (see, § 5). The remaining parts $(\underset{\lambda}{\omega_B^A})$ and $(\underset{\lambda}{\omega_b^A})$ may be chosen arbitrarily. We denote by $D := \omega' - \omega$ the deformation tensor for ω', ω. Hence

$$D_x = \left[\begin{array}{c|c} 0 & 0 \\ \hline \underset{\lambda}{\omega_b'^A} - \underset{\lambda}{\omega_b^A} & \underset{\lambda}{\omega_B'^A} - \underset{\lambda}{\omega_B^A} \end{array}\right]$$

and D_x is a $\ker(f)$-valued tensor.

In the local coordinates $(U\colon x^1, \ldots, x^n)$ $\underset{\lambda}{\omega_j^i} = \Gamma_{jk}^i dx^k$ and the metric tensor field g determined uniquely only the components Γ_{bc}^a and Γ_{Ai}^d (see the paper of Jakubowicz [5]). Hence the remaining components may be chosen arbitrarily and this condition is equivalent to the $\ker(f)$-valuation of the deformation tensor D.

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