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On the Convergence to Zero of Oscillating Solutions of N-th Order Partial Differential Equations

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In this paper we shall prove two theorems on the convergence to zero of some special solutions of partial differential equations. The conception of these solutions is a generalization of a conception of quickly oscillating solutions of ordinary equations in the papers [1] and [2].

Let n be an integer $\geqslant 1$. The small Greek letters α and β denote n-indices: $\alpha = (\alpha_1, \ldots, \alpha_n)$, α_i are integers $\geqslant 0$. We put

$$|\alpha| = \sum_{i=1}^{n} \alpha_i .$$

We introduce the following notation:

$$\alpha \geqslant \beta \Leftrightarrow \alpha_i \geqslant \beta_i \quad \text{for } i = 1, \ldots, n,$$

$$\alpha > \beta \Leftrightarrow \alpha_i > \beta_i \quad \text{for } i = 1, \ldots, n,$$

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$$

(if $\alpha = 0, \ldots, 0$, then $D^\alpha \varphi = \varphi$ for any function φ).

We shall consider the following equation of the N-th order:

$$D^\alpha \varphi = f(x, (D^\beta \varphi)_{\beta \leqslant \alpha, \beta \neq \alpha}) \tag{1}$$

when $x = (x_1, \ldots, x_n)$, $|\alpha| = N$
and f: $\boldsymbol{R}^q \to \boldsymbol{R}$ is a continuous function,

$$f = f(x, (u_\beta)_{\beta \leqslant \alpha, \beta \neq \alpha}), \qquad q = n + \prod_{i=1}^{n} (\alpha_i + 1) - 1 .$$

Let $\{r_\nu\}_{\nu=1}^{\infty}$ be the sequence of positive real numbers, such that $r_\nu \to \infty$ when $\nu \to \infty$ $r_{\nu+1} > r_\nu$

$$\sqrt{r_{\nu+1}^2 - r_\nu^2} \to 0 \text{ when } \nu \to \infty .$$

For $i = 1, \ldots, n$ let H_ν^i be the set contained in the ring $\{x \in R^n\colon r_\nu \leqslant |x| \leqslant r_{\nu+1}\}$. ($|\cdot|$ is the Euclidean norm). We assume that H_ν^i is the boundary of connected set, which contains a ball with the centre in zero and the radius r_ν.

Let k be an integer, $1 \leqslant k \leqslant n$. We say that the solution of the equation (1) is $[k, \{r_\nu\}]$ pseudo-oscillating, if

(a) $D^\gamma \varphi$ is defined and continuous in $[0, \infty)^n$, for $0 \leqslant \gamma \leqslant \alpha$,

(b) $\sup\{|D^\gamma \varphi(x)|\colon x \in H_\nu^{j+1}\} \to 0$, $\nu \to \infty$,

where $\gamma = 0$ for $j = 0$ and $\gamma = (\alpha_1, \ldots, \alpha_j, 0, \ldots, 0)$ for $j = 1, \ldots, k-1$,

(c) $\exists M > 0\colon \sup\{|D^\gamma \varphi(x)|\colon x \in H_\nu^{j+1}\} < M$

for $j = k, \ldots, n-1$ and γ as above.

Remark 1. For $n = 2$ and the hyperbolical equation the problem of the existence of solutions which fulfil (a) was considered in [4]. In this case the conception of a $[1, \{r_\nu\}]$ pseudo-oscillating solution generalizes the conception of an x-regular $\{r_\nu\}$ oscillating solution in [3].

Let $A_k = A_k(\alpha)$ be a set of n-indices β such that

$$\beta_1 = \alpha_1, \ldots, \quad \beta_{s-1} = \alpha_{s-1}, \quad 0 \leqslant \beta_s < \alpha_s, \quad \beta_{s+1} = 0, \ldots, \quad \beta_n = 0$$

for some $s \leqslant k$.

THEOREM 1. *Let k be an integer, $1 \leqslant k \leqslant n$, and $\alpha = (\alpha_1, \ldots, \alpha_n)$ be an n-index such that $\alpha_k \neq 0$. Assume that for all $i = 1, \ldots, n$, $\alpha_i = 0$ implies $\alpha_{i+1} = 0$. Suppose that function f in (1) fulfils the following condition:*

$$\exists C > 0 \quad |f| \leqslant C(1 + \sum_{\beta \in A_k} |u_\beta|)\,. \tag{2}$$

If φ is a $[k, \{r_\nu\}]$ pseudo-oscillating solution of (1) then

$$\sum_{\beta \in A_k} |D^\beta \varphi(x)| \to 0 \text{ when } |x| \to 0 \text{ and } x \in [0, \infty)^n.$$

Now we shall present the second theorem. For $i = 1, \ldots, n$ let $\{w_\nu^i\}_{\nu=1}^\infty$ be sequences of positive real numbers such that $w_\nu^i \to \infty$ when $\nu \to \infty$, $0 < w_{\nu+1}^i - w_\nu^i \to 0$ when $\nu \to \infty$ $W = \{x \in [0, \infty)^n\colon \exists i = 1, \ldots, n, \exists \nu = 1, 2, \ldots\ x_i = w_\nu^i\}$.

We say that the solution φ of (1) is W-oscillating, if

(d) $D^\gamma \varphi$ is defined and continuous in $[0, \infty)^n$, for $0 \leqslant \gamma \leqslant \alpha$,

(e) $\varphi \equiv 0$ on W.

Let $p_i\colon [0, \infty)^n \ni x \to (w_{\nu+\alpha_i+1}^i - w_\nu^i) \in (0, \infty)$
when $w_\nu^i \leqslant x_i < w_{\nu+1}^i$

THEOREM 2. *Let φ be a W-oscillating solution of (1) $\alpha \geqslant (1, \ldots, 1)$. We assume that $\exists C\colon [0, \infty)^n \to (0, \infty)$ such that*

$$|f| \leqslant C(x)(1 + \sum_{\beta < \alpha} |u_\beta|)\,, \tag{3}$$

$$\forall i = 1, \ldots, n \quad C(x) p_i(x) \to 0 \tag{4}$$

when $|x| \to \infty$ uniformly with respect to $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$.

Then $\sum_{\beta<\alpha} |D^\beta \varphi(x)| \to 0$ *when* $|x| \to \infty$ *and* $x \in [0, \infty)^n$.

Remark 2. Condition (4) is fulfilled when C = constant. In the case $n = 1$ both Theorems generalize Corollary 2 in the paper [1]. Theorem 1 generalizes the results of [3].

Now we shall prove some lemmas:

LEMMA 1. *Let* $g: \boldsymbol{R} \to \boldsymbol{R}$ *be a* C^p*-class function for some* $p \geqslant 1$ *and* $\{t_\nu\}_{\nu=1}^{\infty}$ *be a sequence in* $\boldsymbol{R}$ *such that*

$$\exists \lambda > 0,\ \eta > 0\ \ \eta < t_{\nu+1} - t_\nu < 2\eta \quad \text{and} \quad |g(t_\nu)| < \lambda$$

Then $\forall x \in [t_1, t_2) \exists z > t_1$:

$$|z - x| < (2^p + 2^{p-1} - 1)\eta \quad \text{and} \quad |g^{(p)}(z)| < \frac{2^p \lambda}{\eta^p}.$$

Proof of lemma 1: For $p = 1$ this is the immediate consequence of the mean value theorem.

Let us assume that the lemma is true for p. There exists z_1, such that

$$|z_1 - x| < (2^p + 2^{p-1} - 1)\eta \quad \text{and} \quad |g^{(p)}(z_1)| < \frac{2^p \lambda}{\eta^p},\ z_1 > t_1.$$

Let μ be the minimum of the set $\{\nu\colon t_\nu - z_1 > \eta\}$. Then the inequality $x < t_2 \leqslant t_\mu$ holds good. For t_μ we find $z_2 > t_\mu$, such that the following conditions are fulfilled:

$$z_2 - t_\mu < (2^p + 2^{p-1} - 1)\eta \quad \text{and} \quad |g^{(p)}(z_2)| < \frac{2^p \lambda}{\eta^p}.$$

Finally, we find a mean value $z \in (z_1, z_2)$, which has the required properties.

COROLLARY. *Let* δ *be a number,* $0 < \delta < 1$. *Suppose that* g *and* $\{t_\nu\}$ *fulfil the assumption of lemma* 1 *with*

(I) $$\eta = \frac{\delta}{2^{p+1}}$$

or

(II) $$\eta = \frac{\delta}{2^{p+1}} \quad \text{and} \quad \lambda = \frac{\delta^{p+1}}{2^{p(p+2)}}.$$

Then $\exists z_1, \ldots, z_p,\ z_i > t_1$ *for* $i = 1, \ldots, p,\ |z_i - x| < \frac{\delta}{2}$:

(I) $$|g^{(i)}(z_i)| < 2^{2p+1}\lambda$$

or

(II) $$|g^{(i)}(z_i)| < \delta \quad \text{for } i = 1, \ldots, p.$$

LEMMA 2. *Let $g\colon (a, a+\delta) \to \boldsymbol{R}$ be a C^p-function for some $a \in \boldsymbol{R}$ and $0<\delta<1$. Let s be an integer, $0 \leqslant s<p$. Suppose that there exist $\lambda>0$ and points $z_j \in (a, a+\delta)$, $j = s, \ldots, p-1$, such that $|g^{(s)}(z_j)|<\lambda$. Then $\forall x \in (a, a+\delta) \exists \theta \in (a, a+\delta)$: $|g^{(s)}(x)| \leqslant p\lambda + |g^{(p)}(\theta)|\delta$.*

Proof of lemma 2: We apply the mean value theorem and obtain

$$|g^{(s)}(x)| \leqslant |g^{(p)}(\theta)|\delta^{p-s} + \lambda + \ldots + \lambda\delta^{p-s-1} \quad \text{for some } \theta \in (a, a+\delta).$$

Proof of Theorem 1: We fix $\varepsilon>0$ and a positive integer k. Let $Q = 2^{2|\alpha|+1}$. We find $\delta<1$, so that

$$\frac{|\alpha|^2\delta(1+Q)+\delta^{n-k+1}C}{1-C(\alpha_1+\ldots+\alpha_k)\delta^{n-k+1}} < \varepsilon .$$

Let ν_0 be a positive integer such that

$$\sup\{|D^\gamma\varphi(x)|\colon x \in H_\nu^{j+1}\} < \frac{\delta^{|\alpha|+1}}{2^{\alpha^2}}, \qquad \nu>\nu_0 \tag{5}$$

for $0 \leqslant j \leqslant k-1$ and γ as in (b)

$$\sqrt{v_{\nu+1}^2 - v_\nu^2} < \frac{\delta}{2^{|\alpha|+1}}, \qquad \nu>\nu_0 . \tag{6}$$

We shall show that if $x_0 \in [0, \infty)^n$ is not in the open ball with centre in zero and radius ν_{ν_0}, then

$$\sum_{\beta \in A_k} |D^\beta\varphi(x_0)| < \varepsilon .$$

Let J denotes the set $[x_{01}, x_{01}+\delta] \times \ldots \times [x_{0n}, x_{0n}+\delta]$. Now we fix $x \in J$ and $\beta = (\alpha_1, \ldots, \alpha_{s-1}, \beta_s, 0, \ldots, 0) \in A_k$.

We introduce the following function:

$$\psi = D^{(\alpha_1, \ldots, \alpha_{s-1}, 0, \ldots, 0)}\varphi(x_1, \ldots, x_{s-1}, \cdot, x_{s+1}, \ldots, x_n) .$$

By virtue of (6) there exist t_1 such that $0<t_1-x_{0s}<\frac{\delta}{2^{|\alpha|}}$ and $(x_1, \ldots, x_{s-1}, t_1, x_{s+1}, \ldots, x_n) \in H_{\mu_0}^s$ for some μ_0. We choose a sequence of points $(x_1, \ldots, x_{s-1}, t_\nu, x_{s+1}, \ldots, x_n)$ in the intersection of $\bigcup_{\mu>\mu_0} H_\mu^s$ and the straight line $x+\boldsymbol{R}e_s$ (when $e_s = (0, \ldots, \overset{(s)}{1}, \ldots, 0)$), such that $\{t_\nu\}$ fulfils the assumptions of Corollary (II) (for the function ψ). There exist numbers z_j, $j = \beta_s+1, \ldots, \alpha_s$, which fulfil the following conditions:

$$z_j - t_1 < \frac{\delta}{2} \quad \text{and} \quad |\psi(z_j)| < \delta .$$

Since $0<z_j-x_{0s}<\delta$ we can use Lemma 2. As result we obtain $\theta_s \in (x_{0s}, x_{0s}+\delta)$ for which the following inequality holds good:

$$|D^\beta\varphi(x)| \leqslant \alpha_s\delta + |D^{(\alpha_1, \ldots, \alpha_s, 0, \ldots, 0)}\varphi(x_1, \ldots, \theta_s, \ldots, x_n)|\delta .$$

We apply this method several times. Hence we get the numbers $\theta_j \in (x_{0j}, x_{0j}+\delta)$ for $j = s, \ldots, k$, such that

$$|D^\beta \varphi(x)| \leqslant \alpha_s \delta + \ldots + \alpha_k \delta^{k-s+1} + |D^\gamma \varphi(y)| \delta^{k-s+1},$$

where $\gamma = (\alpha_1, \ldots, \alpha_k, 0, \ldots, 0)$ and $y = (x_1, \ldots, x_{s-1}, \theta_s, \ldots, \theta_k, \ldots, x_n)$. Similarly, using Corollary (I) and Lemma 2 we obtain

$$\forall x \in J \exists \theta j \in (x_{0j}, x_{0j}+\delta), \quad j = k+1, \ldots, n,$$

such that

$$|D^\gamma \varphi(x)| \leqslant \alpha_{k+1} Q + \ldots + \alpha_n Q \delta^{n-k+1} + |D^\alpha \varphi(y)| \delta^{n-k},$$

where γ as above and $y = (x_1, \ldots, x_k, \theta_{k+1}, \ldots, \theta_n)$.

Finally, since $\delta < 1$, we obtain

$\forall \beta \in A_k \ \forall x \in J \ \exists \theta \in J$:

$$|D^\beta \varphi(x)| \leqslant (\alpha_1 + \ldots + \alpha_k)\delta + (\alpha_{k+1} + \ldots + \alpha_n) Q\delta + |D^\alpha \varphi(\theta)| \delta^{n-k+1}.$$

The set A_k has $\alpha_1 + \ldots + \alpha_k$ elements. We denote this number by d. By the assumption (2), the following inequalities hold good:

$$\sum_{\beta \in A_k} |D^\beta \varphi(x_0)| \leqslant \sup_{x \in J} \{ \sum_{\beta \in A_k} |D^\beta \varphi(x)| \}$$
$$\leqslant d\big(d\delta + (\alpha_{k+1} + \ldots + \alpha_n) Q\delta + C(1 + \sup_{x \in J} \{ \sum_{\beta \in A_k} |D^\beta \varphi(x)| \}) \delta^{n-k+1}\big).$$

After solving the last inequality we have

$$\sup_{x \in J} \Big\{ \sum_{\beta \in A_k} |D^\beta \varphi(x)| \Big\} \leqslant \frac{d\big(d\delta + (\alpha_{k+1} + \ldots + \alpha_n) Q\delta + C\delta^{n-k+1}\big)}{1 - Cd\delta^{n-k+1}}.$$

This ends the proof.

Proof of Theorem 2: Let us fix $\varepsilon > 0$. To ε we select $\delta > 0$ such that

$$\frac{EL^{|\alpha|}\delta}{1 - EL^{|\alpha|}\delta} < \varepsilon,$$

where $E = \alpha_1 \ldots \alpha_n$ and $L = \max\{w^i_{\nu+1} - w^i_\nu : i = 1, \ldots, n; \nu = 1, 2, \ldots)$ or $L = 1$, if this maximum is less than 1.

The assumptions imply that there exists μ_i, $i = 1, \ldots, n$ such that

$$C(x) p_i(x) < \delta \quad \text{and} \quad p_i(x) < 1, \quad \text{when } x_i \geqslant w^i_{\mu_i}.$$

We introduce the set $K = [0, w^1_{\mu_1}] \times \ldots \times [0, w^n_{\mu_n}]$. We shall prove that if

$$x_0 \in [0, \infty)^n \setminus K \quad \text{then} \sum_{\beta < \alpha} |D^\beta \varphi(x_0)| < \varepsilon.$$

There exist ν_i, $i = 1, \ldots, n$ such that $x_{0i} \in [w^i_{\nu_i}, w^i_{\nu_i+1})$. Since $x_0 \notin K$ there is i such that $x_{0i} > w^i_{\mu_i}$. We may assume that $i = 1$. We put

$$J = [w^1_{\nu_1}, w^1_{\nu_1+\alpha_1+1}] \times \ldots \times [w^n_{\nu_n}, w^n_{\nu_n+\alpha_n+1}].$$

We fix $x \in J$ and $\beta < \alpha$. It follows from (e) that

$$D^{(0,\beta_2,\ldots,\beta_n)}\varphi \equiv 0 \quad \text{in the set } \{w_\nu^1\} \times [0, \infty)^{n-1}.$$

We apply $\alpha_1 - \beta_1$ times Rolle's Theorem and we obtain

$$|D^\beta \varphi(x)| \leqslant |D^{(\alpha_1,\beta_2,\ldots,\beta_n)}\varphi(\theta_1, x_2, \ldots, x_n)| p_1(x_0)$$

for some $\theta_1 \in (w_{\nu_1}^1, w_{\nu_1+\alpha_1+1}^1)$.

We repeat this operation n times. As a result we get

$$|D^\beta \varphi(x)| \leqslant |D^\alpha \varphi(\theta)| p_1(x_0) L^{\alpha_2} \cdot \ldots \cdot L_\nu^{\alpha_n} \quad \text{for some } \theta \in J.$$

The condition (d) implies that

$$\sup_{x \in J} \{\sum_{\beta<\alpha} |D^\beta \varphi(x)|\} = \sum_{\beta<\alpha} |D^\beta \varphi(\bar{x})| \quad \text{for some } \bar{x} \in J.$$

For β and $\bar{x}$ we select θ_β as above. The maximum of $|D^\alpha \varphi(\theta_\beta)|$ for all $\beta < \alpha$ is realized in some $\theta_{\beta_0} = \theta$.

We observe that there are E multi-indices less than α and the function p_1 is constant in the set J. By virtue of (3) we get the following inequality:

$$\sum_{\beta<\alpha} |D^\beta \varphi(\bar{x})| \leqslant L^{|\alpha|} E p_1(x_0) |D^\alpha \varphi(\theta)| \leqslant L^{|\alpha|} E p_1(\bar{x}) C(\bar{x}) (1 + \sum_{\beta<\alpha} |D^\beta \varphi(\bar{x})|.$$

Since $x_0 \in J$ and $p_1(\bar{x}) C(\bar{x}) < \delta$, we prove the thesis after solving the inequality:

$$\sum_{\beta<\alpha} |D^\beta \varphi(\bar{x})| \leqslant \delta E L^{|\alpha|} (1 + \sum_{\beta<\alpha} |D^\beta \varphi(\bar{x})|)$$

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