UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA, FASCICULUS XXV
1985

On Invariant Measures Supported on the Compact Sets

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0. Introduction. The problem of existence of an invariant measure has been often considered since Prodi [6], Foiaş [3] and Hopf [5] stated that the problem an invariant measure corresponds to the problem of turbulence.

This paper is a continuation of the papers [1], [2] in which the existence of invariant measure has been proved for the dynamical system given by some partial differential equation. As far as the applications in biology [4] are concerned, it is not important for the invariant measure to be supported on the whole space of continuous functions, but it is very important for it to be supported on the whole set where this system has the biological sense. This problem is being solved in this paper.

1. The existence of an invariant measure

In this section we consider the equation

$$\frac{\partial u}{\partial t}+x\frac{\partial u}{\partial x}=\gamma u$$

in the set

$$0\leqq x\leqq 1 \qquad t\geqq 0$$

with the initial condition

$$u(0,x)=v(x) \qquad 0\leqq x\leqq 1\,.$$

This problem generates a semi-dynamical system according to the formula

$$(T_t v)(x)=u(t,x) \qquad 0\leqq x\leqq 1\,.$$

This system may be considered on the space V_α of all Lipschitz functions vanishing in 0 and satisfying the condition

$$|v'(x)|\leqq \alpha x^{\gamma-1}$$

THEOREM 1. *If $\gamma>1$, then there exists a non-negative measure μ_α defined on Borel subsets of V_α and satisfying the conditions:*

(i) μ_α is T_t*-invariant i.e. for all $t>0$ and for a Borel subset E of V_α, $\mu_\alpha(T_t^{-1}(E))=\mu_\alpha(E)$,*

(ii) $\mu_\alpha(U)>0$ *for every open non empty subset* U *of* V_α.
(iii) μ_α *is probabilistic i.e.* $\mu_\alpha(V_\alpha) = 1$,
(iv) μ_α *is ergodic i.e. for every* T_t*-invariant set* E
$\mu_\alpha(E) = \mu_\alpha(E)^2$,
(v) $\mu_\alpha(E_0) = 0$, *where* E_0 *is the set of all the periodic points.*
We shall show more, namely that $\mu_\alpha(E_0') = 0$ when $E_0' = \{v|\exists s\colon T_s v \in E_0\}$. We denote the set of all non-negative functions from V_α, by V_α^+.

THEOREM 2. *Theorem* 1 *is true also if we replace the space* V_α *by* V_α^+.
To prove these theorems we shall use some lemmas.

2. Auxiliary lemmas

LEMMA 1. *Let* $\{p_i\}$ *be a sequence satisfying the conditions*
(a) $p_0 = 0$
(b) $\forall i>0 \quad p_i>0$
(c) $\sum_{i=1}^{\infty} p_i = 1$.
Let's define the function $\varrho^*\colon [0, 1] \to [0, 1]$ *by the formula*

$$\begin{cases} \varrho^*(\xi) = p^{-1}(\xi - \sum_{i=1}^{n} p_i) & \text{for } \xi \in (\sum_{i=1}^{n-1} p_i, \sum_{i=1}^{n} p_i] \\ \varrho^*(0) = 0\,. \end{cases}$$

Thus the Lebesgue measure is ergodic with respect to ϱ.

Let's consider now a semi-dynamical system $T_t\colon X \to X$ on a σ-compact metrizable space X.

Definition. A system $\{T_t\}$ is called *surjective* if for every $t>0$, T_t is a surjection.

Let a non-negative Borel measure on X be given, such that for each Borel set E

$$\mu(T_t^{-1}(E)) = E\,.$$

Let's consider the function: $f = f_E\colon [0, 1] \to R$, given by the formula

$$f_E(t) = \mu(T_t^{-1}(E))\,.$$

LEMMA 2. f_E *is measurable.*

By Lip [0, 1] we denote the set of all Lipschitz functions on [0, 1].

LEMMA 3. *Let* n_0 *be a positive integer. There exists such a family* $\mathscr{P}$ *of polynomials with rational coefficients that*
a) $\overline{\mathscr{P}} = \text{Lip}\,[0, 1]$

b) $\forall P \in \mathscr{P}\ \deg P \neq n_0$

c) $\forall P, Q \in \mathscr{P}\ \deg P \neq \deg Q$ for $P \neq Q$.

Proofs of the last three lemmas were presented in [1].

LEMMA 4. *Let $\gamma > 1$. There exists a family Ξ of polynomials with rational coefficients, such that*

1. $\forall P, Q \in \Xi\ \deg P \neq \deg Q$ for $P \neq Q$
2. $\forall P \in \Xi\ P(0) = 0,\ |P'(x)| < (1+x)^{\gamma-1}\ \ x \in [0, 1]$
3. *Ξ is dense in the set*

$$\{f \in AC[0;\ 1] | f(0) = 0\ \ |f'(x)| < (1+x)^{\gamma-1}\ \ x \in [0, 1]\},$$

where $AC[0, 1]$ denotes the set of all absolutely continuous functions on $[0, 1]$.

Proof. Let $f \in AC[0, 1]$, $f(0) = 0$, $|f'(x)| \leqq (1+x)^{\gamma-1}$. Clearly f' is function from L^1. Then, there exists a sequence $\{g_k\}$ of continuous functions convergent to f in L^1. These functions may be chosen in such a way that $|g_n(x)| \leqq (1+x)^{\gamma-1}$. Now let g be a continuous function and let $\varepsilon > 0$. Let's consider the set

$$U(\gamma, g, \varepsilon) = \{v \in C[0, 1] | \, ||v-g|| < \varepsilon,\ |v(x)| < (1+x)^{\gamma-1}.$$

This set is open and nonempty, because the function

$$h(x) = \begin{cases} (1+x)^{\gamma-1} - \dfrac{\varepsilon}{2} & \text{for } g(x) \geqq (1+x)^{\gamma-1} - \dfrac{\varepsilon}{2} \\ g(x) & \text{for } |g(x)| \leqq (1+x)^{\gamma-1} - \dfrac{\varepsilon}{2} \\ -(1+x)^{\gamma-1} + \dfrac{\varepsilon}{2} & \text{for } g(x) \leqq -(1+x)^{\gamma-1} + \dfrac{\varepsilon}{2} \end{cases}$$

belongs to it. Thus, there exists a polynomial from the former lemma which belongs to $U(\gamma, g, \varepsilon)$. Since the uniform convergence implies the convergence in $L^1[0, 1]$ then, there exists a set of polynomials with rational coefficients, satisfying the condition 1 and dense in the set

$$\{g \in L^1 | \, |g(x)| \leqq (1+x)^{\gamma-1}\}.$$

To complete the proof it is sufficient to notice that the antiderivative of a polynomial with rational coefficients is also a polynomial with rational coefficients, and its degree is greater by 1 and that, if

$$f_i(x) = \int_0^x g_i(y)\,dy \qquad (i = 1, 2),$$

then

$$|f_1(x) - f_2(x)| \leqq \int_0^1 |g_1(y) - g_2(y)|\,dy.$$

3. Proof of theorem 1

It is easy to prove that the unique solution of the equation under consideration is given by the formula

$$(T_t v)(x) = u(t, x) = e^{\gamma t} v(xe^{-t})$$

and it is also easy to show that T_t: $V_\alpha \to V_\alpha$ is a continuous semi-dynamical system.

Now let $\{\sigma_n\}$ be a sequence of all polynomials from the set Ξ defined in lemma 4 (the set of all polynomials with rational coefficients is countable). Moreover the sequence $\{\sigma_n\}$ may be chosen in such a way that

$$\forall n \|\sigma_n\| \leqq n \qquad \|\sigma_n'\| \leqq n .$$

Let's define the functions T: $V_\alpha \to V_\alpha$ and S_α: $V_\alpha \to V_\alpha$ by the formulae

$$T = T_{\ln 2}$$

and

$$S_n v(x) = \begin{cases} 2^{-\lambda} v(2x) & 0 \leqq x \leqq \frac{1}{2} \\ 2^{-\lambda} v(1) + 2^{-\lambda} \sigma_n(2x-1) & \frac{1}{2} \leqq x \leqq 1 \end{cases}$$

clearly

$$TS_n = \mathrm{id}_{V_\alpha} .$$

In the paper [1] it was proved that if a sequence of positive integers $\{k_n\}$ satisfies the condition

$$k_n \leqq n \quad \text{for infinitely many } n ,$$

then

$$\operatorname{card} \bigcap_{n=1}^{\infty} S_{k_1} \dots S_{k_n}(V_\alpha) = 1$$

Suppose that we are given a sequence of rational numbers $\{p_i\}$ and a function ϱ^*: $[0, 1] \to [0, 1]$ defined as in lemma 1, such that

$$\sum_{n=1}^{\infty} R_n < \infty ,$$

where

$$R_n = 1 - \sum_{i=1}^{n-1} p_i .$$

Denote by J the set of all irrational numbers in $[0, 1]$ and by ϱ the restriction of ϱ^* to J. Evidently

$$\varrho: J \to J$$

Define functions π: $J \to N$ and k_n: $J \to N$ by the formulae

$$\pi(x) = \sup\{n | \sum_{i=1}^{n-1} p_{i_n} < x\}$$

and

$$k_n(x) = \pi\varrho^n(x) \qquad n = 1, 2, \dots$$

Denote by P the set of all x for which $k_n(x) \leqq x$, for infinitely many n. In the paper [1] it was proved, that $m(P) = 0$.

Let $F = J \setminus P$. For $x \in F$ denote by $\Phi(x)$ the only function which belongs to

$$\bigcap_{n=1}^{\infty} S_{k_1(x)} \dots S_{k_n(x)}(V_\alpha).$$

To prove that Φ is continuous let's consider $x_\nu \to x_0$ and $\varepsilon > 0$. It is obvious that $\Phi(x_\nu) \in V_\alpha$ and $\Phi(x_0) \in V_\alpha$. Hence

$$|\Phi(x_\nu)(y) - \Phi(x_0)(y)| < \frac{2\alpha}{\gamma} y^\gamma$$

and then for all $\varepsilon > 0$ there exists such positive integer n, that

$$|\Phi(x_\nu) - \Phi(x_0)(y)| < \varepsilon \quad \text{for } y \in [0, 2^{-n}]$$

but

$$\Phi(x_\nu) - \Phi(x_0)|[2^{-n}, 1] = \text{const}$$

for ν sufficiently large, and in consequence $\Phi\colon F \to V_\alpha$ is continuous.

We define

$$\bar{\mu}(E) = m(\Phi^{-1}(E))$$

and

$$\mu_\alpha(E) = \frac{1}{\ln 2} \int_0^{\ln 2} \bar{\mu}(T_t^{-1}(E)).$$

The proof that μ_α fulfils (i)—(v) is analogous to the proof presented in the paper [1].

4. Proof of theorem 2

To prove theorem 2 it is sufficient to define the function $h\colon V \to V^+$ by the formula:

$$h(v)(x) = |v(x)|.$$

It is obvious, that the diagram:

$$\begin{array}{ccc} V_\alpha & \xrightarrow{T_t} & V_\alpha \\ \downarrow h & & \downarrow h \\ V_\alpha^+ & \xrightarrow{T_t} & V_\alpha^+ \end{array}$$

is commutative, that $h(V_\alpha) = V_\alpha^+$, and that h is ccotinuous. The measure μ_α^+ may be defined by the formula

$$\mu_\alpha^+(E) = \mu_\alpha(h^{-1}(E)),$$

which complets the proof.

5. Some remarks on the measures generated above

Let's define the set $V_{\alpha,\nu} \subset V$ by the formula:

$$V_{\alpha,\nu} = \left\{ v \in V\colon\ |v'(x)| \leqq \alpha x^{-1} \quad \text{for } x \leqq \frac{1}{\nu} \right\},$$

where V denotes the space of all Lipschitz functions on [0, 1] which vanish in 0. Let's define also the set $V_{\alpha,\infty}$ by the formula:

$$V_{\alpha,\infty} = \{ v \in V\colon\ \exists \delta > 0\colon\ |v'(x)| \leqq \alpha x^{\gamma-1} \quad \text{for } x \leqq \delta \}\,.$$

Clearly

$$V_{\alpha,\infty} = \bigcup_{\nu=1}^{\infty} V_{\alpha,\nu}$$

and this sequence is increasing. Also

$$V_{\alpha,\nu} = T_{\ln \nu}^{-1}(V_\alpha)\,.$$

Now, let $\alpha' > \alpha$. Thus

$$\mu_{\alpha'}(V_{\alpha,\nu}) = \mu_{\alpha'}(V_\alpha)$$

and

$$\mu_{\alpha'}(V_{\alpha,\infty}) = \mu_{\alpha'}(V_\alpha)$$

but

$$T_t^{-1}(V_{\alpha,\infty}) = V_{\alpha,\infty} \quad \text{for } t \geqq 0\,.$$

So

$$\mu_{\alpha'}(V_\alpha) = 0 \quad \text{or } \mu_{\alpha'}(V_\alpha) = 1\,.$$

Let's consider the set:

$$V_{\alpha,\alpha'} = \left\{ v \in V_{\alpha'}\colon\ v\left(\frac{1}{2}\right) > \frac{\alpha + \alpha_{\alpha'}}{2\gamma} \cdot \frac{1}{2} \right\}.$$

This set is open in $V_{\alpha'}$ and disjoint with V_α. Thus

$$\mu_{\alpha'}(V_\alpha) = 0\,.$$

In consequence we have a family of invariant measures

$$\{\mu_\alpha | \alpha \in R\}$$

such that, if $\alpha < \alpha'$, then

$$\operatorname{supp} \mu_\alpha \subset \operatorname{supp} \mu_{\alpha'}$$

and

$$\mu_{\alpha'}(\operatorname{supp} \mu_\alpha) = 0\,.$$

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Received October 15, 1980