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On the Theorem of Fatou for the Biharmonic Problem in a Half-space

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In the paper [1] the solution of the classical biharmonic problem for a half space has been obtained. In our paper we shall generalize this result to the case of boundary conditions given by signed measures. A similar problem for the n-dimensional ball has been solved by E. Keller in the paper [3]. Before the formulation of the problem we shall give necessary definitions and notations. Points of the real n-dimensional Euclidean space $\boldsymbol{R}^n$ $(n \geqslant 2)$ are denoted by $x = (x_1, \ldots, x_{n-1}, x_n) = (x', x_n)$, the inner product and the norm in $\boldsymbol{R}^n$ are defined by $(x|y) := \sum_{i=1}^{n} x_i y_i$ and $||x|| := \sqrt{(x|x)}$, respectively, where $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$. We shall use notations:

$$\boldsymbol{R}_0^n := \{(x_1, \ldots, x_{n-1}, 0) \in \boldsymbol{R}^n\} \quad \text{and} \quad \boldsymbol{R}_+^n := \{(x', x_n) \in \boldsymbol{R}^n,\ x_n > 0\}.$$

$L(\boldsymbol{R}^{n-1})$ denotes the class of Borel subsets of $\boldsymbol{R}^{n-1}$.
Let $\xi = (\xi', 0) \in \boldsymbol{R}_0^n$, then

$$\Gamma(\xi, a) := \left\{x = (x', x_n) \in \boldsymbol{R}_+^n :\ ||(x', 0) - \xi|| < a \cdot x_n,\ 0 < a = \operatorname{tg}\alpha,\ \alpha < \frac{\pi}{2}\right\}$$

and R_ξ denotes the collection of balls $K \subset \boldsymbol{R}^{n-1}$ containing the point $\xi' \in \boldsymbol{R}^{n-1}$. Let μ be a signed measure, defined on $L(\boldsymbol{R}^{n-1})$, which is finite on compact sets and $\mu(\emptyset) = 0$.

The limit

$$\lim_{\substack{K \in R_\xi \\ \mathcal{O}_{n-1}(K) \to 0}} \frac{\mu(K)}{\mathcal{O}_{n-1}(K)},$$

if it exists and is independent of the choice of the balls K, is called *the derivative of measure μ at the point ξ'* and is denoted by $(D\mu)(\xi')$, where $\mathcal{O}_{n-1}(K)$ denotes the $(n-1)$-dimensional Lebesgue measure of set K.

We observe that

$$(D\mu)(\xi') = \frac{1}{\tau_{n-1}} \lim_{\substack{r \to 0 \\ K \in R_\xi}} \frac{\mu(K)}{r^{n-1}},$$

where τ_{n-1} denotes the volume of the $(n-1)$-dimensional unit ball and r denotes the radius of the ball K.

If a point ξ' is the center of balls $K \in R_\xi$, then $(D\mu)(\xi')$ is called *the symmetric derivative* and is denoted by $(D_s\mu)(\xi')$. It is obvious that if $(D\mu)(\xi')$ exists, $(D_s\mu)(\xi')$ exists too, and $(D\mu)(\xi') = (D_s\mu)(\xi')$.

The limit

$$\lim_{\substack{r\to 0 \\ K\in R_\xi K \\ \xi\in R_0^n}} \frac{\mu(K)}{\int ||\xi-(y',0)||\, d\mathcal{O}_{n-1}(y')},$$

if it exists, is called *the second symmetric derivative of measure μ at a point ξ'*, where ξ' is the center of the balls K. It is denoted by $(D_s^2\mu)(\xi')$.

We have (see [4] page 36)

$$(D_s^2\mu)(\xi') = \frac{n}{\theta_{n-1}} \lim_{r\to 0} \frac{\mu(K)}{r^n},$$

where θ_n denotes the surface area of the unit sphere in $\boldsymbol{R}^n$.

The radial limit $R-\lim_{x\to\xi} u(x)$ *of a function u defined on $\boldsymbol{R}^n$ at the point $\xi = (\xi', 0) \in \boldsymbol{R}_0^n$* is defined as follows:

$$R-\lim_{x\to\xi} u(x) := \lim_{x_n\to 0^+} u(\xi', x_n).$$

If $\lim_{\substack{x\to\xi \\ x\in\Gamma(\xi,a)}} u(x)$ exists for every $a>0$ and does not depend on a, then we denote it by $\theta-\lim_{x\to\xi} u(x)$ and we call it a *nontangential limit*.

Let signed measure μ and η be given, defined on $L(\boldsymbol{R}^{n-1})$.

ASSUMPTION A. *μ and η are of bounded variation.*

ASSUMPTION B. *There are non-negative constants C_1, C_2 and constants $\beta_1, \beta_2 \in (n-1, n)$ such that $|\mu(K)| \leqslant C_1 r^{\beta_1}$ $|\eta(K)| \leqslant C_2 r^{\beta_2}$, for every ball $K \subset \boldsymbol{R}^{n-1}$ with sufficiently large radius r.*

Let

$$P(\mu, \eta, \boldsymbol{R}^{n-1})(x) := \frac{x_n^2}{\alpha_n} \int_{\boldsymbol{R}^{n-1}} \frac{1}{||x-(y',0)||^n}\, d\mu(y') + \frac{n}{\alpha_n} x_n^3 \int_{\boldsymbol{R}^{n-1}} \frac{1}{||x-(y',0)||^{n+2}}\, d\eta(y')$$

for $x = (x'\, x_n) \in \boldsymbol{R}_+^n$, where

$$\alpha_n := \int_{\boldsymbol{R}^{n-1}} \frac{dt_1 \ldots dt_{n-1}}{(t_1^2+t_2^2+\ldots+t_{n-1}^2+1)^{n/2}}.$$

We shall prove that under Assumptions A and B and under suitable assumptions with respect to measures μ and η we have:

a) $\Delta^2 P(\mu, \eta, \boldsymbol{R}^{n-1})(x) = 0$ for $x \in \boldsymbol{R}_+^n$,

b) $\theta - \lim_{x \to \xi=(\xi',0)} P(\mu, \eta, \boldsymbol{R}^{n-1})(x) = (D\eta)(\xi')$,

c) $\theta - \lim_{x \to \xi=(\xi',0)} \frac{\partial}{\partial x_n} P(\mu, \eta, \boldsymbol{R}^{n-1})(x) = (D\mu)(\xi')$ for a.e. $\xi \in \boldsymbol{R}_0^n$.

It is easy to check that under Assumptions A and B functions

$$\frac{\partial^k}{\partial x_1^{k_1} \dots \partial x_n^{k_n}} \frac{x_n^2}{||x-(y',0)||^n},$$

$$\frac{\partial^k}{\partial x_1^{k_1} \dots \partial x_n^{k_n}} \frac{x_n^3}{||x-(y',0)||^{n+2}},$$

where $k = 0, 1, 2, 3, 4$, $k = k_1 + \dots + k_n$ and $k_1, \dots, k_n$ non-negative integer, satisfy the assumptions of Theorem 132, Chapter IV, cf [8]. Thus

$$\frac{\partial^k}{\partial x_1^{k_1} \dots \partial x_n^{k_n}} \left(x_n^2 \int_{\boldsymbol{R}^{n-1}} \frac{1}{||x-(y',0)||^n} d\mu(y') \right)$$

$$= \int_{\boldsymbol{R}^{n-1}} \frac{\partial^k}{\partial x_1^{k_1} \dots \partial x_n^{k_n}} \frac{x_n^2}{||x-(y',0)||^n} d\mu(y'),$$

(1)′

$$\frac{\partial^k}{\partial x_1^{k_1} \dots \partial x_n^{k_n}} x_n^3 \int_{\boldsymbol{R}^{n-1}} \frac{1}{||x-(y',0)||^{n+2}} d\eta(y')$$

$$= \int_{\boldsymbol{R}^{n-1}} \frac{\partial^k}{\partial x_1^{k_1} \dots \partial x_n^{k_n}} \frac{x_n^3}{||x-(y',0)||^{n+2}} d\eta(y').$$

Hence and from the fact that functions $\frac{x_n^2}{||x-(y',0)||^n}$ and $\frac{x_n^3}{||x-(y',0)||^{n+2}}$, for fixed $y' \in \boldsymbol{R}^{n-1}$, are biharmonic in $\boldsymbol{R}_+^n$ as functions of variable x, it follows that

$$\Delta^2 P(\mu, \eta, \boldsymbol{R}^{n-1})(x) = 0$$

for $x \in \boldsymbol{R}_+^n$. Then the condition a) is proved.

To prove conditions b) and c) we first show some lemmas. Let us denote

$$P_1(\mu, \boldsymbol{R}^{n-1})(x) := \frac{x_n^2}{\alpha_n} \int_{\boldsymbol{R}^{n-1}} \frac{1}{||x-y||^n} d\mu(y')$$

$$P_2(\eta, \boldsymbol{R}^{n-1})(x) := \frac{n}{\alpha_n} x_n^3 \int_{\boldsymbol{R}^{n-1}} \frac{1}{||x-y||^{n+2}} d\eta(y')$$

$$P_3(\mu, \boldsymbol{R}^{n-1})(x) := \frac{2}{\alpha_n} x_n \int_{\boldsymbol{R}^{n-1}} \frac{1}{||x-y||^n} d\mu(y')$$

$$P_4(\mu, \boldsymbol{R}^{n-1})(x) := \frac{n}{\alpha_n} x_n^3 \int_{\boldsymbol{R}^{n-1}} \frac{1}{\|x-y\|^{n+2}}\, d\mu(y')$$

$$P_5(\eta, \boldsymbol{R}^{n-1})(x) := \frac{3n}{\alpha_n} x_n^2 \int_{\boldsymbol{R}^{n-1}} \frac{1}{\|x-y\|^{n+2}}\, d\eta(y')$$

$$P_6(\eta, \boldsymbol{R}^{n-1})(x) := \frac{n(n+2)}{\alpha_n} x_n^4 \int_{\boldsymbol{R}^{n-1}} \frac{1}{\|x-y\|^{n+4}}\, d\eta(y'),$$

where $y = (y', 0) \in \boldsymbol{R}_0^n$.

Hence

$$P(\mu, \eta, \boldsymbol{R}^{n-1})(x) = P_1(\mu, \boldsymbol{R}^{n-1})(x) + P_2(\eta, \boldsymbol{R}^{n-1})(x)$$

and by (1)′

$$\frac{\partial}{\partial x_n} P(\mu, \eta, \boldsymbol{R}^{n-1})(x) = P_3(\mu, \boldsymbol{R}^{n-1})(x) - P_4(\mu, \boldsymbol{R}^{n-1})(x) + P_5(\eta, \boldsymbol{R}^{n-1})(x) - P_6(\eta, \boldsymbol{R}^{n-1})(x).$$

LEMMA 1. *Let*

$$I(x_n) := x_n^{\alpha} \int_0^{\infty} \frac{r^{n+\gamma}}{(x_n^2 + r^2)^{\frac{n+\beta}{2}}}\, dr,$$

where $\begin{cases} \alpha = 1, 2, 3, 4, \\ \beta = 2, 4, 6, \\ \gamma = 0, 1, \\ \beta - \gamma > 1, \end{cases}$ *then*

(a) $\lim_{x_n \to 0^+} I(x_n) = 0$

for $\alpha - \beta + \gamma + 1 > 0$,

b) $I(x_n) =$ constant

for $\alpha - \beta + \gamma + 1 = 0\gamma$.

Proof. Substituting $r = x_n \cdot z$, we have

$$I(x_n) = x_n^{\alpha-\beta+\gamma+1} \int_0^{\infty} \frac{z^{n+\gamma}}{(1+z^2)^{n+\beta/2}}\, dz,$$

but

$$\int_0^{\infty} \frac{z^{n+\gamma}}{(1+z^2)^{n+\beta/2}}\, \theta z < \infty,$$

so

a) $\lim_{x_n \to 0^+} I(x_n) = 0$ for $\alpha - \beta + \gamma + 1 > 0$,

(b) $I(x_n)$ constant for $\alpha - \beta + \gamma + 1 = 0$.

LEMMA 2. *Let μ and η be signed measures on $L(\boldsymbol{R}^{n-1})$. We assume Assumption* A *and* B *for μ and η and we assume that*

$$(D_s\mu)(\xi') = (D_s\eta)(\xi') = 0\,.$$

Then

$$R - \lim_{x \to \xi} P(\mu, \eta, \boldsymbol{R}^{n-1})(x) = 0\,,$$

for $x \in \boldsymbol{R}^n_+$.

If we assume additionally that $(D_s^2\eta)(\xi') = 0$, then

$$R - \lim_{x \to \xi} \frac{\partial}{\partial x_n} P(\mu, \eta, \boldsymbol{R}^{n-1})(x) = 0\,.$$

Proof. We shall prove the second part of proposition on an example of integrals P_3 and P_5. The proof of the first part follows analogously.

Let ε be an arbitrary positive number. From the assumption $(D_s\mu)(\xi') = 0$ it follows that there exists a positive number $r_0 = r_0(\varepsilon)$ such that

(1) $$\left| \frac{\mu(K)}{\tau_{n-1} r^{n-1}} \right| < \varepsilon$$

for $r < r_0$, where $K = K_{\xi', r}$. Hence we have

(2) $$|\mu(K)| < \varepsilon \tau_{n-1} r^{n-1}$$

for $0 < r < r_0$.

Let

(3) $$P_3(\mu, \boldsymbol{R}^{n-1})(x) = P_3(\mu, K_{\xi', r_0})(x) + P_3(\mu, \boldsymbol{R}^{n-1} \setminus K_{\xi', r_0})(x)$$

and let

$$F(r) := \int_{K_{\xi', r}} d\mu(y') = \mu(K_{\xi', r})\,.$$

Then we have

$$P_3(\mu, K_{\xi', r_0})(\xi', x_n) = \frac{2}{\alpha_n} x_n \int_{K_{\xi', r_0}} \frac{1}{||(\xi', x_n) - y||^n}\, d\mu(y') = \frac{2}{\alpha_n} x_n \int_0^{r_0} \frac{1}{(x_n^2 + r^2)^{n/2}}\, dF(r)\,.$$

Applying the integration by parts we get

$$P_3(\mu, K_{\xi', r_0})(\xi', x_n) = x_n \frac{F(r_0)}{(x_n^2 + r_0^2)^{n/2}} + n \cdot x_n \int_0^{r_0} \frac{rF(r)}{(x_n^2 + r^2)^{n+2/2}}\, dr\,.$$

From (2) it follows that

$$|F(r)| < \tau_{n-1} r^{n-1} \varepsilon$$

for $0 < r < r_0$, so

$$\left| n x_n \int_0^{r_0} \frac{rF(r)}{(x_n^2 + r^2)^{n+2/2}}\, dr \right| \leqslant n x_n \int_0^{r_0} \frac{rF(r)}{(x_n^2 + r^2)^{n+2/2}}\, dr$$

$$\leqslant n\tau_{n-1}\varepsilon x_n \int_0^{r_0} \frac{r^n}{(x_n^2 + r^2)^{n+2/2}}\, dr \leqslant n\tau_{n-1}\varepsilon x_n \int_0^{\infty} \frac{r^n}{(x_n^2 + r^2)^{n+2/2}}\, dr,$$

but in view of Lemma 1

$$x_n \int_0^{\infty} \frac{r^n}{(x_n^2 + r^2)^{n+2/2}}\, dr = \text{const} = C,$$

hence

$$(4) \qquad |P_3(\mu, K_{\xi', r_0})(\xi', x_n)| \leqslant x_n \frac{|F(r_0)|}{(x_n^2 + r^2)^{n/2}} + \varepsilon\tau_{n-1} n C.$$

On the other hand, by Assumption B we get

$$\left| \int_{R^{n-1} \setminus K_{\xi', r_0}} \frac{1}{||(\xi', x_n) - (y', 0)||^n}\, d\mu(y') \right| \leqslant \int_{R^{n-1} \setminus K_{\xi', r_0}} \frac{1}{||(\xi', 0) - (y', 0)||^n}\, d|\mu|(y')$$

$$\leqslant \int_{r_0}^{\infty} \frac{1}{r^n}\, dF^*(r) < C_3,$$

for an arbitrary fixed $x = (\xi', x_n) \in \boldsymbol{R}^n_+$, where $F^*(r) = |\mu|(K_{\xi', r})$ and C_3 is a positive constant. Hence

$$(5) \qquad |P_3(\mu, \boldsymbol{R}^{n-1} \setminus K_{\xi', r_0})(\xi', x_n)| \leqslant x_n C_3.$$

By (3), (4) and (5) we get

$$|P_3(\mu, \boldsymbol{R}^{n-1})(\xi', x_n)| \leqslant \varepsilon \tilde{C}_3$$

for $0 < x_n < \delta$, where $\tilde{C}_3$ is a positive constant and δ is a sufficiently small positive number.

Hence

$$R\text{-}\lim_{x \to \xi} P_3(\mu, \boldsymbol{R}^{n-1})(x) = 0.$$

We shall prove now that

$$R\text{-}\lim_{x \to \xi} P_5(\eta, \boldsymbol{R}^{n-1})(x) = 0.$$

By the assumption $(D_s^2\eta)(\xi')=0$, so

$$\left|\frac{n\eta(K)}{\theta_{n-1}r^n}\right|<\varepsilon$$

i.e.

$$|\eta(K)|<\frac{\varepsilon}{n}\theta_{n-1}r^n$$

for $0<r<r_1$, where r_1 is some positive number. Denoting

$$G(r):=\int_{K_{\xi',r}} d\eta(y')=\eta(K_{\xi',r})$$

we get

$$|G(r)|<\frac{\varepsilon}{n}\theta_{n-1}r^n.$$

We have

$$P_5(\eta, K_{\xi',r_1})(\xi', x_n)=x_n^2\frac{3n}{\alpha_n}\int_{K_{\xi',r_1}}\frac{1}{||x-(y',0)||^{n+2}}\,d\eta(y')=x_n^2\int_0^{r_1}\frac{1}{(x_n^2+r^2)^{n+2/2}}\,dG(r).$$

Applying integration by parts we get

$$P_5(\eta, K_{\xi',r_1})(\xi', x_n)=x_n^2\frac{G(r_1)}{(x_n^2+r_1^2)^{n+2/2}}+(n+2)x_n^2\int_0^{r_1}\frac{rG(r)}{(x_n^2+r^2)^{n+4/2}}\,dr.$$

It is

$$\left|(n+2)x_n^2\int_0^{r_1}\frac{rG(r)}{(x_n^2+r^2)^{n+4/2}}\,dr\right|\leqslant(n+2)x_n^2\int_0^{r_1}\frac{r|G(r)|}{(x_n^2+r^2)^{n+4/2}}\,dr$$

$$\leqslant\frac{n+2}{n}\theta_{n-1}\varepsilon x_n^2\int_0^{\infty}\frac{r^{n+1}}{(x_n^2+r^2)^{n+4/2}}\,dr,$$

but in view of Lemma 1

$$\frac{n+2}{n}x_n^2\theta_{n-1}\int_0^{\infty}\frac{r^{n+1}}{(x_n^2+r^2)^{n+4/2}}\,dr=\text{constant}=C_4,$$

so

$$|P_5(\eta, K_{\xi',r_1})(\xi', x_n)|\leqslant x_n^2\frac{|G(r_1)|}{(x_n^2+r_1^2)^{n+2/2}}+C_4\varepsilon.$$

Further we prove similarly as for integral P_3.

LEMMA 3. *We assume that signed measures μ, η defined on $L(\mathbf{R}^{n-1})$ satisfy Assumptions* **A** *and* **B** *and they are still nonpositive or non-negative. Moreover we assume that* $(D_s\mu)(\xi')=(D_s\eta)(\xi')=0$.

Then

1° $$\theta - \lim_{x \to \xi} P(\mu, \eta, \boldsymbol{R}^{n-1})(x) = 0$$

If we assume additionally that $(D_s^2|\eta|)(\xi') = 0$, *then*

2° $$\theta - \lim_{x \to \xi} \frac{\partial}{\partial \lambda_\eta} P(\mu, \eta, \boldsymbol{R}^{n-1})(x) = 0 .$$

Proof. We shall carry out the proof for integral $P_1(\mu, \boldsymbol{R}^{n-1})$, for other integrals the proof being analogous.

Let measures μ and η non-negative. We are to prove that

$$\theta - \lim_{x \to \xi} P_1(\mu, \boldsymbol{R}^{n-1})(x) = \theta - \lim_{x \to \xi} \left(x_n^2 \int_{\boldsymbol{R}^{n-1}} \frac{1}{||x-(y', 0)||^n} d\mu(y') \right) = 0 .$$

Let us denote the axis of the cone $\Gamma(\xi, a)$ by p and by w we denote the projection of the point x on the line p. We have

$$\frac{||w-x||}{||w-\xi||} \leqslant a$$

for $x \in \Gamma(\xi, a)$, so

$$||w-y|| \leqslant ||w-x|| + ||x-y|| \leqslant a||w-\xi|| + ||x-y|| \leqslant (a+1)||x-y||$$

and therefore there exists a constant $C_5 > 0$ such that

$$\frac{1}{||x-y||} \leqslant C_5 \frac{1}{||w-y||}$$

for every $x \in \Gamma(\xi, a)$ and $y \in \boldsymbol{R}^{n-1}$. We have now

$$\int_{\boldsymbol{R}^{n-1}} \frac{1}{||x-y||^n} d\mu(y') \leqslant C_6 \int_{\boldsymbol{R}^{n-1}} \frac{1}{||w-y||^n} d\mu(y') ,$$

where $C_6 := C_5^n$, hence

$$0 \leqslant \theta - \lim_{x \to \xi} \left(x_n^2 \int_{\boldsymbol{R}^{n-1}} \frac{1}{||x-y||^n} d\mu(y') \right) \leqslant C_6 \theta - \lim_{x \to \xi} \left(x_n^2 \int_{\boldsymbol{R}^{n-1}} \frac{1}{||w-y||^n} d\mu(y') \right)$$

$$= C_6 R - \lim_{w \to \xi} \left(x_n^2 \int_{\boldsymbol{R}^{n-1}} \frac{1}{||w-y||^n} d\mu(y') \right) = 0 ,$$

but in view of Lemma 2 it is

$$\theta - \lim_{x \to \xi} \left(x_n^2 \int_{\boldsymbol{R}^{n-1}} \frac{1}{||x-y||^n} d\mu(y') \right) = 0 .$$

For non-positive measures μ and η the proof is similar.

We say that *measure* η (*defined on* $L(\mathbf{R}^{n-1})$) *fulfils condition* D *at a point* $\xi' \in \mathbf{R}^{n-1}$ if and only if there exists a function f defined on R^{n-1} such that

1° f is differentiable on $\mathbf{R}^{n-1}$ and its first derivatives are continuous at the point $\xi' \in \mathbf{R}^{n-1}$,

2° $f(y') = O(r^{\alpha})$ for $r \to \infty$, where $r = ||y'||$, $\alpha \in (0, 1)$,

3° the measure

$$\lambda(B) := \eta(B) - \int_B f(y')\,dy'$$

$(B \in L(\mathbf{R}^{n-1}), (\eta(B) < \infty)$, satisfies the condition $(D_s^2|\lambda|)(\xi') = 0$.

LEMMA 4. *We assume that*

1° *measure* η *satisfies condition* D *at the point* $\xi' \in \mathbf{R}^{n-1}$,

2° *there exists the bounded symmetric derivative* $(D_s\mu)(\xi') = b$,

3° *measures* μ *and* η *satisfy Assumption A and B,*

4°
$$(D_s|\delta|)(\xi') = 0\,,$$

where $\delta(B) := \mu(B) - b\int_B dy'$, $B \in L(\mathbf{R}^{n-1})$, $\mathcal{O}_{n-1}(B) < \infty$. *Then*

$$\theta - \lim_{x \to \xi} \frac{\partial}{\partial x_n} P(\mu, \eta, \mathbf{R}^{n-1})(x) = b\,.$$

Proof. Let f be the function given by the condition D. Let us denote

$$I := \theta - \lim_{x \to \xi} \frac{\partial}{\partial x_n} P(\mu, \eta, \mathbf{R}^{n-1})(x) - b\,.$$

We need to proof that $I = 0$.

Similarly as in [2] it can be proved that

$$b = \lim_{x \to \xi} \frac{\partial}{\partial x_n} P(f, b, \mathbf{R}^{n-1})(x)\,,$$

where

$$P(f, b, \mathbf{R}^{n-1})(x) = \frac{x_n^2}{\alpha_n} \int_{\mathbf{R}^{n-1}} \frac{f(y')}{||x - (y', 0)||^n}\,dy' + \frac{n}{\alpha_n} x_n^3 \int_{\mathbf{R}^{n-1}} \frac{b}{||x - (y', 0)||^{n+2}}\,dy'$$

therefore

$$\theta - \lim_{x \to \xi} \frac{\partial}{\partial x_n} P(f, b, \mathbf{R}^{n-1})(x) = b\,.$$

$$I = \theta - \lim_{x \to \xi} \frac{\partial}{\partial x_n} P(\mu, \eta, \mathbf{R}^{n-1})(x) - \theta - \lim_{x \to \xi} \frac{\partial}{\partial x_n} P(f, b, \mathbf{R}^{n-1})(x)\,,$$

then

$$(6)\quad I=\theta-\lim_{x\to\xi}\left[\frac{2}{\alpha_n}x_n\int_{R^{n-1}}\left(\frac{1}{||x-(y',0)||^n}\,d\mu(y')\right.\right.$$

$$\left.-b\int_{R^{n-1}}\frac{1}{||x-(y',0)||^n}\,dy'\right)\Bigg]-\theta-\lim_{x\to\xi}\left[\frac{n}{\alpha_n}x_n^3\left(\int_{R^{n-1}}\frac{1}{||x-(y',0)||^{n+2}}\,d\mu(y')\right.\right.$$

$$\left.-b\int_{R^{n-1}}\frac{1}{||x-(y',0)||^{n+2}}\,dy'\right)\Bigg]+\theta-\lim_{x\to\xi}\left[\frac{3n}{\alpha_n}x_n^2\left(\int_{R^{n-1}}\frac{1}{||x-(y',0)||^{n+2}}\,d\eta(y')\right.\right.$$

$$\left.-\int_{R^{n-1}}\frac{f(y')}{||x-(y',0)||^{n+2}}\,dy'\right]-\theta-\lim_{x\to\xi}\left[\frac{n(n+2)}{\alpha_n}x_n^4\left(\int_{R^{n-1}}\frac{1}{||x-(y',0)||^{n+4}}\,d\eta(y')\right.\right.$$

$$\left.\left.-\int_{R^{n-1}}\frac{f(y')}{||x-(y',0)||^{n+4}}\,dy'\right)\right].$$

Let r_0 denote a positive number such that

$$|(K_{\xi',r})|<C_2r^{\beta_2}\quad\text{for } r>r_0\,.$$

By the condition D we have

$$\int_{K_{\xi',r_0}}\frac{1}{||x-(y',0)||^{n+2}}\,d\lambda(y')=\int_{K_{\xi',r_0}}\frac{1}{||x-(y',0)||^{n+2}}\,d\eta(y')-\int_{K_{\xi',r_0}}\frac{1}{||x-(y',0)||^{n+2}}f(y')\,dy'.$$

For the measure δ it is

$$\int_{K_{\xi',r_0}}\frac{1}{||x-(y',0)||^n}\,d\delta(y')=\int_{K_{\xi',r_0}}\frac{1}{||x-(y',0)||^n}\,d\mu(y')-b\int_{K_{\xi',r_0}}\frac{1}{||x-(y',0)||^n}\,dy'.$$

Hence and from (6) we have

$$I=I_1+I_2\,,$$

where

$$I_1:=\theta-\lim_{x\to\xi}\frac{\partial}{\partial x_n}P(\delta,\lambda,K_{\xi',r_0})(x)\,,$$

$$I_2:=\theta-\lim_{x\to\xi}\frac{\partial}{\partial x_n}P(\mu,\eta,R^{n-1}\setminus K_{\xi',r_0})(x)-\theta-\lim_{x\to\xi}\frac{\partial}{\partial x_n}P(f,b,R^{n-1}\setminus K_{\xi',r_0})(x)\,.$$

But

$$|I_1|\leqslant\theta-\lim_{x\to\xi}\left(\frac{2}{\alpha_n}x_n\int_{K_{\xi',r_0}}\frac{1}{||x-(y',0)||^n}\,d|\delta|(y')\right)$$

$$+\theta-\lim_{x\to\xi}\left(\frac{n}{\alpha_n}x_n^3\int\limits_{K_{\xi',r_0}}\frac{1}{||x-(y',0)||^{n+2}}\,d|\delta|(y')\right)$$

$$+\theta-\lim_{x\to\xi}\left(\frac{3n}{\alpha_n}x_n^2\int\limits_{K_{\xi',r_0}}\frac{1}{||x-(y',0)||^{n+2}}\,d|\lambda|(y')\right)$$

$$+\theta-\lim_{x\to\xi}\frac{n(n+2)}{\alpha_n}x_n^4\int\limits_{K_{\xi',r_0}}\frac{1}{||x-(y',0)||^{n+4}}\,d|\lambda|(y')$$

$$=\theta-\lim_{x\to\xi}\frac{\partial}{\partial x_n}P(|\delta|,|\lambda|,K_{\xi',r_0})(x)$$

$$\leqslant\theta-\lim_{x\to\xi}\frac{\partial}{\partial x_n}P(|\delta|,|\lambda|,\boldsymbol{R}^{n-1})(x)\,.$$

Hence and from Lemma 3 $I_1=0$.

In order to prove that $I_2=0$ it is sufficient to check that

$$(7)\quad \theta-\lim_{x\to\xi}\left[\frac{2}{\alpha_n}x_n\left(\int\limits_{\boldsymbol{R}^{n-1}\setminus K_{\xi',r_0}}\frac{1}{||x-(y',0)||^n}\,d\mu(y')-b\int\limits_{\boldsymbol{R}^{n-1}\setminus K_{\xi',r_0}}\frac{1}{||x-(y',0)||^n}\,dy'\right)\right]=0\,,$$

$$\theta-\lim_{x\to\xi}\left[\frac{n}{\alpha_n}x_n^3\left(\int\limits_{\boldsymbol{R}^{n-1}\setminus K_{\xi',r_0}}\frac{1}{||x-(y',0)||^{n+2}}\,d\mu(y')\right.\right.$$

$$\left.\left.-b\int\limits_{\boldsymbol{R}^{n-1}\setminus K_{\xi',r_0}}\frac{1}{||x-(y',0)||^{n+2}}\,dy'\right)\right]=0\,,$$

$$\theta-\lim_{x\to\xi}\left[\frac{3n}{\alpha_n}x_n^2\left(\int\limits_{\boldsymbol{R}^{n-1}\setminus K_{\xi',r_0}}\frac{1}{||x-(y',0)||^{n+2}}\,d\eta(y')\right.\right.$$

$$\left.\left.-\int\limits_{\boldsymbol{R}^{n-1}\setminus K_{\xi',r_0}}\frac{f(y')}{||x-(y',0)||^{n+2}}\,dy'\right)\right]=0\,,$$

$$\theta-\lim_{x\to\xi}\left[\frac{n(n+2)}{\alpha_n}x_n^4\left(\int\limits_{\boldsymbol{R}^{n-1}\setminus K_{\xi',r_0}}\frac{1}{||x-(y',0)||^{n+4}}\,d\eta(y')\right.\right.$$

$$\left.\left.-\int\limits_{\boldsymbol{R}^{n-1}\setminus K_{\xi',r_0}}\frac{f(y')}{||x-(y',0)||^{n+4}}\,dy'\right)\right]=0\,.$$

These equalities are true, because the integrals in these equalities are bounded by a constant, independent of the point x sufficiently near to the point $(\xi',0)$.

LEMMA 5. *If f is absolutely integrable with respect to the Lebesgue measure on $\mathbf{R}^{n-1}$, then*

(a) $$\theta-\lim_{x\to\xi}\left(\frac{x_n}{\alpha_n}\int_{\mathbf{R}^{n-1}}\frac{f(y')}{\|x-(y',0)\|^n}\,dy'\right)=f(\xi') \quad a.e.$$

and

(b) $$\theta-\lim_{x\to\xi}\left(\frac{n}{\alpha_n}x_n^3\int_{\mathbf{R}^{n-1}}\frac{f(y')}{\|x-(y',0)\|^{n+2}}\,dy'\right)=f(\xi') \quad a.e.$$

Proof. (a) is a consequence of Theorem 1, Chapter VII of [7]. We shall prove (b). In virtue of

$$\int_{\mathbf{R}^{n-1}}\frac{1}{(s_1^2+\ldots+s_n^2+1)^{n+2/2}}\,ds_1\ldots ds_{n-1}=\frac{\alpha_n}{n}$$

(see [1], page 228), applying the change of variables $z_i-k_i=x_n s_i$ $(i=1,\ldots,n-1)$ we have, for arbitrary k_i $(i=1,\ldots,n-1)$

(8) $$\int_{\mathbf{R}^{n-1}}\frac{x_n^3}{(x_n^2+\|z-k\|^2)^{n+2/2}}\,dz=\frac{\alpha_n}{n},$$

$z=(z_1,\ldots,z_{n-1})$, $k=(k_1,\ldots,k_{n-1})$, $dz=dz_1\ldots dz_{n-1}$. We have now

$$I:=\frac{n}{\alpha_n}x_n^3\int_{\mathbf{R}^{n-1}}\frac{f(y')}{\|x-(y',0)\|^{n+2}}\,dy'-f(\xi')$$

$$=\frac{n}{\alpha_n}x_n^3\int_{\mathbf{R}^{n-1}}\frac{f(\xi'-z)}{(x_n^2+\|z-(\xi'-x')\|^2)^{n+2/2}}\,dz-f(\xi')$$

and because of (8) we have

$$I=\frac{n}{\alpha_n}x_n^3\int_{\mathbf{R}^{n-1}}\frac{f(\xi'-z)}{(x_n^2+\|z-(\xi'-x')\|^2)^{n+2/2}}\,dz$$

$$-\frac{n}{\alpha_n}x_n^3\int_{\mathbf{R}^{n-1}}\frac{f(\xi')}{(x_n^2+\|z-(\xi'-x')\|^2)^{n+2/2}}\,dz$$

$$=\frac{n}{\alpha_n}\int_{\mathbf{R}^{n-1}}\frac{x_n^3[f(\xi'-z)-f(\xi')]}{(x_n^2+\|z-(\xi'-x')\|^2)^{n+2/2}}\,dz.$$

It is easy to check that for a fixed α, $0\leqslant\alpha<\frac{\pi}{2}$, there exists a constant A_α such that

$$x_n^2+\|z-(\xi-x')\|^2\geqslant A_\alpha(x_n^2+\|z\|^2)$$

for arbitrary $z \in \boldsymbol{R}^{n-1}$ and $||\xi'-x'|| \leqslant ax_n$, where $a = \operatorname{tg}\alpha$. We have

$$|I| \leqslant \frac{n}{\alpha_n} A_\alpha \int\limits_{\boldsymbol{R}^{n-1}} \frac{x_n^3}{(x_n^2+||z||^2)^{n+2/2}} |f(\xi'-z)-f(\xi')|\,dz$$

for $x_n > 0$.

Let ξ' be an arbitrary fixed Lebesgue-point of the function f (a set of such points have full Lebesgue measure, because f is integrable; see [7], § 1, Chapter 1).

Let us fix an arbitrary $\varepsilon > 0$. There is $\delta > 0$ such that

$$\frac{1}{\mathcal{O}_{n-1}(K_{\xi',r})} \int\limits_{K_{\xi',r}} |f(y')-f(\xi')|\,dy' < \frac{\varepsilon}{2A_\alpha}$$

for $0 < r < \delta$.

Let

$$h(y') := \begin{cases} f(y'-f(\xi') & \text{for } ||y'-\xi'|| \leqslant \delta\,, \\ 0 & \text{for } ||y'-\xi'|| > \delta\,. \end{cases}$$

Then we have

$$(Mh)(\xi') := \sup_{r>0}\left(\frac{1}{\mathcal{O}_{n-1}(K_{\xi',r})} \int\limits_{K_{\xi',r}} |h(y')|\,dy'\right) < \frac{\varepsilon}{2A_\alpha}. \tag{9}$$

Let us denote

$$\varphi(z) := \frac{x_n^3}{(x_n^2+||z||^2)^{n+2/2}}$$

and

$$\varphi_\varepsilon(z) := \varepsilon^{-(n-1)}\varphi\left(\frac{z}{\varepsilon}\right)$$

for $\varepsilon > 0$. We have (see [7], Chapter III, Theorem 2a)

$$\sup_{\varepsilon>0}\left|\int\limits_{\boldsymbol{R}^{n-1}} \varphi_\varepsilon(z)|h(\xi'-z)|\,dz\right| \leqslant \frac{\alpha_n}{n}(Mh)(\xi')\,.$$

In particular, for $\varepsilon = 1$ we have

$$\left|\int\limits_{\boldsymbol{R}^{n-1}} \varphi_1(z)|h(\xi'-z)|dz\right| \leqslant \frac{\alpha_n}{n}(Mh)(\xi')$$

i.e.

$$\int\limits_{\boldsymbol{R}^{n-1}} \frac{x_n^3}{(x_n^2+||z||^2)^{n+2/2}} |h(\xi'-z)|\,dz \leqslant \frac{\alpha_n}{n}(Mh)(\xi')\,.$$

Hence and from (9) we get

$$\int_{R^{n-1}} \frac{x_n^3}{(x_n^2+||z||^2)^{n+2/2}} |h(\xi'-z)|\,dz \leqslant \frac{\alpha_n}{n}\frac{\varepsilon}{2A_\alpha}$$

for $x_n > 0$,
therefore

$$\int_{||z||\leqslant\delta} \frac{x_n^3}{(x_n^2+||z||^2)^{n+2/2}} |f(\xi'-z)-f(\xi')|\,dz = \int_{||z||\leqslant\delta} \frac{x_n^3}{(x_n^2+||z||^2)^{n+2/2}} |h(\xi'-z)|\,dz$$

$$\leqslant \int_{R^{n-1}} \frac{x_n^3}{(x_n^2+||z||^2)^{n+2/2}} |h(\xi'-z)|\,dz \leqslant \frac{\alpha_n}{n}\frac{\varepsilon}{2A_\alpha}.$$

Further, we have

$$\int_{||z||>\delta} \frac{x_n^3}{(x_n^2+||z||^2)^{n+2/2}} [f(\xi'-z)-f(\xi')]\,dz = \int_{||z||>\delta} \frac{x_n^3}{(x_n^2+||z||^2)^{n+2/2}} f(\xi'-z)\,dz$$

$$-f(\xi') \int_{||z||>\delta} \frac{x_n^3}{(x_n^2+||z||^2)^{n+2/2}}\,dz,$$

but

$$\left| \int_{||z||>\delta} \frac{x_n^3}{(x_n^2+||z||^2)^{n+2/2}} f(\xi'-z) \right| dz \leqslant \frac{x_n^3}{(x_n^2+\delta^2)^{n+2/2}} \int_{||z||>\delta} |f(\xi'-z)|\,dz$$

$$\leqslant \frac{x_n^3}{(x_n^2+\delta^2)^{n+2/2}} \int_{R^{n-1}} |f(\xi'-z)|\,dz;$$

is integrable and $\dfrac{x_n^3}{(x_n^2+\delta^2)^{n+2/2}} \to 0$ for $x_n \to 0$,

$$\left| \int_{||z||>\delta} \frac{x_n^3}{(x_n^2+||z||^2)^{n+2/2}} f(\xi'-z)\,dz \right| \leqslant \frac{\varepsilon}{4}\frac{\alpha_n}{nA_\alpha}$$

for ient small x_n.
On t. her hand, we have

$$\left| \int_{||z||>\delta} \frac{x_n^3}{(||z||^2)^{n+2/2}} f(\xi')\,dz \right| \leqslant |f(\xi')|\, x_n^3 \int_{||z||>\delta} \frac{1}{||z||^{n+2}}\,dz$$

$$= \xi')|\, x_n^3\theta_{n-1} \int_\delta^\infty \frac{1}{r^{n+2}} r^{n-2}dr = |f(\xi')|\, x_n^3\theta_{n-1} \int_\delta^\infty r^{-4}dr = |f(\xi')|\, x_n^3 \frac{\theta_{n-1}}{3\delta^3},$$

so

$$\left| \int\limits_{||z||>\delta} \frac{x_n^3}{(x_n^2+||z||^2)^{n+2/2}} f(\xi')dz \right| \leqslant \frac{\varepsilon}{4} \frac{\alpha_n}{nA_\alpha}$$

for sufficiently small x_n.
We have finally

$$|I| \leqslant \varepsilon \quad \text{for sufficiently small } x_n$$

i.e.

$$\theta-\lim_{x\to\xi} \frac{n}{\alpha_n} \int\limits_{\boldsymbol{R}^{n-1}} \frac{x_n^3 f(y')}{||x-(y',0)||^{n+2}} dy' = f(\xi')$$

a.e. with respect to the Lebesgue measure.

THEOREM. *Let μ and η be signed measures defined on $L(\boldsymbol{R}^{n-1})$. We assume Assumptions* A *and* B *for them.*

1° *Then*

$$\theta-\lim_{x\to\xi} P(\mu, \eta, \boldsymbol{R}^{n-1})(x) = (D\eta)(\xi')$$

for almost every $\xi' \in \boldsymbol{R}^{n-1}$ with respect to the Lebesgue measure.

2° *If*

$$|\mu_a(K_{\xi',r})| \leqslant D_1 r^{\gamma_1}, \qquad |\eta_a(K_{\xi',r})| \leqslant D_2 r^{\gamma_2}$$

for sufficiently large r, where D_1, D_2 are positive constants and $n-1<\gamma_1, \gamma_2<n$ and μ_a, μ_s denote the absolutely continuous and the singular part of measure μ, respectively, and if measure η satisfies condition D almost everywhere with respect to the Lebesgue measure, then the limit

$$\theta-\lim_{x\to\xi} \frac{\partial}{\partial x_n} P(\mu, \eta, \boldsymbol{R}^{n-1})(x)$$

exists and it is equal $(D\mu)(\xi')$ almost everywhere with respect to the Lebesgue measure.

Proof. From the Lebesgue theorem on decomposition (see [5], pp. 215—217) we have $\mu = \mu_a+\mu_s$ and $\eta = \eta_a+\mu_s$.
Hence

$$P(\mu, \eta, \boldsymbol{R}^{n-1})(x) = \frac{x_n^2}{\alpha_n} \int\limits_{\boldsymbol{R}^{n-1}} \frac{1}{||x-(y',0)||^n} d\mu_a(y') + \frac{n}{\alpha_n} x_n^3 \int\limits_{\boldsymbol{R}^{n-1}} \frac{1}{||x-(y',0)||^{n+2}} d\eta_a(y')$$
$$+ \frac{x_n^2}{\alpha_n} \int\limits_{\boldsymbol{R}^{n-1}} \frac{1}{||x-(y',0)||^n} d\mu_s(y') + \frac{n}{\alpha_n} x_n^3 \int\limits_{\boldsymbol{R}^{n-1}} \frac{1}{||x-(y',0)||^{n+2}} d\eta_s(y') .$$

It is a well-known fact (see [6], page 155) that

$$(D\mu_s)(\xi') = (D\eta_s)(\xi') = 0$$

a.e. with respect to the Lebesgue measure, so from Lemma 3 it follows that

$$\theta-\lim_{x\to\xi} P(\mu_s, \eta_s, \boldsymbol{R}^{n-1})(x) = 0$$

a.e. with respect to the Lebesgue measure.
We shall prove that

$$\theta-\lim_{x\to\xi}\left(\frac{x_n^2}{\alpha_n}\int_{\boldsymbol{R}^{n-1}} \frac{1}{||x-(y',0)||^n}\, d\mu_a(y')\right)$$

$$+\theta-\lim_{x\to\xi}\left(\frac{n}{\alpha_n} x_n^3 \int_{\boldsymbol{R}^{n-1}} \frac{1}{||x-(y',0)||^{n+2}}\, d\eta_a(y')\right) = (D\eta)(\xi').$$

From the Radon-Nikodym theorem (see [5], page 209) it follows that there exist functions f and g defined and integrable on $\boldsymbol{R}^{n-1}$ such that for every $B \in L(\boldsymbol{R}^{n-1})$ we have

$$\mu_a(B) = \int_B g(y')dy', \qquad \eta_a(B) = \int_B f(y')dy'.$$

It is a well-known fact that

$$(D\eta_a)(x) = f(x)$$

a.e. with respect to the Lebesgue measure, so

$$\frac{x_n^2}{\alpha_n}\int_{\boldsymbol{R}^{n-1}} \frac{1}{||x-(y',0)||^n}\, d\mu_a(y') = \frac{x_n^2}{\alpha_n}\int_{\boldsymbol{R}^{n-1}} \frac{1}{||x-(y',0)||^n}\, g(y')dy'$$

and

$$\frac{nx_n^3}{\alpha_n}\int_{\boldsymbol{R}^{n-1}} \frac{1}{||x-(y',0)||^{n+2}}\, d\eta_a(y') = \frac{nx_n^3}{\alpha_n}\int_{\boldsymbol{R}^{n-1}} \frac{1}{||x-(y',0)||^{n+2}}\, f(y')dy'.$$

Hence and from Lemma 5 we have

$$\theta-\lim_{x\to\xi} P(\mu, \eta, \boldsymbol{R}^{n-1})(x) = f(\xi')$$

a.e. with respect to the Lebesgue measure.

Now we shall prove the second part of theorem. It is

$$(D\mu_s^+)(\xi') = (D\mu_s^-)(\xi') = 0$$

a.e. with respect to the Lebesgue measure.

From this and from Lemma 4 we have

$$\theta-\lim_{x\to\xi} \frac{\partial}{\partial x_n} P(\mu_s^+, \eta, \boldsymbol{R}^{n-1})(x) = 0$$

a.e. with respect to the Lebesgue measure and

$$\theta-\lim_{x\to\xi} \frac{\partial}{\partial x_n} P(\mu_s^-, \eta, \boldsymbol{R}^{n-1})(x) = 0$$

a.e. with respect to the Lebesgue measure, then

$$\theta-\lim_{x\to\xi}\frac{\partial}{\partial x_n}P(\mu_s,\eta,\boldsymbol{R}^{n-1})(x)=0$$

a.e. with respect to the Lebesgue measure. We shall prove that

$$\theta-\lim_{x\to\xi}P(\mu_a,\eta,\boldsymbol{R}^{n-1})(x)=0$$

a.e. with respect to the Lebesgue measure.

Let ξ' be such a point that $(D\mu_a)(\xi')=g(\xi')$ (it is true almost everywhere).

Let us define the measure δ as follows:

$$\delta(B):=\mu_a(B)-g(\xi')\int_B dy'$$

for $B\in L(\boldsymbol{R}^{n-1})$, $\mathcal{O}_{n-1}(B)<\infty$.

By the Lebesgue theorem (see [6], Theorem 8.8) we have

$$D|\delta|(\xi')=0\,,$$

then $D|\delta|(\xi')=0$ a.e. with respect to the Lebesgue measure.

The assumptions of Lemma 4 are satisfied by measures μ_a and η almost everywhere, so

$$\theta-\lim_{x\to\xi}\frac{\partial}{\partial x_n}P(\mu_a,\eta,\boldsymbol{R}^{n-1})(x)=(D_s\mu_a)(\xi')=g(\xi')$$

a.e. with respect to the Lebesgue measure.

Remark. Putting in the last theorem

$$\mu(B):=\int_B f(y')dy' \quad\text{and}\quad \eta(B):=\int_B g(y')dy',$$

where $B\in L(\boldsymbol{R}^{n-1})$, while f and g belong to $L^1(\boldsymbol{R}^{n-1})$ and satisfy following assumptions: f is of class C^1, g is continuous,

$$f(y')=O(r^\alpha),\ g(y')=O(r^\alpha),\ \alpha\in(0,1)\,,$$

we get the classical Fatou theorem for the biharmonic problem in half-space.

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