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On Series of Homogeneous Polynomials Noncontinuable Beyond Their Domain of Convergence

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Abstract. Let Ω be a balanced domain of holomorphy. Then functions which are holomorphic in Ω and noncontinuable beyond Ω form a large set in $\mathcal{O}_\Omega$ equipped with a suitable topology.

0. Introduction. Let f be an analytic function of n complex variables defined by a series of homogeneous polynomials

$$f(z) = \sum_{\nu=0}^{\infty} f_\nu(z), \quad (\deg f_\nu = \nu) \tag{0.1}$$

that converges in a neighbourhood of zero in the space C^n.

Let Ω denote the domain of convergence of series (0.1). Since Ω is the domain of holomorphy, there exists at least one function f holomorphic in Ω and noncontinuable beyond Ω. Let us consider the set $\mathcal{O}_\Omega$ of all functions analytic in Ω and a subset of $\mathcal{O}_\Omega$ consisting of those functions which are noncontinuable beyond Ω. The question arises how large this subset is. L. Bieberbach in his book [2] presented the history of this problem and collected many theorems concerning it for $n = 1$.

The aim of this paper is to generalize three of the results inserted in [2]. Two of them, due to Hausdorff and Polya, say that having set up a suitable topology in $\mathcal{O}_\Delta$, the subset under consideration turns out to be dense and open in $\mathcal{O}_\Delta$, Δ being the unit disc in C. The third one, due to Ryll-Nardzewski and Steinhaus, says that in a Banach space consisting of functions holomorphic in the unit disc the functions noncontinuable beyond the disc form a set of second category.

1. Denotations and definitions.

$$\Delta := \{\lambda \in C: |\lambda| < 1\}, \; B(z_0, r) := \{z \in C^n: ||z - z_0|| < r\}, \; z_0 \in C^n, \; r > 0,$$

$||z||$ being the Euclidean norm in C^n.

For a function f: $C^n \supset S \to C$ we write

$$||f||_S = \sup\{|f(z)|: z \in S\}.$$

$P^\nu(C^n, C)$ denotes the set of all homogeneous polynomials from C^n to C of degree ν ($\nu = 0, 1, 2, \dots$).

If $f\colon \Omega \to C$ is analytic in Ω, then $T_a f$ denotes the Taylor series of the function f centred in $a \in \Omega$, $\varrho(T_a f)$ denotes the radius of convergence of this series.

Let f be holomorphic in a domain Ω. We say that f continues through a point $z_0 \in \partial\Omega$ if there exists a point $a \in \Omega$ such that $\varrho(T_a f) > \|z_0 - a\|$. A holomorphic function $f\colon \Omega \to C$ which continues through a point of $\partial\Omega$ is called continuable beyond Ω.

2. Generalization of the theorems of Hausdorff and Polya. Let $\Omega \subset C^n$ be a bounded balanced domain of holomorphy. Let $\mathcal{O}_\Omega = \{f\colon \Omega \to C\colon f \text{ is analytic}\}$. For every $f \in \mathcal{O}_\Omega$ we have the only representation

$$f(z) = \sum_{\nu=0}^{\infty} f_\nu(z), \qquad z \in \Omega, \tag{2.1}$$

where $f_\nu \in P^\nu(C^n, C)$ for $\nu = 0, 1, 2, \dots$

For $f \in \mathcal{O}_\Omega$ we put

$$u_f(z) = \limsup_{\nu\to\infty} \sqrt[\nu]{|f_\nu(z)|}$$

and

$$u_f^*(z) = \limsup_{\zeta\to z} u_f(\zeta).$$

Then $\Omega \subset \{z \in C^n\colon u_f^*(z) < 1\}$.

Following Hausdorff we introduce in $\mathcal{O}_\Omega$ a topology defined by a fundamental system of neighbourhoods. Let $\varepsilon = (\varepsilon_\nu)_{\nu=0}^\infty$ be a sequence of positive real numbers such that $\lim_{\nu\to\infty} \sqrt[\nu]{\varepsilon_\nu} = 1$ and let $f \in \mathcal{O}_\Omega$. By an ε-neighbourhood of f we mean the set

$$V_{\Omega,\varepsilon}(f) = \left\{g \in \mathcal{O}_\Omega\colon \sup_{z\in\bar\Omega}\left(\limsup_{\nu\to\infty} \frac{|f_\nu(z) - g_\nu(z)|}{\varepsilon_\nu}\right) < 1\right\}.$$

Now we define an equivalence relation in $\mathcal{O}_\Omega$ as follows: let two functions $f, g \in \mathcal{O}_\Omega$ be in relation ($f \sim g$) if and only if $f - g \in \mathcal{O}_{\bar\Omega}$; where $\mathcal{O}_{\bar\Omega} = \operatorname{ind\,lim}_{V \supset \bar\Omega,\, V \text{open}} \mathcal{O}_V$.

2.1 Proposition. *Let $f, g \in \mathcal{O}_\Omega$. Then $f \sim g$ if and only if $u_{f-g}^*(z) < 1$ for every $z \in \partial\Omega$.*

Proof. Let $f, g \in \mathcal{O}_\Omega$ and $f - g \in \mathcal{O}_{\bar\Omega}$. Then there exists a number $r > 0$ such that $f - g \in \mathcal{O}_{\bar\Omega^r}$, where

$$\bar\Omega^r = \{z \in C^n\colon \operatorname{dist}(z, \bar\Omega) < r\},$$

$\operatorname{dist}(z, \bar\Omega)$ denotes the distance between the point z and the set $\bar\Omega$ in Euclidean norm in C^n. Since $\bar\Omega^r$ is a balanced neighbourhood of zero in C^n, the inequality $u_{f-g}^*(z) < 1$ holds for $z \in \bar\Omega^r$ and, in particular, for $z \in \partial\Omega$.

Suppose now that $u_{f-g}^*(z) < 1$ for every $z \in \partial\Omega$. Then $\partial\Omega$ so $\bar\Omega$ is contained in the domain of convergence of series $\sum_{\nu=0}^{\infty}(f_\nu - g_\nu)$. Therefore $f \sim g$.

2.2. Now let us consider the topological space $\mathscr{X} = \mathcal{O}_{\Omega/\sim}$ with the quotient topology. Let $\mathscr{N} = \{[f] \in \mathscr{X}: f$ is noncontinuable beyond $\Omega\}$.

2.3. THEOREM (for $n = 1$ see Theorem 4.2.1, in [2]). *Let Ω be a bounded balanced domain of holomorphy in $\mathbf{C}^n$. Suppose that there exists a family P of homogeneous polynomials such that*

$$\Omega = \operatorname{int}\{z \in \mathbf{C}^n: |p(z)| < 1,\ p \in P\}.$$

Then the set $\mathscr{N}$ is dense in $\mathscr{X}$.

Proof. 1°. Choose an arbitrary sequence $\varepsilon = (\varepsilon_\nu)_{\nu=0}^{\infty}$ of positive real numbers such that $\sqrt[\nu]{\varepsilon_\nu} \to 1$, with $\nu \to \infty$. We shall find a series of homogeneous polynomials $\sum_{\nu=0}^{\infty} g_\nu$, $g_\nu \in P^\nu(C^n, C)$, convergent in Ω to a function g noncontinuable beyond Ω and such that $\|g_\nu\|_\Omega \leqslant \varepsilon_\nu$, $\nu = 0, 1, 2, \ldots$

By hypothesis there exists a countable set $A = \{a_1, a_2, \ldots\}$ contained in $\mathbf{C}^n \setminus \Omega$ satisfying the two following conditions:

(i) $\partial\Omega \subset \bar{A}$,

(ii) for every $z \in A$ there exists a homogeneous polynomial p, not identically equal to 1 such that $|p(z)| = 1$ and $\|p\|_\Omega \leqslant 1$.

Let $(b_s)_{s \in N}$ be a sequence of elements of A such that each element of A is repeated in $(b_s)_{s \in N}$ infinitely many times. By virtue of (ii) for every $s \in N$ we can choose a homogeneous polynomial $p_s \neq 1$ such that $\|p_s\|_\Omega \leqslant 1$ and $|p_s(b_s)| = 1$. Given a sequence of positive integers $(m_s)_{s \in N}$ such that $m_s \deg p_s > 2 m_{s-1} \deg p_{s-1}$, $s \geqslant 2$, put

$f_1 = p_1$,

$f_s = p_s^{m_s}$, $s \geqslant 2$

and write $\nu_s := \deg f_s = m_s \deg p_s$. For an arbitrary integer $\nu \geqslant 0$, we define

$$g_\nu := \begin{cases} 0, & \text{if } \nu \notin \{\nu_s:\ s \in N\} \\ \varepsilon_{\nu_s} f_s, & \text{if } \nu = \nu_s \text{ for some } s \in N. \end{cases}$$

Observe that for every $z \in \partial\Omega$

$$\limsup_{\nu \to \infty} \sqrt[\nu]{|g_\nu(z)|} \leqslant \lim_{\nu \to \infty} \sqrt[\nu]{\varepsilon_\nu} = 1.$$

Hence the series $\sum_{\nu=0}^{\infty} g_\nu$ converges in Ω. Moreover,

$$\limsup_{\nu \to \infty} \sqrt[\nu]{|g_\nu(a)|} \geqslant 1,\ a \in A.$$

So Ω is the domain of convergence of $\sum_{\nu=0}^{\infty} g_\nu$. Let

$$g(z) = \sum_{\nu=0}^{\infty} g_\nu(z),\ z \in \Omega.$$

Since the series under consideration is a lacunary one, the function g cannot be continued beyond Ω ([13]).

2°. Let $f \in \mathcal{O}_\Omega$. Given g, constructed in 1°, let us consider the family of functions

$$f_\alpha = \sum_{\nu=0}^{\infty} (f_\nu + \alpha g_\nu), \quad \alpha \in (0, 1).$$

Note that for any $\alpha \in (0, 1)$. $f_\alpha \notin [f]$. Indeed,

$$\limsup_{\nu\to\infty} \sqrt[\nu]{|g_\nu(a)|} \geqslant 1, \; a \in A.$$

Hence, in view of (i),

$$u^*_{f-f_\alpha}(z) = \limsup_{\zeta\to z} (\limsup_{\nu\to\infty} \sqrt[\nu]{\alpha}\sqrt[\nu]{|g_\nu(\zeta)|}) \geqslant 1, \; z \in \partial\Omega.$$

So, by Proposition 2.1, $f_\alpha \notin [f]$. As a consequence of the construction of g we obtain

$$f_\alpha \in V_{\Omega,\varepsilon}(f), \quad \alpha \in (0, 1).$$

We claim that at least one of the functions f_α cannot be continued beyond Ω. Otherwise for every $\alpha \in (0, 1)$ there would exist a point $z \in \Omega$ and a positive integer k such that $\varrho(T_z f_\alpha) \geqslant \operatorname{dist}(z, \partial\Omega) + 1/k$. Let D be a countable subset of Ω, dense in Ω. To every $\alpha \in (0, 1)$ there corresponds a pair $(z, k) \in D \times N$ such that $\varrho(T_z f_\alpha) \geqslant \operatorname{dist}(z, \partial\Omega) + 1/k$. In view of the countability of $D \times N$ and the uncountability of $(0, 1)$ we can find two different numbers $\alpha_1, a_2 \in (0, 1)$ and a pair $(z, k) \in D \times N$ such that

$$\varrho(T_z f_{\alpha_i}) \geqslant \operatorname{dist}(z, \partial\Omega) + 1/k \quad (i = 1, 2).$$

Therefore $\varrho(T_z(f_{\alpha_1} - f_{\alpha_2})) \geqslant \operatorname{dist}(z, \partial\Omega) + 1/k$.

$$\text{But } (f_{\alpha_1} - f_{\alpha_2})(z) = (\alpha_1 - \alpha_2)\sum_{\nu=0}^{\infty} g_\nu(z) = (\alpha_1 - \alpha_2) g(z), \; z \in \Omega.$$

This leads to a contradiction since g is noncontinuable beyond Ω. The proof is ended.

A class of domains which satisfy the hypothesis of Theorem 2.3 is given by the following

2.4. PROPOSITION. *Let Ω be a bounded balanced domain of holomorphy in $\mathbf{C}^n$. Assume that $\partial\Omega$ does not contain a ring (i.e. the intersection of $\partial\Omega$ with any complex vector line is a circle). Then there exists a family P of homogeneous polynomials from $\mathbf{C}^n$ to $\mathbf{C}$ such that*

$$\Omega = \operatorname{int}\{z \in \mathbf{C}^n : |p(z)| < 1, \; p \in P\}.$$

Proof. We define $P := \{p: C^n \to \mathbf{C}: p$ is a homogeneous polynomial, $\|p\|_\Omega = 1$, $\deg p \geqslant 1\}$. Then

$$\Omega \subset \Omega' := \operatorname{int} \bigcap_{p\in P} \{z \in \mathbf{C}^n : |p(z)| < 1\}.$$

We shall prove that $\Omega' \subset \Omega$. With this aim in view fix $z_0 \in \Omega'$ and take a number $r > 1$ such that $rz_0 \in \Omega'$. Then for every $p \in P$ $|p(rz_0)| < 1$. Therefore the point rz_0 belongs to the hull of $\bar{\Omega}$ convex with respect to homogeneous polynomials. By hypothesis $\bar{\Omega}$ is a compact subset of $r\Omega$. But $r\Omega$, as a balanced domain of holomorphy, is convex with respect to homogeneous polynomials. Hence $rz_0 \in r\Omega$, so $z_0 \in \Omega$. The proof is completed.

2.5. Remark. Note that there exist balanced domains of holomorphy of the form

$$\Omega = \operatorname{int} \bigcap_{p \in P} \{z \in C^n : |p(z)| < 1\},$$

the boundaries of Ω contain rings, P being a family of homogeneous polynomials (see Theorems 3.1 and 4.1 in [11]).

2.6. Lemma. *Let Ω be a bounded balanced domain in C^n such that $\partial\Omega$ does not contain a ring. Assume that $f \in \mathcal{O}_\Omega$ is noncontinuable beyond Ω. Put $\mathscr{A} = \{z \in \partial\Omega : f_z$ continues analytically through $1\}$, where $f_z : \lambda \in \lambda \to f(\lambda z) \in C$, $z \in \partial\Omega$. Then $\mathscr{A} = \mathscr{C} \cap \partial\Omega$, $\mathscr{C}$ being a pluripolar cone in C^n.*

Proof. By hypothesis Ω is the Mittag-Leffler star of the function f (for definition see [5]). Let

$$f(z) = \sum_{\nu=0}^{\infty} f_\nu(z), \quad z \in \Omega; \ f_\nu \in P^\nu(C^n, C), \ \nu = 0, 1, 2, \ldots$$

For every $k \in N$ let us consider the k-th function associated with f given by the formula

$$F_k(z) = \sum_{\nu=0}^{\infty} \frac{f_\nu(z)}{\Gamma(1+\nu/k)}, \qquad z \in C^n$$

and its regularized radial indicator

$$H_k^*(z) = \limsup_{\zeta \to z} H_k(\zeta),$$

where

$$H_k(\zeta) = \limsup_{t \to \infty} t^{-k} \ln |F_k(t\zeta)|, \ \zeta \in C^n.$$

Note that for $n = 1$ the function H_k is continuous ([6]), so in that case $H_k = H_k^*$, $k = 1, 2, \ldots$

We claim that $\mathscr{A} = \bigcup_{k=1}^{\infty} A_k$, where $A_k = \{z \in \partial\Omega : H_k(z) < H_k^*(z)\}$. Indeed, given any $z \in \partial\Omega$, we have: f_z continues analytically through 1 if and only if there exists $k \in N$ such that $H_k(z) < 1$. This fact is a consequence of Theorem 4 in [5] for $n = 1$ (since $H_k(z)$ is the value at 1 of the indicator of k-th function associated with f_z). Furthermore, by the assumptions on Ω and by Theorem 4 in [5] we obtain $\Omega = \{z \in C^n : H_k^*(z) < 1\}$, $k = 1, 2, \ldots$ and $H_k^*(z) = 1$, $z \in \partial\Omega$, $k = 1, 2, \ldots$. Hence, for $z \in \partial\Omega$, f_z continues analytically through 1 if and only if there exists $k \in N$ such that $H_k(z) < H_k^*(z)$.

Let $\mathscr{C} = \bigcup_{t \in (0, \infty)} t\mathscr{A}$. Then $\mathscr{C} = \bigcup_{k=1}^{\infty} \{z \in C^n : H_k(z) < H_k^*(z)\}$, since the functions H_k and H_k^* are positively homogeneous (of order k). Hence $\mathscr{C}$ is negligible (see [9]) and thus pluripolar ([1]).

2.7. PROPOSITION. *Let Ω and $f \in \mathcal{O}_\Omega$ satisfy the assumptions of Lemma 2.6. Put $\mathcal{E} := \{z \in \partial\Omega \colon f_z$ continues analytically beyond $\Delta\}$. Then the cone $\bigcup_{t \in (0,\infty)} t\mathcal{E}$ is pluripolar.*

Proof. Let $\{\theta_1, \theta_2, \ldots\} = Q \cap [0, 2\pi)$ where Q denotes the set of rational numbers. Given any $j \in N$ we write

(1) $f_j(z) = f(e^{i\theta_j} z)$, $z \in \Omega$,

(2) $E_j = \{z \in \partial\Omega \colon f_{j,z}$ continues analytically through $1\}$,

where $f_{j,z}\colon \Delta \ni \lambda \to f_j(\lambda z) \in C$.

Then $\mathcal{E} = \bigcup_{j=1}^{\infty} E_j$. By Lemma 2.6 the set $\mathcal{C}_j = \bigcup_{t \in (0,\infty)} tE_j$ is pluripolar for every $j \in N$. Hence $\bigcup_{t \in (0,\infty)} t\mathcal{E} = \bigcup_{j=1}^{\infty} \mathcal{C}_j$ is pluripolar as a countable union of pluripolar sets.

2.8. Remark. Since the set $\mathcal{E}$ is of the type F_σ it is of the first category in $\partial\Omega$. In the case when Ω is a ball in C^n this fact was proved (in another way) by Cima and Globevnik ([4]).

2.9. COROLLARY. *Let Ω satisfy the assumptions of Lemma 2.6 and let $f \in \mathcal{O}_\Omega$. Then f is noncontinuable beyond Ω if and only if there exists a subset Z of $\partial\Omega$ dense in $\partial\Omega$ and such that f_z is noncontinuable beyond Δ for every $z \in Z$.*

2.10. THEOREM (for $n = 1$ see Theorem 4.2.2 in [2]). *Let Ω be a bounded balanced domain of holomorphy in C^n such that $\partial\Omega$ does not contain a ring. Then $\mathcal{N}$ is open in $\mathcal{X}$.*

Proof. Let $f \in \mathcal{O}_\Omega$ be noncontinuable beyond Ω. We shall find a neighbourhood of f consisting of functions which are not continuable beyond Ω.

Let $Z = \{z_1, z_2, \ldots\}$ be a countable subset of $\partial\Omega$, dense in $\partial\Omega$ and such that f_z is noncontinuable beyond Δ for every $z \in Z$. By Theorem 4.2.2 in [2] for each $k \in N$ we can choose a sequence $\varepsilon^{(k)} = (\varepsilon_\nu^{(k)})_{\nu=0}^{\infty}$ of positive real numbers satisfying the following conditions

(1) $\lim_{\nu\to\infty} \sqrt[\nu]{\varepsilon_\nu^{(k)}} = 1$,

(2) for every $\varphi \in V_{\Delta,\varepsilon^{(k)}}(f_{z_k})$ φ is noncontinuable beyond Δ.

Now let us take a strictly increasing sequence $(\mu_k)_{k=1}^{\infty}$ of positive integers such that for any $k \in N$

$$\sqrt[\nu]{\varepsilon_\nu^{(j)}} \geq 1 - \frac{1}{k} \quad \text{if } j \in \{1, 2, \ldots, k\}, \quad \nu \geq \mu_k .$$

Finally we define a new sequence $\varepsilon = (\varepsilon_\nu)_{\nu=0}^{\infty}$ as follows

$$\varepsilon_0 = \varepsilon_1 = \ldots = \varepsilon_{\mu_1 - 1} = 1 ,$$
$$\varepsilon_\nu = \min\{\varepsilon_\nu^{(1)}, \varepsilon_\nu^{(2)}, \ldots, \varepsilon_\nu^{(k)}\}, \ \mu_k \leq \nu < \mu_{k+1}, \ k \in N.$$

Then $\lim_{\nu\to\infty} \sqrt[\nu]{\varepsilon_\nu} = 1$ and for each $k \in N$

$$\varepsilon_\nu \leq \varepsilon_\nu^{(k)}, \ \nu \geq \mu_k . \tag{2.2}$$

We claim that any function $g \in V_{\Omega,\varepsilon}(f)$ is noncontinuable beyond Ω. Indeed, let $g \in V_{\Omega,\varepsilon}(f)$. Then for every $k \in N$ $g_{z_k} \in V_{\Delta,\varepsilon}(f_{z_k})$. By (2.2) $V_{\Delta,\varepsilon}(f_{z_k}) \subset V_{\Delta,\varepsilon(k)}(f_{z_k})$. Hence g_{z_k} does not continue beyond Δ. Since Z is dense in $\partial\Omega$ g is noncontinuable beyond Ω.

3. Generalization of the theorem of Ryll-Nardzewski and Steinhaus. Siciak observed that the n-dimensional version of the theorem of Ryll-Nardzewski and Steinhaus may be proved by the same method as that of one variable.

3.1. Let X be a Fréchet space. Given a domain $\Omega \subset C^n$, let A be a countable subset of Ω, dense in Ω. Assume that a function $F\colon X \times \Omega \to C$ satisfies the following conditions

1) For any $x \in X$ the function $F_x\colon \Omega \ni \zeta \to F(x, \zeta) \in C$ is analytic in Ω.

2) For any $\zeta \in \Omega$ the function $F_\zeta\colon X \ni x \to F(x, \zeta) \in C$ is linear and continuous in X.

We say that a point $(a, p) \in A \times Q$ is regular with respect to F if $p > \operatorname{dist}(a, \partial\Omega)$ and $\varrho(T_a F_x) \geqslant p$ for all $x \in X$.

Let R denote the set of all pairs $(a, p) \in A \times Q$ regular with respect to F. Set

$$T := \{(a, p) \in (A \times Q) \setminus R\colon p > \operatorname{dist}(a, \partial\Omega)\}.$$

We define

$$M_\nu(a, p) := \{x \in X\colon \varrho(T_a F_x) \geqslant p,\ \|T_a F_x\|_{B(a,p)} \leqslant \nu\},\quad \nu \in N,\ (a, p) \in A \times Q,$$

where

$$\|T_a F_x\|_{B(a,p)} = \sup\{|T_a F_x(\zeta)|,\ \zeta \in B(a, p)\};$$

$$\mathscr{P} := \bigcup_{\nu=1}^{\infty} \bigcup_{(a,p)\in T} M_\nu(a, p), \qquad \mathscr{Q} := X \setminus \mathscr{P};$$

$$\mathscr{G} := \bigcup_{(a,p)\in R} B(a, p) \cap \partial\Omega, \qquad \mathscr{H} := (\partial\Omega) \setminus \mathscr{G}.$$

With this denotations we have

3.2. THEOREM (for $n = 1$ Theorem 4.3.1 in [2]; comp. Theorem 4 in [10], Theorem 10.1 in [12] and § 4.5 in [7]).

(i) *The set $\mathscr{P}$ is of the first category in X.*

(ii) *For every $\zeta \in \mathscr{G}$ and $x \in X$ there exists a point $a \in A$ such that $\varrho(T_a F_x) > \|a - \zeta\|$.*

(iii) *For every $\zeta \in \mathscr{H}$, $x \in \mathscr{Q}$, $a \in A$, $\varrho(T_a F_x) \leqslant \|a - \zeta\|$.*

Proof. (i) By virtue of the Vitali Theorem the sets $M_\nu(a, p)$ are closed in X. So it suffices to prove that $\operatorname{int} M_\nu(a, p)$ is empty if $(a, p) \in T$, $\nu \in N$. Suppose it is not true and choose $\nu \in N$ and $(a, p) \in T$ such that $\operatorname{int} M_\nu(a, p) \neq \emptyset$. Let $x_0 \in \operatorname{int} M_\nu(a, p)$. By definition of T we can find a point $x \in X$ such that $\varrho(T_a F_x) < p$. For a suitable $r > 0$ we have $x_0 + rx \in \operatorname{int} M_\nu(a, p)$. As a consequence of 2) we receive $F(x, \zeta) = \frac{1}{r}[F(x_0 + rx, \zeta) - F(x_0, \zeta)]$, $\zeta \in \Omega$. Hence $\varrho(T_a F_x) \geqslant p$. Contradiction.

(ii) This part of the theorem is an immediate consequence of the definition of the set $\mathscr{G}$.

(iii) Suppose that there exist $\zeta \in \mathscr{H}$, $x \in \mathscr{Q}$ and $a \in A$ such that $\varrho(T_a F_x) > \|a - \zeta\|$. Then we can choose such numbers $p \in Q$ and $\nu \in N$ that $x \in M_\nu(a, p)$, $\zeta \in B(a, p)$. Since $x \in \mathscr{Q}$ and $\zeta \in \partial\Omega$ the pair (a, p) belongs to R. Hence $\zeta \in \mathscr{G}$. But this is in contradiction with hypothesis ($\zeta \in \mathscr{H}$). The proof is concluded.

Let Ω be a domain in $\boldsymbol{C}^n$. Let us consider the Fréchet space $\mathcal{O}_\Omega$ with the topology of uniform convergence on compact subsets of Ω. From Theorem 3.2 we derive

3.3. COROLLARY. *Let Ω be a domain in $\boldsymbol{C}^n$ and let A be a countable dense subset of Ω. Assume that for every $\zeta \in \partial\Omega$ there exists a function $g \in \mathcal{O}_\Omega$ such that $\varrho(T_a g) \leqslant \|\zeta - a\|$, $a \in A$. Then Ω is the domain of holomorphy and the set $\{g \in \mathcal{O}_\Omega$: g is noncontinuable beyond $\Omega\}$ is of the second category in $\mathcal{O}_\Omega$.*

Proof. Put $X = \mathcal{O}_\Omega$, $F\colon X \times \Omega \ni (g, \zeta) \to g(\zeta) \in \boldsymbol{C}$. Then F satisfies the hypothesis of Theorem 3.2. In the case under consideration the set $\mathscr{G}$ is empty. Hence $\mathscr{H} = \partial\Omega$ and by (iii) of Theorem 3.2 all functions belonging to $\mathscr{Q}$ are noncontinuable beyond Ω.

3.4. Given a bounded balanced domain of holomorphy $\Omega \subset \boldsymbol{C}^n$, let us consider the two following Banach spaces:

$$X_1 := \{f = \sum_{\nu=0}^{\infty} f_\nu \colon f_\nu \in P^\nu(\boldsymbol{C}^n, \boldsymbol{C}),\ \nu = 0, 1, \ldots;\ \sum_{\nu=0}^{\infty} \|f_\nu\|_\Omega < \infty\},$$

$$X_2 := \{f = \sum_{\nu=0}^{\infty} f_\nu \colon f_\nu \in P^\nu(\boldsymbol{C}^n, \boldsymbol{C}),\ \nu = 0, 1, \ldots;\ \sup_\nu \|f_\nu\|_\Omega < \infty\}$$

with norms

$$\|f\|_1 = \sum_{\nu=0}^{\infty} \|f_\nu\|_\Omega, \quad f \in X_1$$

and

$$\|f\|_2 = \sup_\nu \|f_\nu\|_\Omega, \quad f \in X_2.$$

3.5. THEOREM. *The following conditions are equivalent*

1) *There exists a series $f \in X_1$ that converges in Ω to a function noncontinuable beyond Ω.*
2) *There exists a series $f \in X_2$ that converges in Ω to a function noncontinuable beyond Ω.*
3) *There exists a family P of homogeneous polynomials such that*

$$\Omega = \operatorname{int}\{z \in \boldsymbol{C}^n\colon |p(z)| < 1,\ p \in P\}.$$

Proof. 1) $\Rightarrow$ 2). This implication is obvious. 2) $\Rightarrow$ 3). Suppose that $f \in X_2$ is noncontinuable beyond Ω. Set

$$P_f := \left\{\frac{1}{\|f_\nu\|_\Omega} f_\nu \colon \nu \in N,\ f_\nu \neq 0\right\},$$

$$\Omega' := \operatorname{int}\{z \in \boldsymbol{C}^n\colon |p(z)| < 1,\ p \in P_f\}.$$

We claim that $\Omega = \Omega'$. Suppose it is not true, i.e. $\Omega \subsetneq \Omega'$. Then we can find a point $\zeta_0 \in \partial\Omega$ and a number $r > 0$ such that $B = B(\zeta_0, r) \subset\subset \Omega'$. Now $\|p\|_B \leqslant \|p\|_{\Omega'} \leqslant 1$, $p \in P_f$. Hence

$$\|f_\nu\|_B \leqslant \|f_\nu\|_{\Omega'} \leqslant \|f_\nu\|_\Omega \leqslant \|f\|_2, \quad \nu \in N.$$

Choose $t \in (0, 1)$ so as to $\zeta_0 \in tB$. Then

$$\|f_\nu\|_{tB} \leqslant t^\nu \|f_\nu\|_B \leqslant t^\nu \|f\|_2 .$$

Therefore the series $\sum_{\nu=0}^{\infty} f_\nu$ converges in a neighbourhood tB of the point ζ_0. This leads one to contradiction.

3) $\Rightarrow$ 1). Given a family P of homogeneous polynomials such that

$$\Omega = \operatorname{int}\{z \in C^n: |p(z)| < 1, \ p \in P\}$$

one may obtain a series $f \in X_1$ noncontinuable beyond Ω by repeating the construction from the point 1° of the proof of Theorem 2.3 for the sequence $\varepsilon_\nu = 1/\nu^2$, $\nu \in N$.

Assume that a bounded domain Ω satisfies the condition 3) of Theorem 3.5. Then, as an immediate consequence of Theorems 3.2 and 3.5, we obtain the following corollaries.

3.6. Corollary. *The set $\{f \in X_i : f$ is noncontinuable beyond $\Omega\}$ is of the second category in X_i $(i = 1, 2)$.*

3.7. Corollary. *There exists a function $f \in \mathcal{O}_\Omega$, continuous in $\bar{\Omega}$ and noncontinuable beyond Ω.*

3.8. Remark. There exist bounded balanced domains of holomorphy which do not satisfy the condition 3) of Theorem 3.5. They are the domains of the form

$$\Omega = \{z \in C^n: \Phi(z) < 1\} \tag{3.1}$$

where Φ is plurisubharmonic in C^n with nonpluripolar set of discontinuities and such that $\Phi(\lambda z) = |\lambda| \Phi(z)$ for $\lambda \in C$, $z \in C^n$.

Indeed, let Ω be of the form (3.1) and suppose that Ω satisfies 3) of Theorem 3.5. Then there exists a bounded function $f \in \mathcal{O}_\Omega$ noncontinuable beyond Ω. Let $f = \sum_{\nu=0}^{\infty} f_\nu$, $f_\nu \in P^\nu(C^n, C)$. Then

$$\Phi = (\limsup_{\nu \to \infty} \sqrt[\nu]{|f_\nu|})^*$$

and by boundedness of f

$$\Phi = \left(\sup_{\nu \in N} \sqrt[\nu]{\frac{|f_\nu|}{M}}\right)^*,$$

where $M = \|f\|_\Omega$ (see [14]). But the last function admits discontinuities at most at the points of a pluripolar set.

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