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NORMAL FUNCTORS AND RETRACTORS IN CATEGORIES OF ENDOMORPHISMS

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Abstract. We introduce normal functors in categories of endomorphisms. We prove the isomorphisms–inducing property of normal functors from the category of endomorphisms over an arbitrary category $\mathcal{E}$ into its subcategory of automorphisms. We show the existence of special normal functors called retractors in endomorphisms categories over categories admitting direct or inverse limits. We give further examples in the category of modules over a fixed ring R.

0. Introduction

The Leray functor introduced in [**Mr1**,§4], is defined on the category of endomorphisms over the category of modules and takes values in its full subcategory: the category of automorphisms. An important property of the Leray functor is the isomorphisms inducing theorem, which is used in [**Mr1**] to construct the Conley index for discrete dynamical systems. Another property useful in applications is the fact that, when restricted to the category of automorphisms, it is equivalent to the identity functor.

In the present paper we consider the two mentioned features of the Leray functor purely in the category theory setting. In order to prove the abstract version of the isomorphisms inducing theorem we introduce normal functors in the category of endomorphisms over an arbitrary category $\mathcal{E}$. We show that normal functors exist whenever the category $\mathcal{E}$ admits direct or inverse limits of certain sequences. Moreover, they can be chosen so that, like the Leray functor, they are identity functors, when restricted to the subcategory of isomorphisms. Though interesting in itself, from the point of view of topological dynamics this means that various, non-equivalent Conley indices can be constructed (see [**Mr2**]).

We assume the reader knows only basic notions of the category theory; the general reference being [**ML**]. The concepts of (co)products, direct and inverse

The organisation of the paper is as follows. The first paragraph contains preliminaries. In the second paragraph we define the category of endomorphisms. In the following paragraph we introduce normal functors and prove the general isomorphisms inducing theorem. A special kind of functors, called retractors is introduced in the fourth paragraph. In the next paragraph we study normal retractors in the category of modules. Various retractors are compared in the last paragraph.

1. Preliminaries.

The set of natural numbers, including zero, will be denoted by $\mathbf{N}$ and the set of integers by $\mathbf{Z}$. R will denote a fixed ring. If A, B are modules over R and $\varphi\colon A \to B$ is a morphism of modules, then $\ker\varphi$ and $\operatorname{im}\varphi$ will denote the kernel and the image of φ respectively.

$\mathcal{B}$, $\mathcal{C}$ and $\mathcal{E}$ will stand for categories. We will write $A \in \mathcal{B}$ to denote that A is an object of the category $\mathcal{B}$. We will write $\varphi \in \mathcal{B}(A, B)$ or, more frequently, $\varphi\colon A \to B$ in $\mathcal{B}$ to denote that φ is a morphism in $\mathcal{B}$ from A to B. If A, B, $C \in \mathcal{B}$ and $\varphi\colon A \to B$, $\psi\colon B \longrightarrow C$ are morphisms in $\mathcal{B}$ then their composition will be denoted by $\psi\varphi$ and id_A will stand for the identity morphism in A.

Assume $\mathcal{B}$ is a category and $\{A_j | j \in J\} \subseteq \mathcal{B}$. We recall that $B \in \mathcal{B}$ with a family of morphisms $\{\varrho_j\colon B \to A_j | j \in J\}$ is a product of $\{A_j\}$ in $\mathcal{B}$ iff for every object $X \in \mathcal{B}$ and morphisms $\{\eta_j\colon X \to A_j\}$ there exists exactly one morphism $\theta\colon X \to B$ satisfying $\eta_j = \varrho_j\theta$ for $j \in J$. If $\{A_j | j \in J\}$, $\{B_j | j \in J\}$ are two families of objects in $\mathcal{E}$, A', B' with $\{a_j\}$, $\{b_j\}$ are their products in $\mathcal{E}$ and $\{\varphi_j\colon A_j \to B_j | j \in J\}$ are morphisms in $\mathcal{E}$, then, by the definition of the product, there is exactly one morphism $\varphi\colon A' \to B'$ such that $b_j\varphi_j = \varphi a$ for $j \in J$. It is called the product of morphisms $\{\varphi_j\}$.

In the dual way one defines coproducts.

A direct sequence in $\mathcal{B}$ is a sequence of objects $A_n \in \mathcal{B}$ and morphisms $\alpha_n\colon A_n \to A_{n+1}$ of the form

$$A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \dots \tag{1.1}$$

We say that the family of morphisms $\{\varphi_n\colon A_n \to X | n \in \mathbf{N}\}$ coincides with the sequence (1.1) or is a cone from $\{A_n\}$ to X, iff $\varphi_n = \varphi_{n+1}\alpha_n$ for all $n \in \mathbf{N}$.

The object $A' \in \mathcal{B}$ with the cone $\{a_i\}$ from $\{A_i\}$ to A' is the direct limit of the sequence (1.1) iff it satisfies the following universal factorization property:

(1.2) for each object $B \in \mathcal{B}$ and every family of morphisms $\{\varphi_i\colon A \to B |\ i \in \mathbf{N}\}$ coinciding with (1.1) there exists exactly one morphism $\varphi\colon A' \to B$ such that $\varphi_i = \varphi a_i$ for $i \in \mathbf{N}$

In the dual way one defines dual notions: the inverse sequence and the inverse limit.

The universal factorization property implies that the direct and inverse limits, if exist, are unique up to an isomorphism.

Assume $B \in \mathcal{B}$. A morphism $\tau\colon B \to B_0$ in $\mathcal{B}$ is called a reflect of B with respect to the subcategory $\mathcal{B}_0 \subseteq \mathcal{B}$ iff $B_0 \in \mathcal{B}_0$ and the following property (universal factorization property for reflects) is satisfied

(1.3) for each object $B \in \mathcal{B}_0$ and each morphism $\xi\colon B \to X$ in $\mathcal{B}$ there exists exactly one morphism $\theta\colon B_0 \to X$ in $\mathcal{B}_0$ satisfying $\theta\tau = \xi$.

A functor $\Phi\colon \mathcal{B} \to \mathcal{B}_0$ is called a reflector iff it admits a family of morphisms $\{\tau_B\colon B \to \Phi(B) \mid B \in \mathcal{B}\}$ such that for each $B \in \mathcal{B}$ τ_B is a reflect of B with respect to $\mathcal{B}_0$ and for each morphism $\beta\colon A \to B$ in $\mathcal{B}$ we have $\tau_B\beta = \varphi(\beta)\tau_A$.

The dual definitions of coreflect and coreflector are left to the reader.

The following theorem is a special case of a theorem on preserving (general) limits by functors admitting adjoints (see [ML], Chpt. V.5, Th. 1). However, for the reader not familiar with adjoint functors it may be easier to prove the special case as a not difficult excercise.

THEOREM 1.1. *Reflectors transform coproducts in $\mathcal{B}$ into coproducts in $\mathcal{B}_0$. Coreflectors transform products in $\mathcal{B}$ into products in $\mathcal{B}_0$.* ▮

The following simple facts from the category theory are recalled for reference.

PROPOSITION 1.1. *Assume $\mathcal{B}_0$ is a full subcategory of $\mathcal{B}$ and α is a morphism in $\mathcal{B}_0$. Then α is an isomorphism in $\mathcal{B}_0$ iff α is an isomorphism in $\mathcal{B}$.* ▮

PROPOSITION 1.2. *(see [Ry, Lemma 12.4]). Assume α, β, γ are morphisms in the category $\mathcal{B}$. If $\beta\alpha$ and $\gamma\beta$ are defined and isomorphisms then all three morphisms α, β, γ are isomorphisms.* ▮

2. The category of endomorphisms and limit functors

Let $\mathcal{E}$ be an arbitrary category. The category of endomorphisms over the category $\mathcal{E}$, denoted by Endo($\mathcal{E}$) is defined as follows. The objects of Endo($\mathcal{E}$) are pairs (A, a), where $A \in \mathcal{E}$ and $a \in \mathcal{E}(A, A)$ is a distinguished endomorphism of A. The set of morphisms from $(A, a) \in \mathcal{E}$ to $(B, b) \in \mathcal{E}$ is the subset of $\mathcal{E}(A, B)$ consisting of exactly those morphisms $\varphi \in \mathcal{E}(A, B)$ for which the diagram

$$\begin{array}{ccc} A & \xrightarrow{a} & A \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ B & \xrightarrow[b]{} & B \end{array}$$

commutes.

We will write $\varphi\colon (A, a) \to (B, b)$ to denote that φ is a morphism from (A, a) to (B, b) in $\mathrm{Endo}(\mathcal{E})$.

It is straightforward to verify that $\mathrm{Endo}(\mathcal{E})$ satisfies the axioms of the category. In particular, note that identity $\mathrm{id}_A \in \mathcal{E}(A, A)$ is also the identity in $\mathrm{Endo}(\mathcal{E})$.

We define the category of automorphisms of $\mathcal{E}$ as the full subcategory of $\mathrm{Endo}(\mathcal{E})$ consisting of pairs $(A, a) \in \mathrm{Endo}(\mathcal{E})$ such that $a \in \mathcal{E}(A, A)$ is an automorphism, i.e. both an endomorphism and an isomorphism in $\mathcal{E}$. The category of automorphisms of $\mathcal{E}$ will be denoted by $\mathrm{Auto}(\mathcal{E})$.

There is a functorial embedding

$$\mathcal{E} \ni A \to (A, \mathrm{id}_A) \in \mathrm{Auto}(\mathcal{E}),$$
$$\mathcal{E}(A, B) \ni \varphi \to \varphi \in \mathrm{Auto}(\mathcal{E})(A, B)$$

hence we can consider the category $\mathcal{E}$ as a subcategory of $\mathrm{Auto}(\mathcal{E})$.

The proof of the following not difficult proposition is left to the reader.

PROPOSITION 2.1. *If $\mathcal{E}$ admits (co)products then also* $\mathrm{Endo}(\mathcal{E})$ *admits (co)products. The (co)product of isomorphisms is an isomorphism. Hence the (co)product in* $\mathrm{Auto}(\mathcal{E})$ *is also a (co)product in* $\mathrm{Endo}(\mathcal{E})$. ∎

To each object $(A, a) \in \mathrm{Endo}(\mathcal{E})$ one can assign a direct system of morphisms

$$A \xrightarrow{a} A \xrightarrow{a} A \xrightarrow{a} \dots \tag{2.1}$$

and an inverse systems of morphisms

$$\dots \xrightarrow{a} A \xrightarrow{a} A \xrightarrow{a} A \tag{2.2}$$

If (2.1) (or (2.2)) admits the direct (inverse) limit, it will be called the direct (inverse) limit of (A, a).

Denote by $\mathrm{Endo}_d(\mathcal{E})$ ($\mathrm{Endo}_i(\mathcal{E})$) the full subcategory of $\mathrm{Endo}(\mathcal{E})$ consisting of those objects $(A, a) \in \mathrm{Endo}(\mathcal{E})$ for which the sequence (2.1) (or (2.2)) admits the direct (inverse) limit.

It is straightforward to verify that if $(A, a) \in \mathrm{Auto}(\mathcal{E})$ then A is the direct and inverse limit of $(A. a)$. Hence we have the following

PROPOSITION 2.2. Auto($\mathcal{E}$) *is a full subcategory of both* Endo$_d(\mathcal{E})$ *and* Endo$_i(\mathcal{E})$. ∎

Consider $(A,a),(B,b) \in \mathrm{Endo}_d(\mathcal{E})$ and denote the corresponding direct limits with their cones by A', $\{a_i : A \to A' | i \in \mathbf{N}\}$, B', $\{b_i : B \to B' | i \in \mathbf{N}\}$ respectively. Assume $\varphi : (A,a) \to (B,b)$ is a morphism in Endo$_d(\mathcal{E})$. Then the family $\{b_i\varphi\}$ coincides with (A,a). Thus there exists a unique morphism $\varphi : A' \to B'$ such that $b_i\varphi = \varphi a_i$. It will be called the (direct) limit morphism induced by φ.

It follows from the universal factorization property that the limit morphism induced by identity is the identity in the limit space. Moreover, the limit morphism of a composition of morphisms in Endo$_d(\mathcal{E})$ is the composition of the corresponding limit morphisms. Hence the formulae

$$(A,a) \to A', \quad \varphi \to \varphi'$$

define a functor Ld : Endo$_d(\mathcal{E}) \to \mathcal{E}$, which will be called the direct limit functor. In the dual way one defines the inverse limit functor Li : Endo$_i(\mathcal{E}) \to \mathcal{E}$.

3. Normal functors and the isomorphisms inducing theorem

We begin this paragraph with the following

PROPOSITION 3.1. *Assume* $(A,a),(B,b) \in \mathrm{Endo}(\mathcal{E})$. *Then* $\varphi : A \to B$ *is an isomorphism in* Endo($\mathcal{E}$) *iff* φ *is a morphism in* Endo($\mathcal{E}$) *and an isomorphism in* $\mathcal{E}$.

PROOF: If φ is an isomorphism in Endo($\mathcal{E}$) then obviously it is also an isomorphism in $\mathcal{E}$. Hence assume that $\varphi : A \to B$ is a morphism in Endo($\mathcal{E}$) and an isomorphism in $\mathcal{E}$. Then there exists a morphism $\psi : B \to A$ in $\mathcal{E}$ such that $\varphi\psi = \mathrm{id}_B$, $\psi\varphi = \mathrm{id}_A$. All what we need is to show that ψ is a morphism in Endo($\mathcal{E}$). Since φ is a morphism in Endo($\mathcal{E}$), we have $\varphi a = b\varphi$. It follows that $a\psi = \psi\varphi a\psi = \psi b\varphi\psi = \psi b$. Thus ψ is a morphism in Endo($\mathcal{E}$) and the proof is finished. ∎

PROPOSITION 3.2. *Assume* $(A,a) \in \mathrm{Endo}(\mathcal{E})$. *Then* $a : A \to A$ *is also a morphism in* Endo($\mathcal{E}$).

PROOF: The proof reduces to the trivial equality $aa = aa$ for the edomorphism $a : A \to A$. ∎

Assume $\mathcal{C}$ is a full subcategory of Endo($\mathcal{E}$) and $F : \mathcal{C} \to \mathrm{Auto}(\mathcal{E})$ is a functor. Let $(A,a) \in \mathcal{C}$. Then $F(A,a)$ is an object of Auto($\mathcal{E}$). Let a' denote the automorphism distinguished in $F(A,a)$. By Proposition 3.2 $a\colon (A,a) \to (A,a)$ is a morphism in Endo($\mathcal{E}$) and since $\mathcal{C}$ is a full subcategory of Endo($\mathcal{E}$) it is also a morphism in $\mathcal{C}$. Hence $F(a)$ is defined and it is a morphism from $F(A,a)$ to $F(A,a)$ in Auto($\mathcal{E}$). However, it need not be $F(a) = a'$ in general. $F(a)$ need not be even an isomorphism in Auto($\mathcal{E}$).

DEFINITION 3.1. We will say that $F\colon \mathcal{C} \to$ Auto($\mathcal{E}$) is normal, if for each $(A,a) \in \mathcal{C}$ the morphism $F(a)$ is equal to the automorphism distinguished in $F(A,a)$.

We have now the following

THEOREM 3.1. *(on inducing isomorphisms). Assume $\mathcal{C}$ is a full subcategory of* Endo($\mathcal{E}$)*, $F\colon \mathcal{C} \to$* Auto($\mathcal{E}$) *is a normal functor, (A,a), $(B,b) \in \mathcal{C}$ and a commutative diagram*

$$\begin{array}{ccc} A & \xrightarrow{a} & A \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ B & \xrightarrow[b]{} & B \end{array} \tag{3.1}$$

in $\mathcal{E}$ are given. Then we have also the commutative diagram

$$\begin{array}{ccc} (A,a) & \xrightarrow{a} & (A,a) \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ (B,b) & \xrightarrow[b]{} & (B,b) \end{array} \tag{3.2}$$

in $\mathcal{C}$. Moreover, applying functor F to the above diagram we obtain the commutative diagram

$$\begin{array}{ccc} F(A,a) & \xrightarrow{F(a)} & F(A,a) \\ {\scriptstyle F(\varphi)}\downarrow & & \downarrow{\scriptstyle F(\varphi)} \\ F(B,b) & \xrightarrow[F(b)]{} & F(B,b) \end{array} \tag{3.3}$$

in Auto($\mathcal{E}$)*, in which all morphisms are isomorphisms.*

PROOF: The commutativity of the diagram (3.1) in particular means that φ, ψ are not only morphisms in $\mathcal{E}$ but also in Endo($\mathcal{E}$). Hence the diagram (3.2) makes sense. Its commutativity follows from the commutativity of diagram (3.1). Hence also diagram (3.3) is commutative. Since F is normal, it follows that $F(a)$ coincides with the morphism distinguished in $F(A,a) \in \mathrm{Auto}(\mathcal{E})$. Hence $F(a)$ is an isomorphism in $\mathcal{E}$. By Proposition 3.1 it is an isomorphism in $\mathrm{Endo}(\mathcal{E})$and by Proposition 1.1 also an isomorphism in $\mathrm{Auto}(\mathcal{E})$. The same argument shows that $F(b)$ is an isomorphism in $\mathrm{Auto}(\mathcal{E})$. It follows from Proposition 1.2 that also $F(\varphi)$ and $F(\psi)$ are isomorphisms. ∎

Assume $\mathcal{C} \subseteq \mathrm{Endo}(\mathcal{E})$ is a full subcategory and $L\colon \mathcal{C} \to \mathcal{E}$ is a functor. Assume (A,a), $(B,b) \in \mathrm{Endo}(\mathcal{E})$ and $\varphi\colon (A,a) \to (B,b)$ is a morphism in $\mathrm{Endo}(\mathcal{E})$. Put

$$L'(A,a) := (L(A,a), L(a)), \tag{3.4}$$

$$L'(\varphi) := L(\varphi). \tag{3.5}$$

THEOREM 3.2. *Formulae* (3.4)–(3.5) *define a normal functor*

$$L'\colon \mathcal{C} \to \mathrm{Endo}(\mathcal{E}).$$

PROOF: The only thing we need is to verify that $L(\varphi)$ is a morphism in $\mathrm{Endo}(\mathcal{E})$ but this follows from

$$L(\varphi)L(a) = L(\varphi a) = L(b\varphi) = L(b)L(\varphi).$$

∎

The functor L' given by (3.4) and (3.5) will be called the functor associated with the functor L.

Let $\mathrm{Frgt}\colon \mathrm{End}(\mathcal{E}) \to \mathcal{E}$ denote the forgetful functor which maps $(A,a) \in \mathrm{End}(\mathcal{E})$ into $A \in \mathcal{E}$ and preserves morphisms. The careful reader have probably noted that we have also the following converse of Theorem 3.2, which fully characterizes normal functors.

THEOREM 3.3. *If* $L\colon \mathcal{C} \to \mathrm{End}(\mathcal{E})$ *is a normal functor then it is associated with the composite functor* $\mathrm{Frgt} \circ L$. ∎

4. Retractors in arbitrary categories of endomorphisms

Our aim now is to show that, under suitable assumptions about the category $\mathcal{E}$, normal functors $F\colon \mathrm{Endo}(\mathcal{E}) \to \mathrm{Auto}(\mathcal{E})$ exist. Obviously one can give trivial examples: If $0 \in \mathcal{E}$ is the zero object in $\mathcal{E}$, then one can define the zero functor from $\mathrm{Endo}(\mathcal{E})$ to $\mathrm{Auto}(\mathcal{E})$ by assigning $(0, \mathrm{id})$ to each object $(A,a) \in \mathrm{Endo}(\mathcal{E})$ and the identity map in 0 to each morphism $\varphi\colon A \to B$ in $\mathcal{E}$. It is straightforward to verify that the zero functor is a normal functor. We are interested, however, in less trivial examples.

A good condition for nontriviality, which is also useful in applications, is to assume that F restricted to the subcategory $\mathrm{Auto}(\mathcal{E})$ is naturally equivalent to the identity functor. Hence we introduce the following

DEFINITION 4.1. Assume $\mathcal{B}$ is a category and $\mathcal{B}_0$ is its subcategory. The functor $F\colon \mathcal{B} \to \mathcal{B}_0$ will be called a retractor iff F restricted to $\mathcal{B}_0$ is naturally equivalent to the identity functor on $\mathcal{B}_0$.

From the universal factorization property of reflectors and coreflectors we get the following

PROPOSITION 4.1. *Each reflector (coreflector) is a retractor.*

It turns out that the functors associated with limit functors are normal retractors:

THEOREM 4.1. *The functor* $\mathrm{LD} := (\mathrm{Ld})' : \mathrm{Endo}_d(\mathcal{E}) \to \mathrm{Auto}(\mathcal{E})$ *is a normal reflector. In particular it is a retractor and it maps coproducts in* $\mathrm{Endo}_d(\mathcal{E})$ *into coproducts in* $\mathrm{Auto}(\mathcal{E})$.

PROOF: It follows from Theorem 3.2 that

$$\mathrm{LD} : \mathrm{Endo}_d(\mathcal{E}) \to \mathrm{Endo}(\mathcal{E})$$

is a normal functor. Fix $(A, a) \in \mathrm{Endo}(\mathcal{E})$, put $A' := \mathrm{Ld}(A, a)$ and $a' := \mathrm{Ld}(a)$. We need to show that $\mathrm{LD}(A, a) \in \mathrm{Auto}(\mathcal{E})$, i.e. that a' is an isomorphism in $\mathcal{E}$.

A', as the direct limit of (A, a), admits a cone $\{a\colon A \to A'\}$ with the universal factorization property. The family of morphisms $\{s_i := a_{i+1} | i \in \mathbf{N}\}$ is also a cone, hence the universal factorization property implies that there exists a morphism $u\colon A' \to A'$ such that $s_i = ua_i$ for $i \in \mathbf{N}$. Observe that

$$a'ua_i = a's_i = a'a_{i+1} = a_{i+1}a = a_i.$$

Hence the family $\{a_i\}$ factorizes both through $a'u$ and identity. The uniqueness of factorization implies $a'u = \mathrm{id}$. Similarly,

$$ua'a_i = ua_ia = s_ia = a_{i+1}a = a_i$$

and also $ua' = \mathrm{id}$. This shows that a' is an isomorphism, i.e. LD is indeed a functor from $\mathrm{Endo}_d(\mathcal{E})$ into $\mathrm{Auto}(\mathcal{E})$.

We will show that $\tau_A := a_0\colon (A, a) \to (A', a')$ is a reflect. For this end assume that $(C, c) \in \mathrm{Auto}(\mathcal{E})$ and fix a morphism $\xi\colon (A, a) \to (C, c)$. Put $t_i := c^{-i}\xi$ for $i \in \mathbf{N}$. Then $t_ia = c^{-i}\xi a = c^{-i}c\xi = c^{-i+1}\xi = t_{i-1}$, i.e. the family $\{t_i\}$ coincides with the sequence (2.1) and we have a morphism $\zeta\colon A' \to C$ such that $t_i = \zeta a_i$, in particular $\xi = t_0 = \zeta a_0 = \zeta\tau_A$. Observe that

$$c\zeta a_i = ct_i = cc^{-i}\xi = c^{-i}c\xi = c^{-i}\xi a = t_ia = \zeta a_ia = \zeta a'a_i\ .$$

The uniqueness of factorization implies that $c\zeta = \zeta a'$, i.e. ζ is also a morphism in Endo($\mathcal{E}$). Assume $\theta\colon (A', a') \to (C, c)$ is another morphism such that $\xi = \theta\tau_A$. We have

$$\begin{aligned}\theta a_i &= c^{-i}c^i\theta a_i = c^{-i}\theta(a')^i a_i = c^{-i}\theta a_i a^i \\ &= c^{-i}\theta a_0 = c^{-i}\theta\tau_A = c^{-i}\xi = t_i = \zeta a_i\end{aligned}$$

and the uniqueness of factorization implies that $\theta = \zeta$ which proves that $\tau_A = a_0$ is the reflect of (A, a) with respect to Auto($\mathcal{E}$).

Moreover, if $\varphi\colon (A, a) \to (B, b)$ is a morphism in $\mathrm{Endo}_d(\mathcal{E})$ then, by the definition of Ld(φ), we have

$$\tau_B\varphi = b_0\varphi = \varphi' a_0 = \mathrm{LD}(\varphi)\tau_A,$$

which proves that LD is a reflector. The remaining part of the theorem follows from Proposition 4.1 and Theorem 1.1. ∎

In the dual way we obtain

THEOREM 4.2. *The functor* LI := (Li)$'$: *$\mathrm{Endo}_i(\mathcal{E}) \to$* Auto($\mathcal{E}$) *is a normal coreflector. In particular it is a retractor and it maps products in* $\mathrm{Endo}_i(\mathcal{E})$ *into products in* Auto($\mathcal{E}$). ∎

From Proposition 2.1 we obtain the following

COROLLARY 4.1. *If $\mathcal{E}$ admits products (coproducts) then* LI(LD) *maps products (coproducts) in* $\mathrm{Endo}_i(\mathcal{E})(\mathrm{Endo}_d(\mathcal{E}))$ *into products (coproducts) in* Endo($\mathcal{E}$). ∎

5. Retractors in the category of modules

From now on we assume that $\mathcal{E} = \mathrm{Mod}(R)$, the category of modules over a fixed ring R. First we recall the concept of the generalized kernel (see [**Le**]). Assume an endomorphism $a\colon A \to A$ of module A is given. Put

$$\mathrm{gker}(a) := \bigcup \{ker(a^n) \mid n \in \mathrm{N}\}.$$

The dual notion is the generalized image of a:

$$\mathrm{gim}(a) := \bigcap \{\mathrm{im}(a^n) \mid n \in \mathrm{N}\}.$$

From the point of view of Theorem 5.1 below, however, the role dual to gker(a) plays the set

$$\mathrm{sim}(a) := \{x \in A \mid \exists \{x_n\}_{n=1}^{\infty} \subseteq A \ s.t. \ a(x_{n+1}) = x_n \text{ for } n \in \mathrm{N}, x_0 = x\}.$$

which will be called the sequential image of a.

We define the category Mono($\mathcal{E}$) as the full subcategory of Endo($\mathcal{E}$) which objects have a monomorphisms as the distinguished endomorphism. Similarly, by distinguishing epimorphisms, we define the category Epi($\mathcal{E}$).

Assume $(A, a), (B, b) \in \text{Endo}(\mathcal{E})$ and $\varphi\colon (A, a) \to (B, b)$ is a morphism in Endo($\mathcal{E}$). Put

$$\text{Lm}(A, a) := A/\,\text{gker}(a), \tag{5.1}$$

$$\text{Lm}(\varphi) := (A/\,\text{gker}(a) \ni [x] \to [\varphi(x)] \in B/\,\text{gker}(b)), \tag{5.2}$$

$$\text{Le}(A, a) := \text{gim}(a), \tag{5.3}$$

$$\text{Le}(\varphi) := (\text{gim}(a) \ni x \to \varphi(x) \in \text{gim}(a)), \tag{5.4}$$

$$\text{Ls}(A, a) := \text{sim}(a), \tag{5.5}$$

$$\text{Ls}(\varphi) := (\text{sim}(a) \ni x \to \varphi(x) \in \text{sim}(b)). \tag{5.6}$$

One can easily verify that formulae (5.1) – (5.6) define three functors $\text{Lm}, \text{Le}, \text{Ls} : \text{Endo}(\mathcal{E}) \to \mathcal{E}$. By Theorem 3.2 we have also functors

$$\text{LM} := (\text{Lm})', \;\; \text{LE} := (\text{Le})', \;\; \text{LS} := (\text{Ls})' : \text{Endo}(\mathcal{E}) \to \text{Endo}(\mathcal{E}).$$

Assume $(A_i, a_i) \in \text{Endo}(\mathcal{E})$ for $i = 1, 2, \ldots n$. It is straightforward to verify that

$$\text{gker}(a_1 \times a_2 \times \cdots \times a_n) = \text{gker}(a_1) \times \text{gker}(a_2) \times \cdots \times \text{gker}(a_n) \tag{5.7}$$

and

$$\text{sim}(a_1 \times a_2 \times \cdots \times a_n) = \text{sim}(a_1) \times \text{sim}(a_2) \times \cdots \times \text{sim}(a_n). \tag{5.8}$$

From the above formulae one can easily obtain the following

PROPOSITION 5.1. *Both* LM *and* LS *map finite (co)products in* Endo($\mathcal{E}$) *into (co)products in* Endo($\mathcal{E}$). ∎

Also the following proposition is not difficult to verify.

PROPOSITION 5.2. Lm(a) *is a monomorphism in $\mathcal{E}$ and it is an isomorphism in $\mathcal{E}$ whenever a is an epimorphism in $\mathcal{E}$.* Ls(a) *is an epimorphism in $\mathcal{E}$ and it is an isomorphism in $\mathcal{E}$ whenever a is a monomorphism.* ∎

It follows that LM takes values in Mono($\mathcal{E}$) and LS in Epi($\mathcal{E}$). More precisely we have the following

THEOREM 5.1. LM: Endo($\mathcal{E}$) $\to$ Mono($\mathcal{E}$) *is a normal reflector and* LS: Endo($\mathcal{E}$) $\to$ Epi($\mathcal{E}$) *is a normal coreflector.*

PROOF: Fix $(A,a) \in$ Endo($\mathcal{E}$) and put $a' :=$ Lm(a), $a'' :=$ $Ls(a)$. By Proposition 3.2 a' is a monomorphism and a'' is an epimorphism. Hence

$$\mathrm{LM}(A,a) = (\mathrm{Lm}(A,a), \mathrm{Lm}(a)) \in \mathrm{Mono}(\mathcal{E})$$

and

$$\mathrm{LS}(A,a) = (\mathrm{Ls}(A,a), \mathrm{Ls}(a)) \in \mathrm{Epi}(\mathcal{E}).$$

By Theorem 3.2 LM and LS are normal functors. We shall show that LM is a reflector. For this end denote by τ_A the morphism

$$\tau_A \colon A \ni x \to [x] \in \mathrm{Lm}(A,a).$$

It is straightforward that τ_A is also a morphism in Endo($\mathcal{E}$). To prove the universal factorization property consider $(B,b) \in$ Mono($\mathcal{E}$) and assume a morphism $\xi \colon (A,a) \to (B,b)$ in Endo($\mathcal{E}$) is given. We shall prove that there exists exactly one morphism $\zeta \colon \mathrm{LM}(A,a) \to (B,b)$ in Endo($\mathcal{E}$) such that $\zeta\tau_A = \xi$, i.e.

$$\zeta([x]) = \xi(x) \quad \text{for all } x \in A. \tag{5.9}$$

Uniqueness follows at once from (5.9). Assume $x, x' \in A$ and $[x] = [x']$. Then $a^n(x) = a^n(x')$ for some $n \in \mathrm{N}$ and

$$(b^n\xi)(x - x') = (\xi a^n)(x - x') = \xi(0) = 0.$$

Since $(B,b) \in$ Mono($\mathcal{E}$), b is a monomorphism. It follows that $\xi(x - x') = 0$, i.e. $\xi(x) = \xi(x')$. This shows that formula (5.9) defines a morphism $\zeta \colon \mathrm{Lm}(A,a) \to B$ in $\mathcal{E}$. We have

$$\zeta(a'([x])) = \zeta([a(x)]) = \xi(a(x)) = b(\xi(x)) = b(\zeta([x])),$$

which implies that ζ is a morphism in Endo($\mathcal{E}$). Hence we have proved that τ_A is the reflect of (A,a) with respect to Mono($\mathcal{E}$). It is straightforward to verify that if $\varphi \colon (A,a) \to (B,b)$ is a morphism in Endo($\mathcal{E}$) then $\mathrm{LM}(\varphi)\tau_A = \tau_A\varphi$, which proves that LM is a reflector.

Now consider LS. Let $\sigma_A \colon \mathrm{Ls}(A,a) \to A$ denote the inclusion. Take $(B,b) \in$ Epi($\mathcal{E}$) and assume a morphism $\xi \colon (B,b) \to (A,a)$ in Endo($\mathcal{E}$) is given. We shall prove that there exists exactly one morphism $\zeta \colon (B,b) \to \mathrm{LS}(A,a)$ such that $\sigma_A z = \xi$, i.e.

$$\zeta(x) = \xi(x) \text{ for all } x \in B. \tag{5.10}$$

Again uniqueness is straightforward. To show that (5.10) defines a morphism $\zeta\colon (B,b) \to \mathrm{LS}(A,a)$ it suffices to prove that $\xi(x) \in \mathrm{Ls}(A,a)$ for all $x \in B$. Hence take $x \in B$. Since b is an epimorphism, we can define recursively a sequence $(x_n)_{n=0}^{\infty}$ such that $b(x_{n+1}) = x_n$ for $i \in \mathbf{N}$ and $x_0 = x$. Put $y_n := \xi(x_n)$. Then

$$a(y_n) = a\xi(x_n) = \xi b(x_n) = \xi(x_{n-1}) = y_{n-1}.$$

This shows that $y_0 = \xi(x_0) = \xi(x) \in \mathrm{sim}(a)$. Thus (5.10) indeed defines a morphism $\zeta\colon B \to \mathrm{Ls}(A,a)$ in $\mathcal{E}$. Since $a''\xi(x) = \xi b(x)$ for all $x \in B$, we have also $a''\zeta(x) = \zeta b(x)$, which shows that ζ is also a morphism in $\mathrm{Endo}(\mathcal{E})$. Hence σ_A is the coreflect of (A,a) with respect to $\mathrm{Epi}(\mathcal{E})$. It is straightforward that $\varphi\sigma_A = \sigma_A\,\mathrm{LS}(\varphi)$ for any morphism $\varphi\colon (A,a) \to (B,b)$ in $\mathrm{Endo}(\mathcal{E})$. This shows that LS is a coreflector and finishes the proof. ∎

Remark 5.1. It should be noted that $\mathrm{Le}(a)$ for an endomorphism $a\colon A \to A$ need not be an epimorphism (see Example 6.3). This causes that the functor LE can be considered only as a functor from $\mathrm{Endo}(\mathcal{E})$ into $\mathrm{Endo}(\mathcal{E})$. However, it is obvious that LE and LS are equal on the subcategory $\mathrm{Mono}(\mathcal{E})$. In particular $\mathrm{LE} \circ \mathrm{LM} = \mathrm{LS} \circ \mathrm{LM}$. The composite functor $\mathrm{LE} \circ \mathrm{LM}$ was introduced in [**Mr1**] and called the Leray functor, since the idea of a generalized kernel comes from the paper [**Le**] of Leray.

From Theorem 5.1 we get the following

Corollary 5.1. *The composite functors* $\mathrm{LMS} := \mathrm{LM} \circ \mathrm{LS}$ *and* $\mathrm{LSM} := \mathrm{LS} \circ \mathrm{LM}$ *are retractors from* $\mathrm{Endo}(\mathcal{E})$ *to* $\mathrm{Auto}(\mathcal{E})$. *They map finite (co)products in* $\mathrm{Endo}(\mathcal{E})$ *into (co)products in* $\mathrm{Endo}(\mathcal{E})$.

Proof: Let $(A,a) \in \mathrm{Endo}(\mathcal{E})$. Then $\mathrm{LS}(a)$ is an epimorphism in $\mathcal{E}$ and, by Proposition 5.2, $\mathrm{LM}(\mathrm{LS}(a))$ is an isomorphism in $\mathcal{E}$. Hence $\mathrm{LM}(\mathrm{LS}(A,a)) \in \mathrm{Auto}(\mathcal{E})$. It is straightforward to verify that $\mathrm{LM} \circ \mathrm{LS}$ restricted to $\mathrm{Auto}(\mathcal{E})$ is naturally equivalent to the identity functor. Hence LMS is a retractor from $\mathrm{Endo}(\mathcal{E})$ into $\mathrm{Auto}(\mathcal{E})$. Similarly one can prove that LSM is a retractor.

The remaining part of the assertion follows from Proposition 5.1. ∎

6. Comparision of various retractors

In case of the category $\mathcal{E} = \mathrm{Mod}(R)$ we defined four retractors LI, LD, LMS, LSM from $\mathrm{Endo}(\mathcal{E})$ into *Auto*$(\mathcal{E})$. The following three examples show that they are all different. ($R(A)$ in the examples below denotes the free module over R with basis A).

EXAMPLE 6.1. Let A be a free module over R with basis $\mathbf{N}$. Let $a\colon A \to A$ be an endomorphism of A defined on the basis by $a(n) := n+1$ for $n \in \mathbf{N}$ (comp. Fig. 1a). One can easily verify that $\mathrm{LD}(A)$ is non-zero (precisely $\mathrm{Ld}(A,a) = R(\mathbf{Z})$ and $\mathrm{Ld}(a)$ is an endomorphism induced by a shift), whereas

$$\mathrm{LI}(A,a) = \mathrm{LMS}(A,a) = \mathrm{LSM}(A,a) = 0.$$

EXAMPLE 6.2. Let A be as in the above example but take the endomorphism $a\colon A \to A$ defined by (comp. Fig. 1b)

$$a(n) := \begin{cases} 0 & \text{for } n = 0 \\ n-1 & \text{for } n > 0. \end{cases}$$

Then $\mathrm{Li}(A,a) = R(\mathbf{Z}), \mathrm{Li}(a)$ is again a shift automorphism, hence $\mathrm{LI}(A,a) \neq 0$ but $\mathrm{LD}(A,a) = \mathrm{LMS}(A,a) = \mathrm{LSM}(A,a) = 0$.

EXAMPLE 6.3. Take

$$S := \{(m,n) \in \mathbf{Z} \times \mathbf{N} \mid n = 0 \text{ and } m > 0 \text{ or } n > 0 \text{ and } -n \le m \le 0\},$$

$A := R(S)$ and define $a\colon A \to A$ on S by (comp. Fig. 1c)

$$a(m,n) := \begin{cases} (m+1,n) & \text{for } m \neq 0 \\ (1,0) & \text{for } m = 0. \end{cases}$$

Then $\mathrm{LSM}(A,a) = \mathrm{LD}(A,a) = (R(\mathbf{Z}), s)$ with s denoting a shift automorphism and $\mathrm{LMS}(A,a) = \mathrm{LI}(A,a) = 0$.

Note that $\mathrm{Le}(A,a) = \{(n,0) \mid n \in \{1,2,\ldots\}\}$ and $(1,0) \notin \mathrm{im}(\mathrm{Le}(a))$, which shows that $\mathrm{Le}(a)$ need not be an epimorphism.

Despite the above examples, under some restrictions all four retractors LI, LP, LMS, LSM coincide, as shows the following

THEOREM 6.1. *Assume R is a field and $(A,a) \in \mathrm{Endo}(\mathcal{E})$ is a finite dimensional vector space with a distinguished endomorphism. Then $\mathrm{LI}(A,a)$, $\mathrm{LP}(A,a)$, $\mathrm{LMS}(A,a)$, $\mathrm{LSM}(A,a)$ are all isomorphic.*

PROOF: Fix $(A,a) \in \mathrm{Endo}(\mathcal{E})$. Denote $a' := \mathrm{LM}(a)$, $a'' := \mathrm{LS}(a)$, i.e.

$$a'\colon \mathrm{LM}(A,a) = A/\operatorname{gker}(a) \ni [x] \to [a(x)] \in \mathrm{LM}(A,a).$$
$$a''\colon \mathrm{LS}(A,a) = \operatorname{sim}(a) \ni x \to a(x) \in \mathrm{LS}(A,a).$$

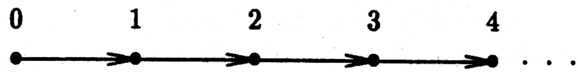

Fig. 1.a

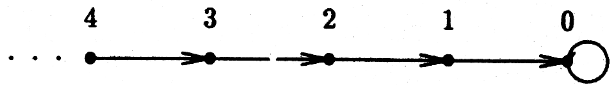

Fig. 1.b

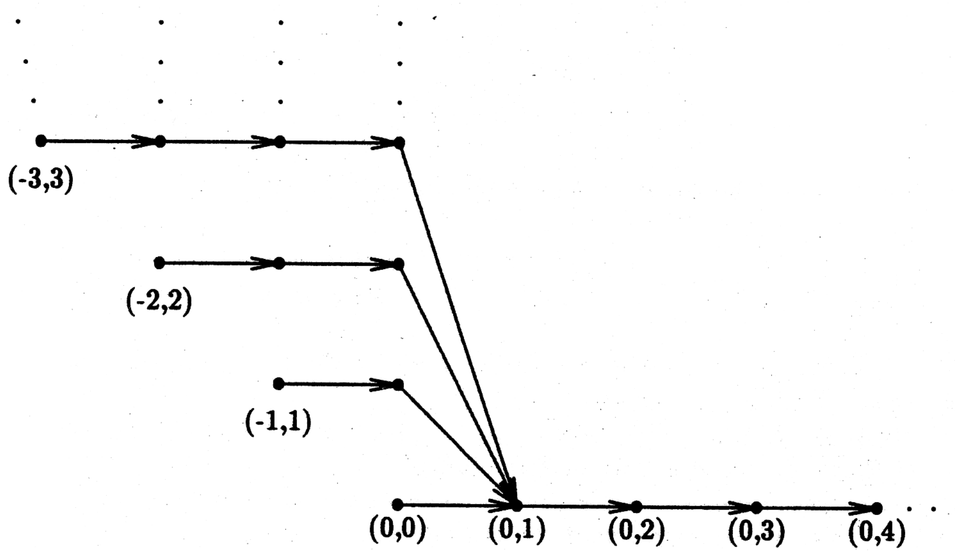

Fig. 1.c

Fig. 1. Graphs of the endomorphism a restricted to basis in Examples 6.1, 6.2 and 6.3 (on Fig. 1.a, 1.b and 1.c respectively)

Since A is finite dimensional, so are $\mathrm{LS}(A, a)$ and $\mathrm{LM}(A, a)$. Moreover a' and a'' are isomorphisms in $\mathcal{E}$ as a monomorphism or epimorphism of finite dimensional vector spaces into itself. It follows that

$$\text{(6.1)} \qquad \mathrm{LSM}(A, a) = \mathrm{LM}(A, a) \quad \text{and} \quad \mathrm{LMS}(A, a) = \mathrm{LS}(A, a).$$

One can easily verify that $\{\ker a^n\}$, $\{\operatorname{im} a^n\}$ are monotone sequences of subspaces of the finite dimensional vector space A. It follows that there exists $p \in \mathbf{N}$ such that $\ker a^n = \ker a^p$ and $\operatorname{im} a^n = \operatorname{im} a^p$ for all $n \geq p$. In particular

$$\operatorname{gker} a = \bigcup \ker a^n = \ker a^p,$$
$$\operatorname{gim} a = \bigcap \operatorname{im} a^n = \operatorname{im} a^p.$$

We will show that under the assumptions of the theorem

$$\text{(6.2)} \qquad \operatorname{sim} a = \operatorname{gim} a.$$

Obviously $\operatorname{sim} a \subseteq \operatorname{gim} a$. To show the opposite inclusion take $x_0 \in \operatorname{gim} a$. Then there exists $w_1 \in A$ such that $x_0 = a^{p+1}(w_1)$. Put $x_1 := a^p(w_1)$. Then $a(x_1) = x_0$ and $x_1 \in \operatorname{im} a^p = \operatorname{gim} a$. Hence we can find $w_2 \in A$ such that $x_1 = a^{p+1}(w_2)$. Put $x_2 := a^p(w_2)$. Proceeding recursively we define a sequence $\{x_n\}$ such that $a(x_{n+1}) = x_n$ for $n \in \mathbf{N}$. This shows that $x_0 \in \operatorname{sim} a$ and (6.2) is proved.

Put $\varphi := a^p$. Since $\ker(\varphi) = \operatorname{gker}(a)$, we have an induced isomorphism

$$\varphi' \colon A/\operatorname{gker}(a) = A/\ker(\varphi) \ni [x] \to \varphi(x) \in \operatorname{im} \varphi = \operatorname{sim}(a).$$

Moreover, one can easily verify that the diagram

$$\begin{array}{ccc} A/\operatorname{gker}(a) & \xrightarrow{\ \varphi'\ } & \operatorname{sim}(a) \\ {\scriptstyle a'}\downarrow & & \downarrow{\scriptstyle a''} \\ A/\operatorname{gker}(a) & \xrightarrow[\ \varphi'\ ]{} & \operatorname{sim}(a) \end{array}$$

is commutative, which means that φ' is a morphism in $\mathrm{Endo}(\mathcal{E})$. It follows from Proposition 3.1 that φ is an isomorphism in $\mathrm{Endo}(\mathcal{E})$, i.e. $\mathrm{LM}(A, a)$ and $\mathrm{LS}(A, a)$ are isomorphic. In view of (6.1) we see that also $\mathrm{LMS}(A, a)$ and $\mathrm{LSM}(A, a)$ are isomorphic.

We will show that

$$\text{(6.3)} \qquad A = \operatorname{gker}(a) \oplus \operatorname{gim}(a) \quad \text{(direct sum)}.$$

First observe that $x \in \text{gker}(a) \cap \text{gim}(a)$ implies $0 = a^m(x) = (a'')^m(x)$. But a'' is an isomorphism, hence $x = 0$. Thus $\text{gker}(a) \cap \text{gim}(a) = 0$ and

$$\begin{aligned}\dim[\text{gker}(a) + \text{gim}(a)] &= \dim(\text{gker}(a)) + \dim(\text{gim}(a)) \\ &= \dim(\ker(a^p)) + \dim(im(a^p)) = \dim A.\end{aligned}$$

It follows that $\text{gker}(a) + \text{gim}(a) = A$ and (6.3) is proved.

In order to show that $\text{LD}(A, a)$ is isomorphic to $\text{LM}(A, a)$ define the family of maps

$$\varphi_n \colon A \ni x \to (a')^{-n}([x]) \in A/\,\text{gker}(a) \quad \text{for } n \in \mathbf{N}.$$

It is straightforward to verify that

$$\varphi_{n+1} a = \varphi_k \quad \text{for } k = 0, 1, 2, \ldots$$

i.e. the family $\{\varphi_k\}$ coincides with the sequence (2.1).

Assume $Z \in \mathcal{E}$ and $\gamma_i \colon A \to Z$ is a cone in $\mathcal{E}$ from (2.1) to Z. If $\Theta \colon A/\,\text{gker}(a) \to Z$ is a morphism in $\mathcal{E}$ such that

$$\Theta \varphi_i = \gamma_i \quad \text{for } i = 0, 1, 2, \ldots \tag{6.4}$$

then, in particular, for each $x \in A$ we have

$$\Theta([x]) = (\Theta \varphi_0)(x) = \gamma_0(x), \tag{6.5}$$

i.e. Θ is uniquely determined by (6.4).

Assume that $[x] = [x']$. Then there exists $m \in \mathbf{N}$ such that $a^m(x) = a^m(x')$ and $\gamma_0(x) = (\gamma_m a^m)(x) = (\gamma_m a^m)(x') = \gamma_0(x')$. This shows that (6.5) defines a morphism $\Theta \colon A/\,\text{gker}(a) \to Z$. We will show that Θ satisfies (6.4). Let $x \in A$. By (6.3) we can find $w \in \text{gker}(a) = \ker(a^p)$ and $y \in A$ such that $x = w + a^k(y)$. Then $[x] = [a^k(y)]$, $a^p(w) = 0$,

$$\Theta \varphi_k(x) = \Theta((a')^{-k}([x]) = \Theta((a')^{-k}([a^k(y)])) = \Theta([y]) = \gamma_0(y)$$

and

$$\gamma_k(x) = \gamma_k(w + a^k(y)) = \gamma_k(w) + \gamma_0(y) = \gamma_{k+p}(a^p(w)) + \gamma_0(y) = \gamma_0(y).$$

Hence $\Theta \varphi_k(x) = \gamma_k(x)$ for all $x \in A$. This shows that $A/\,\text{gker}(a)$ is the direct limit of the sequence (2.1). Moreover, we have

$$a' \varphi_n = \varphi_n a \quad \text{for all } n \in \mathbf{N}$$

and it follows from the uniqueness of factorization that

$$\mathrm{LM}(A,a) = (A/\,\mathrm{gker}(a), a') = \mathrm{LD}(A,a).$$

It remains to show that $\mathrm{LS}(A,a) = \mathrm{LI}(A,a)$. For this end define the family of maps

$$\varrho\colon \mathrm{gim}(a) \ni x \to (a'')^{-k}(x) \in \mathrm{gim}(a).$$

Then $a\varrho_{k+1} = \varrho_k$ for $k \in \mathbf{N}$, i.e. the family $\{\varrho_k\}$ coincides with the sequence (2.2). Assume $Z \in \mathcal{E}$ and $\tau_i\colon Z \to A$ is another family of morphisms which coincide with (2.2). If $\xi\colon Z \to \mathrm{gim}(a)$ is a morphism in $\mathcal{E}$ such that

(6.6) $$\varrho_k \xi = \tau_k \quad \text{for } k \in \mathbf{N}$$

then, in particular for $x \in Z$

(6.7) $$\xi(x) = \varrho_0(\xi(x)) = \tau_0(x),$$

i.e. ξ is uniquely determined by (6.6). Observe that for each $x \in A$, $\tau_k(x) \in \mathrm{gim}(a)$, because $\tau_k(x) = a^l(\tau_{k+l}(x))$ for all $l \in \mathbf{N}$. Hence (6.7) determines a morphism $\xi\colon Z \to \mathrm{gim}(a)$ and

$$\tau_0(x) = a^k(\tau_k(x)) = (a'')^k(\tau_k(x))$$

implies

$$\varrho_k\xi(x) = (a'')^{-k}(\tau_0(x)) = \tau_k(x).$$

This shows that $\mathrm{gim}(a)$ is the inverse limit of (2.2). Since also

$$\varrho_k a'' = a\varrho_k \quad \text{for all } k \in \mathbf{N},$$

the uniqueness of factorization applied to $\{a\varrho_k\}$ implies

$$\mathrm{LS}(A,a) = (\mathrm{gim}(a), a'') \sim \mathrm{LI}(A,a),$$

which finishes the proof. ∎

Note that in view of (6.2) we have also

THEOREM 6.2. *Functors* LS *and* LE *are equal on finite dimensional vector spaces. In particular, the same applies to compositions* $\mathrm{LM} \circ \mathrm{LS}$ *and* $\mathrm{LM} \circ \mathrm{LE}$. ∎

We finish the paper with some open questions. The results of §5–6 are restricted to the category of modules over a fixed ring. It would be interesting to know whether they can be carried over to more general categories, for instance to abelian categories.

Another question is whether retracts and coretracts could provide retractors in general categories in a way similar to that described in §5. Finally we would like to have a classification of retractors, at least in certain categories, though it may be a difficult task.

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