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THE FINITE DIFFERENCE METHOD IN PARTIAL DIFFERENTIAL - FUNCTIONAL EQUATIONS

BY KATARZYNA PRZĄDKA

1. Introduction

For $a > 0$, $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbf{R}^n$, $\mathbf{M} = (M_1, \ldots, M_n) \in \mathbf{R}^n$, where $b_i > 0$, $M_i > 0$, $aM_i \leq b_i$ for $i = 1, \ldots, n$, denote by $E \subset \mathbf{R}^{1+n}$ the Haar pyramid

$$E = \left\{ (x, \mathbf{y}) = (x, y_1, \ldots, y_n) \colon x \in [0, a], \quad \mathbf{y} \in [-\mathbf{b} + \mathbf{M}x, \mathbf{b} - \mathbf{M}x] \right\}.$$

Let

$$E_0 = [-\tau_0, 0] \times [-\mathbf{b}, \mathbf{b}], \qquad \text{where } \tau_0 \geq 0.$$

Denote by $C\left(E_0 \cup E, \mathbf{R}^k\right)$ the space of continuous functions with domain $E_0 \cup E$ and range in $\mathbf{R}^k$. Let

$$\mathbf{f} = (f_1, \ldots, f_k) \colon E \times \mathbf{R}^k \times C\left(E_0 \cup E, \mathbf{R}^k\right) \times \mathbf{R}^n \longrightarrow \mathbf{R}^k .$$

For a function $\mathbf{z} = (z_1, \ldots, z_k) \colon E_0 \cup E \longrightarrow \mathbf{R}^k$ of the variables $(x, \mathbf{y})$ we denote

$$D_x \mathbf{z} = (D_x z_1, \ldots, D_x z_k), \quad D_{\mathbf{y}} z_j = (D_{y_1} z_j, \ldots, D_{y_n} z_j)$$

for $j = 1, \ldots, k$ and

$$\mathbf{f}\left(x, \mathbf{y}, \mathbf{z}(x, \mathbf{y}), \mathbf{z}, D_{\mathbf{y}} \mathbf{z}(x, \mathbf{y})\right) = \left(f_1(x, \mathbf{y}, \mathbf{z}(x, \mathbf{y}), \mathbf{z}, D_{\mathbf{y}} z_1(x, \mathbf{y})), \ldots, f_k\left(x, \mathbf{y}, \mathbf{z}(x, \mathbf{y}), \mathbf{z}, D_{\mathbf{y}} z_k(x, \mathbf{y})\right)\right).$$

We consider the initial problem

$$\begin{aligned} D_x \mathbf{z}(x, \mathbf{y}) &= \mathbf{f}\left(x, \mathbf{y}, \mathbf{z}(x, \mathbf{y}), \mathbf{z}, D_{\mathbf{y}} \mathbf{z}(x, \mathbf{y})\right), \\ \mathbf{z}(x, \mathbf{y}) &= \omega(x, \mathbf{y}), \qquad \text{for } (x, \mathbf{y}) \in E_0, \end{aligned} \tag{1}$$

where $\omega\colon E_0 \longrightarrow \mathbf{R}^k$ is a given function. We assume that $\mathbf{f}$ satisfies the Volterra condition, i.e. if $\mathbf{z}, \overline{\mathbf{z}} \in C\left(E_0 \cup E, \mathbf{R}^k\right)$ and $\mathbf{z}(\xi,\eta) = \overline{\mathbf{z}}(\xi,\eta)$ for $(\xi,\eta) \in E_0 \cup E$, $\xi \leq x$, then $\mathbf{f}(x,\mathbf{y},\mathbf{p},\mathbf{z},\mathbf{q}) = \mathbf{f}(x,\mathbf{y},\mathbf{p},\overline{\mathbf{z}},\mathbf{q})$, where $(x,\mathbf{y},\mathbf{p},\mathbf{q}) \in E \times \mathbf{R}^{k+n}$. The problem (1) includes as special cases the initial-value problems for partial differential equations, retarded partial differential problems and Volterra integro–differential problems.

Suppose that there exists a solution $\overline{\mathbf{u}}$ of (1). In this paper we give sufficient conditions for the convergence of the sequence $\{\mathbf{v_h}\}$, where $\mathbf{v_h}$ are solutions of a difference problem corresponding to (1), to the solution $\overline{\mathbf{u}}$ of (1). We also give an error estimate of the method, in particular as a function of power of the step h_0. indicating the order of the method.

The main problem in these investigations is to find such a difference equation which is stable and satisfies a consistency condition with respect to a differential (or differential–functional) problem. The problems of the stability of finite difference equations generated by first order partial differential (or differential–functional) equations are investigated in [7], [10], [11], [12] by the difference or difference – functional inequalities method. In [5], [8] the stability of difference problems is established by means of comparison theorems for linear recurrent inequalities.

In the present paper we investigate the stability of one–step difference method for (1) using a comparison theorem for recurrent inequalities. Therefore we do not assume the monotonicity of an increment function $\Phi_{\mathbf{h}}$ with respect to the functional argument and we do not assume the quasi–monotonicity of $\Phi_{\mathbf{h}}$ (see Remark 1). Instead of it, the Lipschitz condition is assumed. We introduce some general difference operators which enable us to get a general class of difference schemes which are convergent to a solution of (1).

The problems of existence and uniqueness of solutions of differential – functional equations with first order partial derivatives were considered by many authors and under various assumptions: [2], [3], [4], [6], [14], [15]. Difference methods for (1) in the case when differential equation does not contain a functional argument were discussed in [1], [7], [8], [9], [10].

2. Discretization

We shall be using vectorial inequalities freely, with the understanding that the same inequalities hold between the corresponding components of vectors. For $\mathbf{y} = (y_1,\ldots,y_n)$, $\overline{\mathbf{y}} = (\overline{y}_1,\ldots,\overline{y}_n)$, $\mathbf{y},\overline{\mathbf{y}} \in \mathbf{R}^n$, we define $\mathbf{y} * \overline{\mathbf{y}} = (y_1\overline{y}_1,\ldots,y_n\overline{y}_n) \in \mathbf{R}^n$.

We introduce a mesh in $E_0 \cup E$. Let $\mathbf{c} = (c_0,c_1,\ldots,c_n) \in \mathbf{R}^{1+n}$, $c_i > 0$ for $i = 0,1,\ldots,n$. For $\mathbf{h} = (h_0,h_1,\ldots,h_n) \in (0,\mathbf{c}]$ define $\mathbf{h}' = (h_1,\ldots,h_n)$ and $\mathbf{h} = (h_0,\mathbf{h}')$. Assume that for a $\mathbf{h} = (h_0,\mathbf{h}') \in (0,\mathbf{c}]$ there exists

$\overline{\mathbf{n}} = (\overline{n}_0, \overline{n}_1, \ldots, \overline{n}_n)$ such that $\overline{n}_i$, $i = 0, 1, \ldots, n$, are natural numbers and $a + \tau_0 = h_0 \overline{n}_0$, $\mathbf{h}' * \overline{\mathbf{n}}' = \mathbf{b}$, where $\overline{\mathbf{n}}' = (\overline{n}_1, \ldots, \overline{n}_n)$. Denote by $\mathbf{I_c}$ the subset of $(0, \mathbf{c}]$ having the above property. In the further part of the paper we introduce some additional assumptions concerning the set $\mathbf{I_c}$. For $\mathbf{h} \in \mathbf{I_c}$ we define $|\mathbf{h}| = h_0 + h_1 + \cdots + h_n$. Let n_0 and n^* be integers chosen so that $n_0 + n^* = \overline{n}_0$ and the sequence $x^{(-n_0+i)} = -\tau_0 + ih_0$, $i = 0, 1, \ldots, n_0 + n^*$, satisfies the condition $x^{(0)} \leq 0 < x^{(1)}$. For $\mathbf{m} = (m_0, m_1, \ldots, m_n)$, where m_i are integers, $i = 0, 1, \ldots, n$, we denote $\mathbf{m}' = (m_1, \ldots, m_n)$ and $\mathbf{y}^{(\mathbf{m}')} = (y_1^{(m_1)}, \ldots, y_n^{(m_n)})$, where $\mathbf{y}^{(\mathbf{m}')} = \mathbf{m}' * \mathbf{h}'$ for $-\overline{\mathbf{n}}' \leq \mathbf{m}' \leq \overline{\mathbf{n}}'$, $-\overline{\mathbf{n}}' = (-\overline{n}_1, \ldots, -\overline{n}_n)$. We define the following sets for $\mathbf{h} \in \mathbf{I_c}$:

$$\Gamma_{\mathbf{h}} = \left\{\mathbf{m} : (x^{(m_o)}, \mathbf{y}^{(\mathbf{m}')}) \in E\right\}, \qquad E_{\mathbf{h}} = \left\{(x^{(m_0)}, \mathbf{y}^{(\mathbf{m}')}) : \mathbf{m} \in \Gamma_{\mathbf{h}}\right\},$$

$$\Gamma_{0,h} = \left\{\mathbf{m} : (x^{(m_o)}, \mathbf{y}^{(\mathbf{m}')}) \in E_0\right\}, \quad E_{o,\mathbf{h}} = \left\{(x^{(m_0)}, \mathbf{y}^{(\mathbf{m}')}) : \mathbf{m} \in \Gamma_{0,h}\right\}$$

and

$$\widetilde{\Gamma}_{\mathbf{h}} = \{(m_0, \mathbf{m}') \in \Gamma_{\mathbf{h}} : m_0 < n^*\}.$$

Denote by $\mathcal{F}(X, Y)$ the set of all functions defined on X taking values in Y. For $w \in \mathcal{F}(E_{o,\mathbf{h}} \cup E_{\mathbf{h}}, \mathbf{R})$ we denote $w^{(\mathbf{m})} = w\left(x^{(m_0)}, \mathbf{y}^{(\mathbf{m}')}\right)$ and $\mathbf{z}^{(\mathbf{m})} = \left(z_1^{(\mathbf{m})}, \ldots, z_k^{(\mathbf{m})}\right)$ if $\mathbf{z} \in \mathcal{F}\left(E_{o,\mathbf{h}} \cup E_{\mathbf{h}}, \mathbf{R}^k\right)$.

For $\mathbf{p} = (p_1, \ldots, p_k) \in \mathbf{R}^k$ we define $\|\mathbf{p}\| = \max\{|p_j| : 1 \leq j \leq k\}$ and $[\![\mathbf{p}]\!] = (|p_1|, \ldots, |p_k|)$.

Let $E_{\mathbf{h}}^{(m_0)} = \left\{(x^{(i)}, \mathbf{y}^{(\mathbf{m}')}) \in E_{o,\mathbf{h}} \cup E_{\mathbf{h}} : i \leq m_0\right\}$.

For $z \in \mathcal{F}(E_{o,\mathbf{h}} \cup E_{\mathbf{h}}, \mathbf{R})$ we define:

$$|z|_{\mathbf{h}}^{(m_0)} = \max\left\{|z^{(i,\mathbf{m}')}| : (x^{(i)}, \mathbf{y}^{(\mathbf{m}')}) \in E_{\mathbf{h}}^{(m_0)}\right\}.$$

For $\mathbf{w} = (w_1, \ldots, w_k) \in \mathcal{F}\left(E_{o,\mathbf{h}} \cup E_{\mathbf{h}}, \mathbf{R}^k\right)$ we define:

$$[\![\mathbf{w}]\!]_{\mathbf{h}}^{(m_0)} = \left(|w_1|_{\mathbf{h}}^{(m_0)}, \ldots, |w_k|_{\mathbf{h}}^{(m_0)}\right)$$

and

$$\|\mathbf{w}\|_{\mathbf{h}}^{(m_0)} = \max\left\{\left\|\mathbf{w}^{(i,\mathbf{m}')}\right\| : (x^{(i)}, \mathbf{y}^{(\mathbf{m}')}) \in E_{\mathbf{h}}^{(m_0)}\right\}.$$

For a matrix $C = [c_{ij}]_{i,j=1,\ldots,k}$ we define

$$\|C\| = \max_{1 \leq i \leq k} \sum_{j=1}^{k} |c_{ij}|.$$

For an integer $r > 0$ we denote

$$N_r = \Big\{ \mathbf{s} = (s_1, \ldots, s_n): \\ s_i \in \{-r, -r+1, \ldots, -1, 0, 1, \ldots, r\} \quad \text{for } i = 1, \ldots, n \Big\}.$$

Let $w \in \mathcal{F}(E_{o,\mathbf{h}} \cup E_{\mathbf{h}}, \mathbf{R})$. We define the family of the operators in the following way:

$$A^{(j)} w^{(\mathbf{m})} = \sum_{s \in N_r} a_s^{(j)} w^{(m_0, \mathbf{m}'+\mathbf{s})},$$

$$B_i^{(j)} w^{(\mathbf{m})} = \sum_{s \in N_r} b_{s,i}^{(j)} w^{(m_0, \mathbf{m}'+\mathbf{s})},$$

(2) and

$$\Delta_0^{(j)} w^{(\mathbf{m})} = h_0^{-1} \left[w^{(m_0+1, \mathbf{m}')} - A^{(j)} w^{(\mathbf{m})} \right],$$

$$\Delta_\tau^{(j)} w^{(\mathbf{m})} = h_\tau^{-1} \sum_{s \in N_r} c_{s,\tau}^{(j)} w^{(m_0, \mathbf{m}'+\mathbf{s})},$$

where $i, j = 1, \ldots, k$, $\tau = 1, \ldots, n$. We denote

$$A w^{(\mathbf{m})} = \left(A^{(1)} w^{(\mathbf{m})}, \ldots, A^{(k)} w^{(\mathbf{m})} \right),$$

$$\Delta^{(j)} w^{(\mathbf{m})} = \left(\Delta_1^{(j)} w^{(\mathbf{m})}, \ldots, \Delta_n^{(j)} w^{(\mathbf{m})} \right).$$

We shall approximate $D_x z_j \left(x^{(m_0)}, \mathbf{y}^{(\mathbf{m}')} \right)$, $D_{\mathbf{y}} z_j \left(x^{(m_0)}, \mathbf{y}^{(\mathbf{m}')} \right)$ and $\mathbf{z} \left(x^{(m_0)}, \mathbf{y}^{(\mathbf{m}')} \right)$ by means of $\Delta_0^{(j)} z_j^{(\mathbf{m})}$, $\Delta^{(j)} z_j^{(\mathbf{m})}$ and $\left(B_1^{(j)} z_1^{(\mathbf{m})}, \ldots, B_k^{(j)} z_k^{(\mathbf{m})} \right)$, $j = 1, \ldots, k$, respectively.

Let $\mathbf{Z_h} \subset \mathcal{F}\left(E_{o,\mathbf{h}} \cup E_{\mathbf{h}}, \mathbf{R}^k\right)$ and $\Sigma_{\mathbf{h}} = E_{\mathbf{h}} \times \mathbf{R}^k \times \mathbf{Z_h} \times \mathbf{R}^n$. Assume that for each $\mathbf{h} \in \mathbf{I_c}$, we have $\Phi_{\mathbf{h}} = \left(\Phi_{\mathbf{h}}^{(1)}, \ldots, \Phi_{\mathbf{h}}^{(k)} \right) \in \mathcal{F}\left(\Sigma_{\mathbf{h}}, \mathbf{R}^k\right)$.
We denote:

$$\Phi_{\mathbf{h}} \left(x^{(m_0)}, \mathbf{y}^{(\mathbf{m}')}, B\mathbf{z}^{(\mathbf{m})}, \mathbf{z}, \Delta \mathbf{z}^{(\mathbf{m})} \right) = \\ \left(\Phi_{\mathbf{h}}^{(1)} \left(x^{(m_0)}, \mathbf{y}^{(\mathbf{m}')}, B^{(1)} \mathbf{z}^{(\mathbf{m})}, \mathbf{z}, \Delta^{(1)} z_1^{(\mathbf{m})} \right), \right. \\ \left. \ldots, \Phi_{\mathbf{h}}^{(k)} \left(x^{(m_0)}, \mathbf{y}^{(\mathbf{m}')}, B^{(k)} \mathbf{z}^{(\mathbf{m})}, \mathbf{z}, \Delta^{(k)} z_k^{(\mathbf{m})} \right) \right),$$

where $B^{(j)}\mathbf{z}^{(\mathbf{m})} = \left(B_1^{(j)} z_1^{(\mathbf{m})}, \ldots, B_k^{(j)} z_k^{(\mathbf{m})}\right)$ for $\mathbf{z} = (z_1, \ldots, z_k) \in \mathbf{Z_h}$.
We write that $\Phi_\mathbf{h} \in \mathcal{F}\left(\Sigma_\mathbf{h}, \mathbf{R}^k\right)$ satisfies the Volterra condition if for arbitrary $\mathbf{z}, \overline{\mathbf{z}} \in \mathbf{Z_h}$ such that $\mathbf{z}(x^{(i)}, \mathbf{y}^{(\mathbf{m}')}) = \overline{\mathbf{z}}(x^{(i)}, \mathbf{y}^{(\mathbf{m}')})$ on $E_\mathbf{h}^{(m_0)}$, we have

$$\Phi_\mathbf{h}\left(x^{(m_0)}, \mathbf{y}^{(\mathbf{m}')}, \mathbf{p}, \mathbf{z}, \mathbf{q}\right) = \Phi_\mathbf{h}\left(x^{(m_0)}, \mathbf{y}^{(\mathbf{m}')}, \mathbf{p}, \overline{\mathbf{z}}, \mathbf{q}\right),$$

where $\left(x^{(m_0)}, \mathbf{y}^{(\mathbf{m}')}, \mathbf{p}, \mathbf{q}\right) \in E_\mathbf{h} \times \mathbf{R}^{k+n}$.

We consider the following difference method for the Cauchy problem (1):

$$\begin{aligned} \Delta_0 \mathbf{z}^{(\mathbf{m})} &= \Phi_\mathbf{h}\left(x^{(m_0)}, \mathbf{y}^{(\mathbf{m}')}, B\mathbf{z}^{(\mathbf{m})}, \mathbf{z}, \Delta\mathbf{z}^{(\mathbf{m})}\right), \\ \mathbf{z}^{(\mathbf{m})} &= \omega^{(\mathbf{m})} + \delta_\mathbf{h}^{(\mathbf{m})}, \quad \text{for } \mathbf{m} \in \Gamma_{0,h}, \end{aligned} \tag{3}$$

where

$$\begin{aligned} \delta_\mathbf{h} &= \left(\delta_\mathbf{h}^{(1)}, \ldots, \delta_\mathbf{h}^{(k)}\right) \in \mathcal{F}\left(E_{o,\mathbf{h}}, \mathbf{R}^k\right), \\ \Delta_0 \mathbf{z}^{(\mathbf{m})} &= \left(\Delta_0^{(1)} z_1^{(\mathbf{m})}, \ldots, \Delta_0^{(k)} z_k^{(\mathbf{m})}\right). \end{aligned}$$

3. Stability

We introduce

Assumption H_0. Suppose that

1° for each $\mathbf{h} \in \mathbf{I_c}$ the function $\Phi_\mathbf{h} = \left(\Phi_\mathbf{h}^{(1)}, \ldots, \Phi_\mathbf{h}^{(k)}\right) \in \mathcal{F}\left(\Sigma_\mathbf{h}, \mathbf{R}^k\right)$ of the variables $(x, \mathbf{y}, \mathbf{p}, \mathbf{z}, \mathbf{q})$ satisfies the Volterra condition,

2° for each $(x, \mathbf{y}, p_1, \ldots, p_{j-1}, p_{j+1}, \ldots, p_k, \mathbf{z}) \in E_\mathbf{h} \times \mathbf{R}^{k-1} \times \mathbf{Z_h}$, $\mathbf{h} \in \mathbf{I_c}$ and for each $j = 1, \ldots, k$, the function
$\Phi_\mathbf{h}^{(j)}(x, \mathbf{y}, p_1, \ldots, p_{j-1}, \cdot, p_{j+1}, \ldots, p_k, \mathbf{z}, \cdot)$ is continuous on $\mathbf{R}^{1+n}$,

3° for each $P = (x, \mathbf{y}, \mathbf{p}, \mathbf{z}, \mathbf{q}) \in \Sigma_\mathbf{h}$ there exist the partial derivatives $D_{p_j}\Phi_\mathbf{h}^{(j)}(P)$, $D_{q_i}\Phi_\mathbf{h}^{(j)}(P)$ for $j = 1, \ldots, k$, $i = 1, \ldots, n$, $\mathbf{h} \in \mathbf{I_c}$ and the functions

$$\begin{aligned} &D_{p_j}\Phi_\mathbf{h}^{(j)}(x, \mathbf{y}, p_1, \ldots, p_{j-1}, \cdot, p_{j+1}, \ldots, p_k, \mathbf{z}, \cdot), \\ &D_{q_i}\Phi_\mathbf{h}^{(j)}(x, \mathbf{y}, p_1, \ldots, p_{j-1}, \cdot, p_{j+1}, \ldots, p_k, \mathbf{z}, \cdot), \end{aligned}$$

for $j = 1, \ldots, k$, $i = 1, \ldots, n$, $\mathbf{h} \in \mathbf{I_c}$, are continuous on $\mathbf{R}^{1+n}$,

4° there exist matrices $\Lambda = [\lambda_{ij}]_{i,j=1,\ldots,k}$, $\widetilde{\Lambda} = [\widetilde{\lambda}_{ij}]_{i,j=1,\ldots,k}$, with $\lambda_{ij}, \widetilde{\lambda}_{ij} \geq 0$, $i, j = 1, \ldots, k$, such that

$$
\begin{aligned}
&\left[\!\left[\Phi_{\mathbf{h}}(x^{(m_0)}, \mathbf{y}^{(\mathbf{m}')}, \mathbf{p}, \mathbf{z}, \mathbf{q}) - \Phi_{\mathbf{h}}\left(x^{(m_0)}, \mathbf{y}^{(\mathbf{m}')}, \overline{\mathbf{p}}, \overline{\mathbf{z}}, \mathbf{q}\right) \right]\!\right]^{\mathbf{T}} \\
&\qquad \leq \Lambda \left[\!\left[\mathbf{p} - \overline{\mathbf{p}} \right]\!\right]^{\mathbf{T}} + \widetilde{\Lambda}\left(\left[\!\left[\mathbf{z} - \overline{\mathbf{z}} \right]\!\right]_{\mathbf{h}}^{(m_0)}\right)^{\mathbf{T}},
\end{aligned} \tag{4}
$$

5° for each $\mathbf{s} \in N_r$, $P \in \Sigma_{\mathbf{h}}$, $j = 1, \ldots, k$, we have

$$
a_s^{(j)} + h_0\, b_{s,j}^{(j)}\, D_{p_j} \Phi_{\mathbf{h}}^{(j)}(P) + h_0 \sum_{i=1}^{n} h_i^{-1}\, c_{s,i}^{(j)}\, D_{q_i} \Phi_{\mathbf{h}}^{(j)}(P) \geq 0 \tag{5}
$$

and

$$
\sum_{s \in N_r} a_s^{(j)} = \sum_{s \in N_r} b_{s,j}^{(j)} = 1, \qquad \sum_{s \in N_r} c_{s,i}^{(j)} = 0 \tag{6}
$$

for $j = 1, \ldots, k$, $i = 1, \ldots, n$,

6° $r\, h_i \leq M_i\, h_0$, $\quad i = 1, \ldots, n$,

7° a function $\sigma_{o,h} = (\sigma_{o,h}^{(1)}, \ldots, \sigma_{o,h}^{(k)}) \in \mathcal{F}(\mathbf{I_c}, \mathbf{R})$ satisfies the condition $\left[\!\left[\delta_{\mathbf{h}}^{(\mathbf{m})} \right]\!\right] \leq \sigma_{o,h}$ on $\Gamma_{0,h}$, $\mathbf{h} \in \mathbf{I_c}$.

We define the matrix $L = [l_{ij}]_{i,j=1,\ldots,k}$ by

$$
l_{ij} = \begin{cases} \lambda_{ij} + \widetilde{\lambda}_{ij}, & \text{if } i = j, \\ \lambda_{ij} \sum\limits_{\mathbf{s} \in N_r} |b_{s,i}^{(j)}| + \widetilde{\lambda}_{ij}, & \text{if } i \neq j. \end{cases}
$$

THEOREM 1. *Suppose that*

1° *Assumption* H_0 *is satisfied,*

2° $\widetilde{\mathbf{u}} = (\widetilde{u}_1, \ldots, \widetilde{u}_k) \in \mathbf{Z_h}$ *and there exists a function* $\widetilde{\gamma} = (\widetilde{\gamma}_1, \ldots, \widetilde{\gamma}_k) \in \mathcal{F}\left(\mathbf{I_c}, \mathbf{R}_+^k\right)$ *such that*

$$
\begin{aligned}
&\left[\!\left[\widetilde{\mathbf{u}}^{(m_0+1,\mathbf{m}')} - A\widetilde{\mathbf{u}}^{(\mathbf{m})} - \right. \\
&\left. h_0 \Phi_{\mathbf{h}}\left(x^{(m_0)}, \mathbf{y}^{(\mathbf{m}')}, B\widetilde{\mathbf{u}}^{(\mathbf{m})}, \widetilde{\mathbf{u}}, \Delta\widetilde{\mathbf{u}}^{(\mathbf{m})}\right) \right]\!\right] \leq h_0 \widetilde{\gamma}(\mathbf{h}), \\
&\qquad \text{for } \mathbf{m} \in \widetilde{\Gamma}_{\mathbf{h}},\ \mathbf{h} \in \mathbf{I_c},
\end{aligned} \tag{7}
$$

3° $\widetilde{\mathbf{u}}^{(\mathbf{m})} = \omega^{(\mathbf{m})}$ *for* $\mathbf{m} \in \Gamma_{0,h}$ *and* $\mathbf{v_h} = (v_{h,1}, \ldots, v_{h,k}) \in \mathbf{Z_h}$ *is a solution of the difference method* (3).

Under these assumptions we have

$$\left(\left\| \tilde{u} - v_{\mathbf{h}} \right\|_{\mathbf{h}}^{(m_0)}\right)^{\mathbf{T}}$$

$$\leq (I + h_0 L)^{m_0} \sigma_{o,h}{}^{\mathbf{T}} + h_0 \sum_{i=0}^{m_0 - 1} (I + h_0 L)^i \left[\tilde{\gamma}(\mathbf{h})\right]^{\mathbf{T}}, \tag{8}$$

$$m_0 = 0, 1, \ldots, n^*.$$

PROOF: Let $\mathbf{w_h} = (w_{h,1}, \ldots, w_{h,k}) \in \mathcal{F}\left(E_{\mathbf{h}}, \mathbf{R}^k\right)$ be the function defined by $\mathbf{w_h}\left(x^{(m_0)}, \mathbf{y}^{(\mathbf{m}')}\right) = \tilde{\mathbf{u}}^{(\mathbf{m})} - \mathbf{v}_{\mathbf{h}}^{(\mathbf{m})}$. We have

$$\begin{aligned}
\mathbf{w_h}\left(x^{(m_0+1)}, \mathbf{y}^{(\mathbf{m}')}\right) &= \tilde{\mathbf{u}}^{(m_0+1,\mathbf{m}')} - \mathbf{v}_{\mathbf{h}}^{(m_0+1,\mathbf{m}')} \\
&= \left[\tilde{\mathbf{u}}^{(m_0+1,\mathbf{m}')} - A\tilde{\mathbf{u}}^{(\mathbf{m})} - h_0 \Phi_{\mathbf{h}}\left(x^{(m_0)}, \mathbf{y}^{(\mathbf{m}')}, B\tilde{\mathbf{u}}^{(\mathbf{m})}, \tilde{\mathbf{u}}, \Delta\tilde{\mathbf{u}}^{(\mathbf{m})}\right)\right] \\
&+ A\tilde{\mathbf{u}}^{(\mathbf{m})} - A\mathbf{v}_{\mathbf{h}}^{(\mathbf{m})} \\
&+ h_0\left[\Phi_{\mathbf{h}}\left(x^{(m_0)}, \mathbf{y}^{(\mathbf{m}')}, B\tilde{\mathbf{u}}^{(\mathbf{m})}, \tilde{\mathbf{u}}, \Delta\tilde{\mathbf{u}}^{(\mathbf{m})}\right)\right. \\
&\quad \left. - \Phi_{\mathbf{h}}\left(x^{(m_0)}, \mathbf{y}^{(\mathbf{m}')}, B\mathbf{v}_{\mathbf{h}}^{(\mathbf{m})}, \tilde{\mathbf{u}}, \Delta\mathbf{v}_{\mathbf{h}}^{(\mathbf{m})}\right)\right] \\
&+ h_0\left[\Phi_{\mathbf{h}}\left(x^{(m_0)}, \mathbf{y}^{(\mathbf{m}')}, B\mathbf{v}_{\mathbf{h}}^{(\mathbf{m})}, \tilde{\mathbf{u}}, \Delta\mathbf{v}_{\mathbf{h}}^{(\mathbf{m})}\right)\right. \\
&\quad \left. - \Phi_{\mathbf{h}}\left(x^{(m_0)}, \mathbf{y}^{(\mathbf{m}')}, B\mathbf{v}_{\mathbf{h}}^{(\mathbf{m})}, \mathbf{v_h}, \Delta\mathbf{v}_{\mathbf{h}}^{(\mathbf{m})}\right)\right].
\end{aligned} \tag{9}$$

By Taylor's expansion we have for $j = 1, \ldots, k$ ($\widetilde{P}_j$ denote intermediate points):

$$\begin{aligned}
&\Phi_{\mathbf{h}}^{(j)}\left(x^{(m_0)}, \mathbf{y}^{(\mathbf{m}')}, B^{(j)}\tilde{\mathbf{u}}^{(\mathbf{m})}, \tilde{\mathbf{u}}, \Delta^{(j)}\tilde{u}_j^{(\mathbf{m})}\right) \\
&- \Phi_{\mathbf{h}}^{(j)}\left(x^{(m_0)}, \mathbf{y}^{(\mathbf{m}')}, B_1^{(j)}\tilde{u}_1^{(\mathbf{m})},\right. \\
&\quad \left. \ldots, B_{j-1}^{(j)}\tilde{u}_{j-1}^{(\mathbf{m})}, B_j^{(j)}v_{h,j}^{(\mathbf{m})}, B_{j+1}^{(j)}\tilde{u}_{j+1}^{(\mathbf{m})}, \ldots, B_k^{(j)}\tilde{u}_k^{(\mathbf{m})}, \tilde{\mathbf{u}}, \Delta^{(j)}v_{h,j}^{(\mathbf{m})}\right) \\
&= D_{p_j}\Phi_{\mathbf{h}}^{(j)}(\widetilde{P}_j) \sum_{\mathbf{s}\in N_r} b_{s,j}^{(j)}\left[\tilde{u}_j^{(m_0,\mathbf{m}'+\mathbf{s})} - v_{h,j}^{(m_0,\mathbf{m}'+\mathbf{s})}\right] \\
&+ \sum_{\tau=1}^{n} D_{q_\tau}\Phi_{\mathbf{h}}^{(j)}(\widetilde{P}_j)\,\mathbf{h}_\tau^{-1} \sum_{\mathbf{s}\in N_r} c_{s,\tau}^{(j)}\left[\tilde{u}_j^{(m_0,\mathbf{m}'+\mathbf{s})} - v_{h,j}^{(m_0,\mathbf{m}'+\mathbf{s})}\right].
\end{aligned}$$

By (4) the following estimation holds true: $|D_{p_j}\Phi_{\mathbf{h}}^{(j)}(P)| \leq \lambda_{jj}$ on $\Sigma_{\mathbf{h}}$. Then by (4), (5), (6) we obtain

$$\left| A^{(j)}\tilde{u}_j^{(\mathbf{m})} - A^{(j)}v_{h,j}^{(\mathbf{m})} + h_0\left[\Phi_{\mathbf{h}}^{(j)}\left(x^{(m_0)}, \mathbf{y}^{(\mathbf{m}')}, B^{(j)}\tilde{\mathbf{u}}^{(\mathbf{m})}, \tilde{\mathbf{u}}, \Delta^{(j)}\tilde{u}_j^{(\mathbf{m})}\right)\right.\right.$$

$$
\begin{aligned}
&\quad - \Phi_{\mathbf{h}}^{(j)}\left(x^{(m_0)}, \mathbf{y}^{(\mathbf{m}')}, B^{(j)}\mathbf{v}_{\mathbf{h}}^{(\mathbf{m})}, \tilde{\mathbf{u}}, \Delta^{(j)} v_{h,j}^{(\mathbf{m})}\right)\Big]\Big| \\
&= \Bigg| \sum_{\mathbf{s}\in N_\tau} a_{\mathbf{s}}^{(j)}\, w_{h,j}^{(m_0,\mathbf{m}'+\mathbf{s})} + h_0\, D_{p_j}\Phi_{\mathbf{h}}^{(j)}(\widetilde{P}_j) \sum_{\mathbf{s}\in N_\tau} b_{\mathbf{s},j}^{(j)}\, w_{h,j}^{(m_0,\mathbf{m}'+\mathbf{s})} \\
&\qquad + \sum_{\tau=1}^{n} D_{q_\tau}\Phi_{\mathbf{h}}^{(j)}(\widetilde{P}_j)\, h_\tau^{-1} \sum_{\mathbf{s}\in N_\tau} c_{\mathbf{s},\tau}^{(j)}\, w_{h,j}^{(m_0,\mathbf{m}'+\mathbf{s})} \\
&\quad + h_0\Big[\Phi_{\mathbf{h}}^{(j)}(x^{(m_0)}, \mathbf{y}^{(\mathbf{m}')}, B_1^{(j)}\tilde{u}_1^{(\mathbf{m})}, \ldots, B_{j-1}^{(j)}\tilde{u}_{j-1}^{(\mathbf{m})}, B_j^{(j)} v_{h,j}^{(\mathbf{m})}, B_{j+1}^{(j)}\tilde{u}_{j+1}^{(\mathbf{m})}, \\
&\qquad \ldots, B_k^{(j)}\tilde{u}_k^{(\mathbf{m})}, \tilde{\mathbf{u}}, \Delta^{(j)} v_{h,j}^{(\mathbf{m})}) \\
&\qquad - \Phi_{\mathbf{h}}^{(j)}\left(x^{(m_0)}, \mathbf{y}^{(\mathbf{m}')}, B^{(j)}\mathbf{v}_{\mathbf{h}}^{(\mathbf{m})}, \tilde{\mathbf{u}}, \Delta^{(j)} v_{h,j}^{(\mathbf{m})}\right)\Big]\Bigg| \\
&\le \sum_{\mathbf{s}\in N_\tau} \left| w_{h,j}^{(m_0,\mathbf{m}'+\mathbf{s})} \right| \\
&\qquad \cdot \left[a_{\mathbf{s}}^{(j)} + h_0\, b_{\mathbf{s},j}^{(j)}\, D_{p_j}\Phi_{\mathbf{h}}^{(j)}(\widetilde{P}_j) + h_0 \sum_{\tau=1}^{n} h_\tau^{-1}\, c_{\mathbf{s},\tau}^{(j)}\, D_{q_\tau}\Phi_{\mathbf{h}}^{(j)}(\widetilde{P}_j) \right] \\
&\qquad + h_0 \sum_{\substack{\tau=1 \\ \tau\neq j}}^{k} \lambda_{j\tau} \sum_{\mathbf{s}\in N_\tau} |b_{\mathbf{s},\tau}^{(j)}| \left| \tilde{u}_\tau^{(m_0,\mathbf{m}'+\mathbf{s})} - v_{h,\tau}^{(m_0,\mathbf{m}'+\mathbf{s})} \right| \\
&\qquad \le |w_{h,j}|_{\mathbf{h}}^{(m_0)}\, (1 + h_0\lambda_{jj}) + h_0 \sum_{\substack{\tau=1 \\ \tau\neq j}}^{k} \lambda_{j\tau}\, |w_{h,\tau}|_{\mathbf{h}}^{(m_0)} \sum_{\mathbf{s}\in N_\tau} |b_{\mathbf{s},\tau}^{(j)}| \,.
\end{aligned}
$$

By (9), (4) we get

$$
\begin{aligned}
&\left| w_{h,j}\left(x^{(m_0+1)}, \mathbf{y}^{(\mathbf{m}')}\right)\right| \le h_0\, \widetilde{\gamma}_j(\mathbf{h}) + |w_{h,j}|_{\mathbf{h}}^{(m_0)}\, (1 + h_0\lambda_{jj}) \\
&\qquad + h_0 \sum_{\substack{\tau=1 \\ \tau\neq j}}^{k} \lambda_{j\tau}\, |w_{h,\tau}|_{\mathbf{h}}^{(m_0)} \sum_{\mathbf{s}\in N_\tau} |b_{\mathbf{s},\tau}^{(j)}| + h_0 \sum_{\tau=1}^{k} \widetilde{\lambda}_{j\tau}\, |w_{h,\tau}|_{\mathbf{h}}^{(m_0)} \\
&= |w_{h,j}|_{\mathbf{h}}^{(m_0)} \left(1 + h_0[\lambda_{jj} + \widetilde{\lambda}_{jj}]\right) \\
&\qquad + h_0 \sum_{\substack{\tau=1 \\ \tau\neq j}}^{k} |w_{h,\tau}|_{\mathbf{h}}^{(m_0)} \left(\lambda_{j\tau} \sum_{\mathbf{s}\in N_\tau} |b_{\mathbf{s},\tau}^{(j)}| + \widetilde{\lambda}_{j\tau} \right) + h_0\widetilde{\gamma}_j(\mathbf{h})
\end{aligned}
$$

$$= |w_{h,j}|_{\mathbf{h}}^{(m_0)} + h_0 \sum_{r=1}^{\kappa} |w_{h,r}|_{\mathbf{h}}^{(m_0)} l_{jr} + h_0 \tilde{\gamma}_j(\mathbf{h})$$

and then we obtain the inequality

$$\left([\![\mathbf{w_h}]\!]_{\mathbf{h}}^{(m_0+1)} \right)^{\mathbf{T}} \leq (I + h_0 L) \left([\![\mathbf{w_h}]\!]_{\mathbf{h}}^{(m_0)} \right)^{\mathbf{T}} + h_0 (\gamma(\mathbf{h}))^{\mathbf{T}} .$$

By this the thesis (8) follows. ∎

REMARK 1. The problem of convergence of the difference method (3) is investigated in [11]. There the stability is established by the difference–functional inequalities method. The conditions of the monotonicity of $\Phi_{\mathbf{h}}$ with respect to the functional argument and the quasimonotonicity of $\Phi_{\mathbf{h}}$ with respect to p are the basic assumptions. In Theorem 1 in this paper we omit those assumptions. We assume that $\Phi_{\mathbf{h}}$ satisfies the Lipschitz condition (4).

In [11] instead of the Lipschitz condition, the following estimation is supposed:

$$[\![\Phi_{\mathbf{h}} \left(x^{(m_0)}, \mathbf{y}^{(\mathbf{m}')}, \mathbf{p} + \overline{\mathbf{p}}, \mathbf{z} + \overline{\mathbf{z}}, \mathbf{q} \right) - \Phi_{\mathbf{h}} \left(x^{(m_0)}, \mathbf{y}^{(\mathbf{m}')}, \mathbf{p}, \mathbf{z}, \mathbf{q} \right)]\!]^{\mathbf{T}}$$
$$\leq \Lambda \left[\overline{\mathbf{p}} + [\![\overline{\mathbf{z}}]\!]_{\mathbf{h}}^{(m_0)} \right]^{\mathbf{T}} \qquad \text{on } \Sigma_{\mathbf{h}},\ \mathbf{h} \in \mathbf{I_c},$$

where $\overline{\mathbf{p}} \in \mathbf{R}_+^k$, $\overline{\mathbf{z}} \in \mathcal{F}\left(E_{o,\mathbf{h}} \cup E_{\mathbf{h}}, \mathbf{R}_+^k\right)$ and $\overline{\mathbf{z}}$ depends only on the first argument. The above condition is more general than (4).

There exist functions that satisfy the condition given above and do not satisfy the Lipschitz condition (4) (see [13]).

REMARK 2. If assumptions of Theorem 1 are satisfied, we obtain from (8):

$$\|\widetilde{\mathbf{u}} - \mathbf{v_h}\|_{\mathbf{h}}^{(m_0)} \leq \begin{cases} \|\sigma_{o,h}\| \left(1 + h_0 \widetilde{L}\right)^{(m_0)} & \\ \quad + \|\widetilde{\gamma}(\mathbf{h})\| \widetilde{L}^{-1} \left[\left(1 + h_0 \widetilde{L}\right)^{(m_0)} - 1 \right], & \text{for } \widetilde{L} > 0, \\ \|\sigma_{o,h}\| + m_0 h_0 \|\widetilde{\gamma}(\mathbf{h})\|, & \text{for } \widetilde{L} = 0, \end{cases}$$

where $\widetilde{L} = \|L\|$, $m_0 = 0, 1, \ldots, n^*$.

Since $\left(1 + h_0 \widetilde{L}\right)^{(m_0)} \leq e^{a\widetilde{L}}$, $m_0 = 0, 1, \ldots, n^*$, we have

$$\|\widetilde{\mathbf{u}} - \mathbf{v_h}\|_{\mathbf{h}}^{(m_0)} \leq \begin{cases} \|\sigma_{o,h}\| \, e^{a\widetilde{L}} + \|\widetilde{\gamma}(\mathbf{h})\| \widetilde{L}^{-1} [e^{a\widetilde{L}} - 1], & \text{for } \widetilde{L} > 0, \\ \|\sigma_{o,h}\| + a \|\widetilde{\gamma}(\mathbf{h})\|, & \text{for } \widetilde{L} = 0. \end{cases}$$

4. Convergence

We introduce

Assumption H_1. Suppose that

1° $\overline{\mathbf{u}} = (\overline{u}_1, \ldots, \overline{u}_k) \in C\left(E_0 \cup E, \mathbf{R}^k\right)$ is a solution of (1), $\overline{\mathbf{u}}|_E$ is a function of class C^2,

2° $\mathbf{f} = (f_1, \ldots, f_k) \in C(E \times \mathbf{R}^k \times C\left(E_0 \cup E, \mathbf{R}^k\right) \times \mathbf{R}^n, \mathbf{R}^k)$,

3°

$$\sum_{s \in N_\tau} a_s^{(j)} s_\tau = 0, \quad \sum_{s \in N_\tau} c_{s,i}^{(j)} s_\tau = \delta_{i\tau},$$

where $\delta_{i\tau}$ is the Kronecker symbol, $i, \tau = 1, \ldots, n$, $j = 1, \ldots, k$,

4° there exists a $c_0 > 0$ such that $h_i h_\tau^{-1} \leq c_0$ for $i, \tau = 1, \ldots, n$.

Now we prove that if the difference method (3) is stable and satisfies the consistency condition (10),then it is convergent. We denote by $\overline{\mathbf{u}}_\mathbf{h}$ the restriction $\overline{\mathbf{u}}|_{E_{o,\mathbf{h}} \cup E_\mathbf{h}}$.

THEOREM 2. *Suppose that*

1° *Assumptions* H_0, H_1 *are satisfied,*

2° $\mathbf{v}_\mathbf{h} \in \mathbf{Z}_\mathbf{h}$ *is a solution of the difference method* (3),

3° *there exists a function* $\gamma_0 = (\gamma_{0,1}, \ldots, \gamma_{0,k}) \in \mathcal{F}\left(\mathbf{I}_\mathbf{c}, \mathbf{R}_+^k\right)$ *such that*

$$\begin{aligned} &\left[\!\left[\Phi_\mathbf{h}\left(x^{(m_0)}, \mathbf{y}^{(\mathbf{m}')}, B\overline{\mathbf{u}}^{(\mathbf{m})}, \overline{\mathbf{u}}_\mathbf{h}, \Delta\overline{\mathbf{u}}^{(\mathbf{m})}\right)\right.\right. \\ &\left.\left. -\mathbf{f}\left(x^{(m_0)}, \mathbf{y}^{(\mathbf{m}')}, \overline{\mathbf{u}}^{(\mathbf{m})}, \overline{\mathbf{u}}, \Delta\overline{\mathbf{u}}^{(\mathbf{m})}\right)\right]\!\right] \leq \gamma_0(\mathbf{h}), \end{aligned} \tag{10}$$

for $\mathbf{m} \in \Gamma_\mathbf{h}$, $\mathbf{h} \in \mathbf{I}_\mathbf{c}$ *and* $\lim_{|\mathbf{h}| \to 0} \gamma_0(\mathbf{h}) = 0$.

Under these assumptions there exists a function $\widetilde{\gamma} = (\widetilde{\gamma}_1, \ldots, \widetilde{\gamma}_k) \in \mathcal{F}\left(\mathbf{I}_\mathbf{c}, \mathbf{R}_+^k\right)$ *such that*

$$\begin{aligned} &\left(\left[\!\left[\overline{\mathbf{u}} - \mathbf{v}_\mathbf{h} \right]\!\right]_\mathbf{h}^{(m_0)}\right)^\mathbf{T} \\ &\leq (I + h_0 L)^{(m_0)} \sigma_{o,h}^\mathbf{T} + h_0 \sum_{i=0}^{m_0 - 1} (I + h_0 L)^i \left[\widetilde{\gamma}(\mathbf{h})\right]^\mathbf{T} \end{aligned} \tag{11}$$

for $m_0 = 0, 1, \ldots, n^*$ *and*

$$\lim_{|\mathbf{h}| \to 0} \widetilde{\gamma}(\mathbf{h}) = 0. \tag{12}$$

In particular, if

$$\lim_{|\mathbf{h}| \to 0} \sigma_{o,h} = 0, \tag{13}$$

then we have

$$\lim_{|\mathbf{h}|\to 0} [\![\overline{\mathbf{u}}^{(\mathbf{m})} - \mathbf{v}_{\mathbf{h}}^{(\mathbf{m})}]\!] = 0 \quad \textit{for } \mathbf{m} \in \Gamma_{\mathbf{h}}. \tag{14}$$

PROOF: (it is similar to that in [11]).

Adopt the following notations:

$$R_0^{(\mathbf{m})}(\mathbf{h}) = \Phi_{\mathbf{h}}\left(x^{(m_0)}, \mathbf{y}^{(\mathbf{m}')}, B\overline{\mathbf{u}}^{(\mathbf{m})}, \overline{\mathbf{u}}_{\mathbf{h}}, \Delta\overline{\mathbf{u}}^{(\mathbf{m})}\right) - \mathbf{f}\left(x^{(m_0)}, \mathbf{y}^{(\mathbf{m}')}, \overline{\mathbf{u}}^{(\mathbf{m})}, \overline{\mathbf{u}}, \Delta\overline{\mathbf{u}}^{(\mathbf{m})}\right),$$

$$R_1^{(\mathbf{m})}(\mathbf{h}) = \mathbf{f}\left(x^{(m_0)}, \mathbf{y}^{(\mathbf{m}')}, \overline{\mathbf{u}}^{(\mathbf{m})}, \overline{\mathbf{u}}, \Delta\overline{\mathbf{u}}^{(\mathbf{m})}\right) - \mathbf{f}\left(x^{(m_0)}, \mathbf{y}^{(\mathbf{m}')}, \overline{\mathbf{u}}^{(\mathbf{m})}, \overline{\mathbf{u}}, D_{\mathbf{y}}\overline{\mathbf{u}}^{(\mathbf{m})}\right),$$

$$R_2^{(\mathbf{m})}(\mathbf{h}, \xi) = \mathbf{f}\left(x^{(m_0)}, \mathbf{y}^{(\mathbf{m}')}, \overline{\mathbf{u}}^{(\mathbf{m})}, \overline{\mathbf{u}}, D_{\mathbf{y}}\overline{\mathbf{u}}^{(\mathbf{m})}\right) - \mathbf{f}\Big(x^{(m_0)} + \xi h_0, \mathbf{y}^{(\mathbf{m}')}, \overline{\mathbf{u}}(x^{(m_0)} + \xi h_0, \mathbf{y}^{(\mathbf{m}')}), \overline{\mathbf{u}}, D_{\mathbf{y}}\overline{\mathbf{u}}(x^{(m_0)} + \xi h_0, \mathbf{y}^{(\mathbf{m}')})\Big)\Big),$$

$$R_{3,j}^{(\mathbf{m})}(\mathbf{h}, \xi) = -(2h_0)^{-1} \sum_{s\in N_\tau} \sum_{i,\tau=1}^{n} a_s^{(j)} s_i s_\tau h_i h_\tau \cdot D_{\mathbf{y}_i\mathbf{y}_\tau}\overline{u}_j(x^{(m_0)}, y_1^{(m_1)} + \xi s_1 h_1, \ldots, y_n^{(m_n)} + \xi s_n h_n),$$

$$R_3^{(\mathbf{m})} = (R_{3,1}^{(\mathbf{m})}, \ldots, R_{3,k}^{(\mathbf{m})})$$

and

$$\gamma_i(\mathbf{h}) = \max\left\{ [\![R_i^{(\mathbf{m})}(\mathbf{h})]\!] : \mathbf{m} \in \Gamma_{\mathbf{h}} \right\}, \qquad i = 0, 1,$$

$$\gamma_i(\mathbf{h}) = \max\left\{ [\![R_i^{(\mathbf{m})}(\mathbf{h}, \xi)]\!] : \mathbf{m} \in \Gamma_{\mathbf{h}}, \xi \in [0,1] \right\}, \quad i = 2, 3.$$

By our assumptions we conclude that $\lim_{|\mathbf{h}|\to 0} \gamma_i(\mathbf{h}) = 0$, $i = 0, 1, 2, 3$. Using Taylor's expansion for function $\overline{\mathbf{u}}$ we obtain by Assumption H_1:

$$h_0^{-1}\left[\overline{u}_j^{(m_0+1, \mathbf{m}')} - \sum_{s\in N_\tau} a_s^{(j)} \overline{u}_j^{(m_0, \mathbf{m}'+\mathbf{s})}\right] = D_x\overline{u}_j\left(x^{(m_0)} + \theta_0 h_0, \mathbf{y}^{(\mathbf{m}')}\right) + R_{3,j}^{(\mathbf{m})}\left(\mathbf{h}, \widetilde{\theta}\right),$$

$\mathbf{m} \in \widetilde{\Gamma}_{\mathbf{h}}$, $j = 1, \ldots, k$, where $\theta_0, \widetilde{\theta} \in (0,1)$. Thus we get

$$\Phi_{\mathbf{h}}\left(x^{(m_0)}, \mathbf{y}^{(\mathbf{m}')}, B\overline{\mathbf{u}}^{(\mathbf{m})}, \overline{\mathbf{u}}_{\mathbf{h}}, \Delta\overline{\mathbf{u}}^{(\mathbf{m})}\right) - h_0^{-1}\left[\overline{\mathbf{u}}^{(m_0+1,\mathbf{m}')} - A\overline{\mathbf{u}}^{(\mathbf{m})}\right]$$
$$= R_0^{(\mathbf{m})}(\mathbf{h}) + R_1^{(\mathbf{m})}(\mathbf{h}) + R_2^{(\mathbf{m})}(\mathbf{h}, \theta_0) - R_3^{(\mathbf{m})}\left(\mathbf{h}, \widetilde{\theta}\right),$$

for $\mathbf{m} \in \widetilde{\Gamma}_{\mathbf{h}}$, $\mathbf{h} \in \mathbf{I}_{\mathbf{c}}$. By our assumptions we conclude

$$\left[\!\left[\, \overline{\mathbf{u}}^{(m_0+1,\mathbf{m}')} - A\overline{\mathbf{u}}^{(\mathbf{m})} - h_0\, \Phi_{\mathbf{h}}\left(x^{(m_0)}, \mathbf{y}^{(\mathbf{m}')}, B\overline{\mathbf{u}}^{(\mathbf{m})}, \overline{\mathbf{u}}_{\mathbf{h}}, \Delta\overline{\mathbf{u}}^{(\mathbf{m})}\right) \right]\!\right]$$
$$\leq \mathbf{h}_0\widetilde{\gamma}(\mathbf{h}),$$

where $\widetilde{\gamma}(\mathbf{h}) = \sum_{i=0}^{3} \gamma_i(\mathbf{h})$.

Then we obtain (12). Using Theorem 1 we get the estimation (11). Since $m_0 h_0 \leq a$ for $\mathbf{m} \in \Gamma_{\mathbf{h}}$, by (11) – (13) we see that (14) holds true, which completes the proof. ∎

REMARK 3. If the assumptions of Theorem 2 are satisfied then we have (see Remark 2):

$$\left\|\overline{\mathbf{u}}_{\mathbf{h}} - \mathbf{v}_{\mathbf{h}}\right\|_{\mathbf{h}}^{(m_0)} \leq \begin{cases} \|\sigma_{o,h}\|\, e^{a\widetilde{L}} + \|\widetilde{\gamma}(\mathbf{h})\|\, \widetilde{L}^{-1}\left[e^{a\widetilde{L}} - 1\right], & \text{for } \widetilde{L} > 0, \\ \|\sigma_{o,h}\| + a\, \|\widetilde{\gamma}(\mathbf{h})\|, & \text{for } \widetilde{L} = 0, \end{cases}$$

$m_0 = 0, 1, \ldots, n^*$.

REMARK 4. Suppose that

1° Assumption H_0 and the conditions 1°, 2° of Assumption H_1 are satisfied,

2° $\mathbf{v}_{\mathbf{h}} \in \mathbf{Z}_{\mathbf{h}}$ is a solution of the difference method (3),

3° there exist constants $\alpha, \beta > 0$, $c \geq 0$ and functions $\gamma^*, \gamma_0^* \in \mathcal{F}\left(\mathbf{I}_{\mathbf{c}}, \mathbf{R}_+^k\right)$ such that

$$\left[\!\left[\, \overline{\mathbf{u}}^{(m_0+1,\mathbf{m}')} - A\overline{\mathbf{u}}^{(\mathbf{m})} - h_0\, \Phi_{\mathbf{h}}\left(x^{(m_0)}, \mathbf{y}^{(\mathbf{m}')}, B\overline{\mathbf{u}}^{(\mathbf{m})}, \overline{\mathbf{u}}_{\mathbf{h}}, \Delta\overline{\mathbf{u}}^{(\mathbf{m})}\right) \right]\!\right]$$
$$\leq h_o^{\alpha+1}\, \gamma^*(\mathbf{h}),$$
$$\left[\!\left[\, \sigma_{o,h} \,\right]\!\right] \leq h_0^{\beta}\, \gamma_0^*(\mathbf{h}),$$

for $\mathbf{m} \in \widetilde{\Gamma}_{\mathbf{h}}$, $\mathbf{h} \in \mathbf{I}_{\mathbf{c}}$, where $\|\gamma^*(\mathbf{h})\|, \|\gamma_0^*(\mathbf{h})\| \leq c$ on $\mathbf{I}_{\mathbf{c}}$.

Under these assumptions we have

$$\left(\left[\!\left[\, \overline{\mathbf{u}} - \mathbf{v}_{\mathbf{h}} \,\right]\!\right]_{\mathbf{h}}^{(m_0)}\right)^{\mathbf{T}} \leq h_0^{\delta}\Big\{ h_0^{\beta-\delta}\, (I + h_0 L)^{(m_0)}\, [\gamma_0^*(\mathbf{h})]^{\mathbf{T}}$$
$$+ h_0^{\alpha-\delta+1} \sum_{i=0}^{\mathbf{m}_0-1} (I + h_0 L)^{i}\, [\gamma^*(\mathbf{h})]^{\mathbf{T}} \Big\}$$

for $m_0 = 0, 1, \ldots, n^*$, where $\delta = \min\{\alpha, \beta\}$. In particular, the following estimates hold:

$$\|\overline{\mathbf{u}} - \mathbf{v}\|_{\mathbf{h}}^{(m_0)} \leq \begin{cases} h_0^{\delta} c \left[h_0^{\beta-\delta} e^{a\widetilde{L}} + \widetilde{L}^{-1} h_0^{\alpha-\delta} \left(e^{a\widetilde{L}} - 1 \right) \right], \\ \quad \text{for } \widetilde{L} > 0, \\ h_0^{\delta} c \left[h_0^{\beta-\delta} + a h_0^{\alpha-\delta} \right], \quad \text{for } \widetilde{L} = 0, \end{cases}$$

$m_0 = 0, 1, \ldots, n^*$.

PROOF: The thesis follows from Theorem 1 for $\widetilde{\gamma}(\mathbf{h}) = h_0^{\alpha} \gamma^*(\mathbf{h})$ and from Remark 2. ∎

EXAMPLE 1. Consider the difference method:

$$\begin{aligned} \mathbf{z}^{(m_0+1, \mathbf{m}')} &= \mathbf{z}^{(\mathbf{m})} + h_0 \Phi_{\mathbf{h}} \left(x^{(m_0)}, \mathbf{y}^{(\mathbf{m}')}, \mathbf{z}^{(\mathbf{m})}, \mathbf{z}, \Delta \mathbf{z}^{(\mathbf{m})} \right), \\ \mathbf{z}^{(\mathbf{m})} &= \omega^{(\mathbf{m})} + \delta_{\mathbf{h}}^{(\mathbf{m})}, \quad \text{for } \mathbf{m} \in \Gamma_{0,h}, \end{aligned} \tag{15}$$

where

$$\begin{aligned} \Delta_i^{(j)} w^{(\mathbf{m})} &= h_i^{-1} \left[w^{(\mathbf{m})} - w^{(-i(\mathbf{m}))} \right] \qquad \text{for } i \in I_j^{(+)}, \\ \Delta_i^{(j)} w^{(\mathbf{m})} &= h_i^{-1} \left[w^{(i(\mathbf{m}))} - w^{(\mathbf{m})} \right] \qquad \text{for } i \in I_j^{(-)} \end{aligned}$$

and $I_j^{(-)}, I_j^{(+)}$ are sets such that $I_j^{(-)} \cup I_j^{(+)} = \{1, \ldots, n\}$, $I_j^{(-)} \cap I_j^{(+)} = \emptyset$, $j = 1, \ldots, k$. For $\mathbf{m} = (m_0, m_1, \ldots, m_n)$ we denote:

$$\begin{aligned} -i(\mathbf{m}) &= (m_0, \ldots, m_{i-1}, m_i - 1, m_{i+1}, \ldots, m_n), \\ i(\mathbf{m}) &= (m_0, \ldots, m_{i-1}, m_i + 1, m_{i+1}, \ldots, m_n). \end{aligned}$$

The condition (5) is the basic assumption in investigating of stability. We introduce an example of realisation of the condition (5) for the operators Δ given above. Suppose that the function $\Phi_{\mathbf{h}}$ satisfies the conditions $1^\circ - 4^\circ$ of Assumption H_0. Then the condition (5) is equivalent to the following one:

$$\begin{aligned} D_{q_\tau} \Phi_{\mathbf{h}}^{(j)}(P) &\leq 0 \qquad \text{for } \tau \in I_j^{(-)}, \\ D_{q_\tau} \Phi_{\mathbf{h}}^{(j)}(P) &\geq 0 \qquad \text{for } \tau \in I_j^{(+)} \end{aligned} \tag{16}$$

and

$$1 + h_0 D_{p_j} \Phi_{\mathbf{h}}^{(j)}(P) - h_0 \sum_{\tau=1}^{n} h_\tau^{-1} \left| D_{q_\tau} \Phi_{\mathbf{h}}^{(j)}(P) \right| \geq 0.$$

Others examples of difference methods are introduced in [11],[12].

5. Convergence of the difference method for almost linear systems

Now we consider the difference – functional problem

$$
\begin{aligned}
D_x z_j(x,\mathbf{y}) &= \tilde{f}_j(x,\mathbf{y},\mathbf{z}(x,\mathbf{y}),\mathbf{z}) + \sum_{i=1}^{n} g_{ji}(x,\mathbf{y})\, D_{y_i} z_j(x,\mathbf{y}), \\
& \qquad j = 1,\dots,k, \\
\mathbf{z}(x,\mathbf{y}) &= \omega(x,\mathbf{y}) \qquad \text{for } (x,\mathbf{y}) \in E_0,
\end{aligned} \tag{17}
$$

where

$$
\tilde{\mathbf{f}} = \left(\tilde{f}_1,\dots,\tilde{f}_k\right) \in \mathcal{F}\left(E \times \mathbf{R}^k \times C\left(E_0 \cup E, \mathbf{R}^k\right), \mathbf{R}^k\right),
$$

$$
g_j = (g_{j1},\dots,g_{jn}) \in \mathcal{F}(E, \mathbf{R}^n), \quad j = 1,\dots,k.
$$

Let $\Sigma_{\mathbf{h}} = E_{\mathbf{h}} \times \mathbf{R}^k \times \mathbf{Z}_{\mathbf{h}}$.

Suppose that $\tilde{\Phi}_{\mathbf{h}} = \left(\tilde{\Phi}_{\mathbf{h}}^{(1)},\dots,\tilde{\Phi}_{\mathbf{h}}^{(k)}\right) \in \mathcal{F}\left(\Sigma_{\mathbf{h}}, \mathbf{R}^k\right)$ and $\sigma_{o,h} \in \mathcal{F}\left(E_{o,\mathbf{h}}, \mathbf{R}^k\right)$, where $\mathbf{h} \in \mathbf{I}_c$.

Define the difference operators $\Delta^{(j)} = \left(\Delta_1^{(j)},\dots,\Delta_n^{(j)}\right)$ as follows:

$$
\begin{aligned}
\Delta_i^{(j)} w^{(\mathbf{m})} &= h_i^{-1}\left[w^{(\mathbf{m})} - w^{(-i(\mathbf{m}))}\right] \quad \text{if } g_{ji}^{(\mathbf{m})} < 0, \\
\Delta_i^{(j)} w^{(\mathbf{m})} &= h_i^{-1}\left[w^{(i(\mathbf{m}))} - w^{(\mathbf{m})}\right] \quad \text{if } g_{ji}^{(\mathbf{m})} \geq 0,
\end{aligned} \tag{18}
$$

where $w \in \mathcal{F}(E_{o,\mathbf{h}} \cup E_{\mathbf{h}}, \mathbf{R})$. Consider the difference method

$$
\begin{aligned}
\Delta_0^{(j)} z_j^{(\mathbf{m})} &= \tilde{\Phi}_{\mathbf{h}}^{(j)}(x^{(m_0)}, \mathbf{y}^{(\mathbf{m}')}, B^{(j)}\mathbf{z}^{(\mathbf{m})}, \mathbf{z}) + \sum_{i=1}^{n} g_{ji}^{(\mathbf{m})} \Delta_i^{(j)} z_j^{(\mathbf{m})}, \\
& \qquad j = 1,\dots,k, \\
\mathbf{z}^{(\mathbf{m})} &= \omega^{(\mathbf{m})} + \delta_h^{(\mathbf{m})}, \quad \text{for } \mathbf{m} \in \Gamma_{0,h},
\end{aligned} \tag{19}
$$

where $\Delta_0^{(j)}, B^{(j)}$, $j = 1,\dots,k$, are given by (2) for $r = 1$.

In Example 1 we have assumed that the derivatives $D_{q_\tau}\Phi_{\mathbf{h}}^{(j)}$, $\tau = 1,\dots,n$, $j = 1,\dots,k$, satisfy (16). Now we omit this assumption for the method (18),(19).

Assumption H_2. Suppose that

1° the function $\Phi_{\mathbf{h}} = \left(\Phi_{\mathbf{h}}^{(1)},\dots,\Phi_{\mathbf{h}}^{(k)}\right)$ defined by

$$
\Phi_{\mathbf{h}}^{(j)}(x,\mathbf{y},\mathbf{p},\mathbf{z},\mathbf{q}) = \tilde{\Phi}_{\mathbf{h}}^{(j)}(x,\mathbf{y},\mathbf{p},\mathbf{z}) + \sum_{i=1}^{n} g_{ji}(x,\mathbf{y})\, q_i, \ \mathbf{h} \in \mathbf{I}_c,
$$

satisfies the conditions 1° – 4° of Assumption H_0,

2° the condition 7° of Assumption H_0 is satisfied and $h_i \leq M_i h_0$, $i = 1, \ldots, n$, $\mathbf{h} \in \mathbf{I_c}$,

3° for each $P = (x, \mathbf{y}, \mathbf{p}, \mathbf{z}) \in \Sigma_{\mathbf{h}}$, $\mathbf{h} \in \mathbf{I_c}$, $j = 1, \ldots, k$,

$$a_\theta^{(j)} + h_0 b_{\theta,j}^{(j)} D_{p_j} \tilde{\Phi}_{\mathbf{h}}^{(j)}(P) - h_0 \sum_{\tau=1}^{n} h_\tau^{-1} |g_{j\tau}(x, \mathbf{y})| \geq 0,$$

$$a_s^{(j)} + h_0 b_{s,j}^{(j)} D_{p_j} \tilde{\Phi}_{\mathbf{h}}^{(j)}(P) \geq 0$$

$$\text{for } s \in N_1,\ s \neq \theta,\quad \theta = (0, \ldots, 0) \in \mathbf{R}^n$$

and the condition (6) is satisfied.

Let $N_1^{(0)} = \{\mathbf{s} \in N_1 : \mathbf{s} \neq \theta\}$, $I_{j,m}^{(-)} = \left\{i \in \{1, \ldots, n\} : g_{ji}^{(\mathbf{m})} < 0\right\}$ and $I_{j,m}^{(+)} = \left\{i \in \{1, \ldots, n\} : g_{ji}^{(\mathbf{m})} \geq 0\right\}$ for $j = 1, \ldots, k$, $\mathbf{m} \in \Gamma_{\mathbf{h}}$.

THEOREM 3. *Suppose that*

1° *Assumption H_2 and the conditions 3°, 4° of Assumption H_1 are satisfied,*

2° $\tilde{\mathbf{f}} = \left(\tilde{f}_1, \ldots, \tilde{f}_k\right) \in C(E \times \mathbf{R}^k \times C(E_0 \cup E, \mathbf{R}^k), \mathbf{R}^k)$,
$g_j = (g_{j1}, \ldots, g_{jn}) \in C(E, \mathbf{R}^n)$, $j = 1, \ldots, k$,
$\overline{\mathbf{u}} = (\overline{u}_1, \ldots, \overline{u}_k) \in C\left(E_0 \cup E, \mathbf{R}^k\right)$ *is a solution of* (17),
$\overline{\mathbf{u}}|_E$ *is a function of class* C^2, $\mathbf{v_h} = (v_{h,1}, \ldots, v_{h,k}) \in \mathbf{Z_h}$ *is a solution of the difference method* (19),

3° *there exists a function* $\gamma_0 = (\gamma_{0,1}, \ldots, \gamma_{0,k}) \in \mathcal{F}\left(\mathbf{I_c}, \mathbf{R}_+^k\right)$ *such that*

$$\left|\tilde{\Phi}_{\mathbf{h}}^{(j)}\left(x^{(m_0)}, \mathbf{y}^{(\mathbf{m}')}, B^{(j)}\overline{\mathbf{u}}^{(\mathbf{m})}, \overline{\mathbf{u}}_{\mathbf{h}},\right) \tilde{f}^{(j)}\left(x^{(m_0)}, \mathbf{y}^{(\mathbf{m}')}, \overline{\mathbf{u}}^{(\mathbf{m})}, \overline{\mathbf{u}}\right)\right| \leq \gamma_{0,j}(\mathbf{h}),$$

$$\text{for } \mathbf{m} \in \Gamma_{\mathbf{h}},\ \mathbf{h} \in \mathbf{I_c},\ j = 1, \ldots, k$$

and $\lim_{|\mathbf{h}| \to 0} \gamma_0(\mathbf{h}) = 0$.

Under these assumptions there exists a function $\tilde{\gamma} = (\tilde{\gamma}_1, \ldots, \tilde{\gamma}_k) \in \mathcal{F}\left(\mathbf{I_c}, \mathbf{R}_+^k\right)$ *such that*

$$\left(\left[\!\left[\overline{\mathbf{u}} - \mathbf{v_h}\right]\!\right]_{\mathbf{h}}^{(m_0)}\right)^{\mathbf{T}} \leq (I + h_0 L)^{(m_0)} \sigma_{o,h}^{\mathbf{T}} + h_0 \sum_{i=0}^{m_0 - 1} (I + h_0 L)^i \left[\tilde{\gamma}(\mathbf{h})\right]^{\mathbf{T}}$$

for $m_0 = 0, 1, \ldots, n^*$ *and* $\lim_{|\mathbf{h}| \to 0} \tilde{\gamma}(\mathbf{h}) = 0$.

In particular, if $\lim_{|\mathbf{h}| \to 0} \sigma_{o,h} = 0$, *then* $\lim_{|\mathbf{h}| \to 0} \left[\!\left[\overline{\mathbf{u}}^{(\mathbf{m})} - \mathbf{v}_{\mathbf{h}}^{(\mathbf{m})}\right]\!\right] = 0$ *for* $\mathbf{m} \in \Gamma_{\mathbf{h}}$.

PROOF: Let $\mathbf{w_h} = (w_{h,1}, \ldots, w_{h,k}) \in \mathcal{F}\left(E_\mathbf{h}, \mathbf{R}^k\right)$ be the function defined by $\mathbf{w_h}\left(x^{(m_0)}, \mathbf{y}^{(\mathbf{m}')}\right) = \overline{\mathbf{u}}^{(\mathbf{m})} - \mathbf{v}_\mathbf{h}^{(\mathbf{m})}$. We have

$$
\begin{aligned}
(20)\quad w_{h,j}\left(x^{(m_0+1)}, \mathbf{y}^{(\mathbf{m}')}\right) &= \overline{u}_j^{(m_0+1,\mathbf{m}')} - v_{h,j}^{(m_0+1,\mathbf{m}')} \\
&= \Bigg[\overline{u}_j^{(m_0+1,\mathbf{m}')} - A^{(j)}\overline{u}_j^{(\mathbf{m})} \\
&\quad - h_0\,\tilde{\Phi}_\mathbf{h}^{(j)}\left(x^{(m_0)}, \mathbf{y}^{(\mathbf{m}')}, B^{(j)}\overline{\mathbf{u}}^{(\mathbf{m})}, \overline{\mathbf{u}}_\mathbf{h}\right) - h_0 \sum_{i=1}^{n} g_{ji}^{(\mathbf{m})}\,\Delta_i^{(j)}\overline{u}_j^{(\mathbf{m})}\Bigg] \\
&\quad + A^{(j)}\overline{u}_j^{(\mathbf{m})} - A^{(j)}v_{h,j}^{(\mathbf{m})} \\
&\quad + h_0\Bigg[\tilde{\Phi}_\mathbf{h}^{(j)}\left(x^{(m_0)}, \mathbf{y}^{(\mathbf{m}')}, B^{(j)}\overline{\mathbf{u}}^{(\mathbf{m})}, \overline{\mathbf{u}}_\mathbf{h}\right) \\
&\qquad - \tilde{\Phi}_\mathbf{h}^{(j)}\left(x^{(m_0)}, \mathbf{y}^{(\mathbf{m}')}, B^{(j)}\mathbf{v}_\mathbf{h}^{(\mathbf{m})}, \overline{\mathbf{u}}_\mathbf{h}\right)\Bigg] \\
&\quad + h_0 \sum_{i=1}^{n} g_{ji}^{(\mathbf{m})}\left[\Delta_i^{(j)}\overline{u}_j^{(\mathbf{m})} - \Delta_i^{(j)}v_{h,j}^{(\mathbf{m})}\right] \\
&\quad + h_0\Bigg[\tilde{\Phi}_\mathbf{h}^{(j)}\left(x^{(m_0)}, \mathbf{y}^{(\mathbf{m}')}, B^{(j)}\mathbf{v}_\mathbf{h}^{(\mathbf{m})}, \overline{\mathbf{u}}_\mathbf{h}\right) \\
&\qquad - \tilde{\Phi}_\mathbf{h}^{(j)}\left(x^{(m_0)}, \mathbf{y}^{(\mathbf{m}')}, B^{(j)}\mathbf{v}_\mathbf{h}^{(\mathbf{m})}, \mathbf{v}_\mathbf{h}\right)\Bigg].
\end{aligned}
$$

Since

$$
\begin{aligned}
h_0^{-1}\left[\overline{u}_j^{(m_0+1,\mathbf{m}')} - \sum_{\mathbf{s}\in N_r} a_s^{(j)}\,\overline{u}_j^{(m_0,\mathbf{m}'+\mathbf{s})}\right] \\
= D_x\overline{u}_j\left(x^{(m_0)} + \theta_0 h_0, \mathbf{y}^{(\mathbf{m}')}\right) + R_{3,j}^{(\mathbf{m})}(\mathbf{h}, \tilde{\theta}),
\end{aligned}
$$

$\mathbf{m} \in \tilde{\Gamma}_\mathbf{h}$, $j = 1, \ldots, k$, $\theta_0, \tilde{\theta} \in (0,1)$

($R_{3,j}^{(\mathbf{m})}(\mathbf{h}, \tilde{\theta})$ is defined in the proof of Theorem 2),
we get

$$
\overline{u}_j^{(m_0+1,\mathbf{m}')} - A^{(j)}\overline{u}_j^{(\mathbf{m})} - h_0\,\tilde{\Phi}_\mathbf{h}^{(j)}\left(x^{(m_0)}, \mathbf{y}^{(\mathbf{m}')}, B^{(j)}\overline{\mathbf{u}}^{(\mathbf{m})}, \overline{\mathbf{u}}_\mathbf{h}\right)
$$

$$
\begin{aligned}
&- h_0 \sum_{i=1}^{n} g_{ji}^{(\mathbf{m})}\, \Delta_i^{(j)} \overline{u}_j^{(\mathbf{m})} \\
&= h_0 \Bigg\{ \Big[\tilde{f}_j \Big(x^{(m_0)} + \theta_0 h_0, \mathbf{y}^{(\mathbf{m}')}, \overline{\mathbf{u}} \Big(x^{(m_0)} + \theta_0 h_0, \mathbf{y}^{(\mathbf{m}')} \Big), \overline{\mathbf{u}} \Big) \\
&\qquad - \tilde{f}_j \Big(x^{(m_0)}, \mathbf{y}^{(\mathbf{m}')}, \overline{\mathbf{u}}^{(\mathbf{m})}, \overline{\mathbf{u}} \Big) \Big] \\
&\quad + \Big[\tilde{f}_j \Big(x^{(m_0)}, \mathbf{y}^{(\mathbf{m}')}, \overline{\mathbf{u}}^{(\mathbf{m})}, \overline{\mathbf{u}} \Big) - \tilde{\Phi}_{\mathbf{h}}^{(j)} \Big(x^{(m_0)}, \mathbf{y}^{(\mathbf{m}')}, B^{(j)} \overline{\mathbf{u}}^{(\mathbf{m})}, \overline{\mathbf{u}}_{\mathbf{h}} \Big) \Big] \\
&\quad + \sum_{i=1}^{n} g_{ji} \Big(x^{(m_0)} + \theta_0 h_0, \mathbf{y}^{(\mathbf{m}')} \Big) \\
&\qquad \cdot \Big[\Delta_i^{(j)} \overline{u}_j \Big(x^{(m_0)} + \theta_0 h_0, \mathbf{y}^{(\mathbf{m}')} \Big) - \Delta_i^{(j)} \overline{u}_j^{(\mathbf{m})} \Big] \\
&\quad + \sum_{i=1}^{n} \Delta_i^{(j)} \overline{u}_j^{(\mathbf{m})} \Big[g_{ji} \Big(x^{(m_0)} + \theta_0 h_0, \mathbf{y}^{(\mathbf{m}')} \Big) - g_{ji}^{(\mathbf{m})} \Big] + R_{3,j}^{(\mathbf{m})} \Big(\mathbf{h}, \widetilde{\theta} \Big) \Bigg\}
\end{aligned}
$$

and in similar way like in Theorem 2 we prove that there exists a function $\widetilde{\gamma} = (\widetilde{\gamma}_1, \ldots, \widetilde{\gamma}_k) \in \mathcal{F}\left(\mathbf{I}_{\mathbf{c}}, \mathbf{R}_+^k\right)$ such that

$$
\begin{aligned}
&\Big| \overline{u}_j^{(m_0+1, \mathbf{m}')} - A^{(j)} \overline{u}_j^{(\mathbf{m})} - \\
&\quad - h_0\, \tilde{\Phi}_{\mathbf{h}}^{(j)} \big(x^{(m_0)}, \mathbf{y}^{(\mathbf{m}')}, B^{(j)} \overline{\mathbf{u}}^{(\mathbf{m})}, \overline{\mathbf{u}}_{\mathbf{h}} \big) \\
&\quad - h_0 \sum_{i=1}^{n} g_{ji}^{(\mathbf{m})}\, \Delta_i^{(j)} \overline{u}_j^{(\mathbf{m})} \Big| \leq h_0\, \widetilde{\gamma}_j(\mathbf{h})
\end{aligned} \tag{21}
$$

for $\mathbf{m} \in \widetilde{\Gamma}_{\mathbf{h}}$, $\mathbf{h} \in \mathbf{I}_{\mathbf{c}}$, $j = 1, \ldots, k$ and $\lim\limits_{|\mathbf{h}| \to 0} \widetilde{\gamma}(\mathbf{h}) = 0$.

Using Assumption H_2 and the mean value theorem we obtain

$$
\begin{aligned}
&\Big| A^{(j)} \overline{u}_j^{(\mathbf{m})} - A^{(j)} v_{h,j}^{(\mathbf{m})} + h_0 \Big[\tilde{\Phi}_{\mathbf{h}}^{(j)} \Big(x^{(m_0)}, \mathbf{y}^{(\mathbf{m}')}, B^{(j)} \overline{\mathbf{u}}^{(\mathbf{m})}, \overline{\mathbf{u}}_{\mathbf{h}} \Big) \\
&\qquad - \tilde{\Phi}_{\mathbf{h}}^{(j)} \Big(x^{(m_0)}, \mathbf{y}^{(\mathbf{m}')}, B^{(j)} \mathbf{v}_{\mathbf{h}}^{(\mathbf{m})}, \overline{\mathbf{u}}_{\mathbf{h}} \Big) \Big] \\
&\qquad + h_0 \sum_{i=1}^{n} g_{ji}^{(\mathbf{m})} \Big[\Delta_i^{(j)} \overline{u}_j^{(\mathbf{m})} - \Delta_i^{(j)} v_{h,j}^{(\mathbf{m})} \Big] \Big| \\
&\leq \Big| \sum_{s \in N_1} a_s^{(j)}\, w_{h,j}^{(m_0, \mathbf{m}'+\mathbf{s})} + h_0\, D_{p_j} \tilde{\Phi}_{\mathbf{h}}^{(j)}(\widetilde{P}_j) \sum_{s \in N_1} b_{s,j}^{(j)}\, w_{h,j}^{(m_0, \mathbf{m}'+\mathbf{s})}
\end{aligned}
$$

$$+ h_0 \sum_{i \in I_{j,m}^{(-)}} h_i^{-1} g_{ji}^{(\mathbf{m})} \left[w_{h,j}^{(\mathbf{m})} - w_{h,j}^{(-i(\mathbf{m}))} \right]$$

$$+ h_0 \sum_{i \in I_{j,m}^{(+)}} h_i^{-1} g_{ji}^{(\mathbf{m})} \left[w_{h,j}^{(i(\mathbf{m}))} - w_{h,j}^{(\mathbf{m})} \right] \Bigg|$$

$$+ h_0 \Big| \tilde{\Phi}_{\mathbf{h}}^{(j)}(x^{(m_0)}, \mathbf{y}^{(\mathbf{m}')}, B_1^{(j)} \overline{u}_1^{(\mathbf{m})}, \ldots, B_{j-1}^{(j)} \overline{u}_{j-1}^{(\mathbf{m})}, B_j^{(j)} v_{h,j}^{(\mathbf{m})}, B_{j+1}^{(j)} \overline{u}_{j+1}^{(\mathbf{m})},$$

$$\ldots, B_k^{(j)} \overline{u}_k^{(\mathbf{m})}, \overline{\mathbf{u}}_{\mathbf{h}}) - \tilde{\Phi}_{\mathbf{h}}^{(j)} \left(x^{(m_0)}, \mathbf{y}^{(\mathbf{m}')}, B^{(j)} \mathbf{v}_{\mathbf{h}}^{(\mathbf{m})}, \overline{\mathbf{u}}_{\mathbf{h}} \right) \Big|$$

$$\leq |w_{h,j}^{(\mathbf{m})}| \left[a_\theta^{(j)} + h_0 \, b_{\theta,j}^{(j)} \, D_{p_j} \tilde{\Phi}_{\mathbf{h}}^{(j)}(\widetilde{P}_j) - h_0 \sum_{\tau=1}^{n} h_\tau^{-1} \, |g_{j\tau}(x, \mathbf{y})| \right]$$

$$+ \sum_{\mathbf{s} \in N_1^{(0)}} \left| w_{h,j}^{(m_0, \mathbf{m}'+\mathbf{s})} \right| \left[a_s^{(j)} + h_0 \, b_{s,j}^{(j)} \, D_{p_j} \tilde{\Phi}_{\mathbf{h}}^{(j)}(\widetilde{P}_j) \right]$$

$$+ h_0 \sum_{i \in I_{j,m}^{(-)}} h_i^{-1} \, |g_{ji}^{(\mathbf{m})}| |w_{h,j}^{(-i(\mathbf{m}))}| + h_0 \sum_{i \in I_{j,m}^{(+)}} h_i^{-1} \, |g_{ji}^{(\mathbf{m})}| |w_{h,j}^{(i(\mathbf{m}))}|$$

$$+ h_0 \sum_{\substack{\tau=1 \\ \tau \neq j}}^{k} \lambda_{j\tau} \sum_{\mathbf{s} \in N_1} |b_{s,\tau}^{(j)}| \, |\overline{u}_\tau^{(m_0, \mathbf{m}'+\mathbf{s})} - v_{h,\tau}^{(m_0, \mathbf{m}'+\mathbf{s})}|$$

$$\leq |w_{h,j}|_{\mathbf{h}}^{(m_0)} \left[a_\theta^{(j)} + h_0 \, b_{\theta,j}^{(j)} \, D_{p_j} \tilde{\Phi}_{\mathbf{h}}^{(j)}(\widetilde{P}_j) \right.$$

$$\left. + \sum_{\mathbf{s} \in N_1^{(0)}} \left(a_s^{(j)} + h_0 \, b_{s,j}^{(j)} \, D_{p_j} \tilde{\Phi}_{\mathbf{h}}^{(j)}(\widetilde{P}_j) \right) \right]$$

$$+ h_0 \sum_{\substack{\tau=1 \\ \tau \neq j}}^{k} |w_{h,\tau}|_{\mathbf{h}}^{(m_0)} \, \lambda_{j\tau} \sum_{\mathbf{s} \in N_1} |b_{s,\tau}^{(j)}|$$

$$\leq |w_{h,j}|_{\mathbf{h}}^{(m_0)} (1 + h_0 \lambda_{jj}) + h_0 \sum_{\substack{\tau=1 \\ \tau \neq j}}^{k} |w_{h,\tau}|_{\mathbf{h}}^{(m_0)} \, \lambda_{j\tau} \sum_{\mathbf{s} \in N_1} |b_{s,\tau}^{(j)}| \, .$$

Then from the above estimation, (20) and (21) we get

$$\left|w_{h,j}\left(x^{(m_0+1)},\mathbf{y}^{(\mathbf{m}')}\right)\right| \leq h_0\tilde{\gamma}_j(\mathbf{h}) + |w_{h,j}|_{\mathbf{h}}^{(m_0)}(1+h_0\lambda_{jj})$$
$$+h_0\sum_{\substack{\tau=1\\ \tau\neq j}}^{k}|w_{h,\tau}|_{\mathbf{h}}^{(m_0)}\lambda_{j\tau}\sum_{s\in N_1}\left|b_{s,\tau}^{(j)}\right| + h_0\sum_{\tau=1}^{k}|w_{h,\tau}|_{\mathbf{h}}^{(m_0)}\tilde{\lambda}_{j\tau}.$$

In a similar way like in Theorem 1 we obtain the thesis. ∎

REMARK 5. Suppose that

1° Assumption H_2 and the condition 2° of Theorem 3 are satisfied,

2° there exist constants $\alpha,\beta>0$, $c\geq 0$ and functions $\gamma^*,\gamma_0^*\in\mathcal{F}\left(\mathbf{I}_c,\mathbf{R}_+^k\right)$ such that

$$\left|\overline{u}_j^{(m_0+1,\mathbf{m}')} - A^{(j)}\overline{u}_j^{(\mathbf{m})} - h_0\,\tilde{\Phi}_{\mathbf{h}}^{(j)}\left(x^{(m_0)},\mathbf{y}^{(\mathbf{m}')},B^{(j)}\overline{\mathbf{u}}^{(\mathbf{m})},\overline{\mathbf{u}}_{\mathbf{h}}\right)\right.$$
$$\left.-\sum_{i=1}^{n}g_{ji}^{(\mathbf{m})}\,\Delta_i^{(j)}\overline{u}_j^{(\mathbf{m})}\right| \leq h_0^{\alpha+1}\gamma_j^*(\mathbf{h}),\quad j=1,\ldots,k$$

and

$$[\![\sigma_{o,h}]\!] \leq h_0^{\beta}\gamma_0^*(\mathbf{h}),$$

for $\mathbf{m}\in\widetilde{\Gamma}_{\mathbf{h}}$, $\mathbf{h}\in\mathbf{I}_c$, where $\|\gamma^*(\mathbf{h})\|$, $\|\gamma_0^*(\mathbf{h})\|\leq c$ on $\mathbf{I}_c$.
Under these assumptions we have

$$\left([\![\overline{\mathbf{u}}-\mathbf{v}_{\mathbf{h}}]\!]_{\mathbf{h}}^{(m_0)}\right)^{\mathbf{T}} \leq h_0^{\delta}\left\{h_0^{\beta-\delta}\,(I+h_0L)^{(m_0)}\,[\gamma_0^*(\mathbf{h})]^{\mathbf{T}}\right.$$
$$\left.+\,h_0^{\alpha-\delta+1}\sum_{i=0}^{m_0-1}(I+h_0L)^i\,[\gamma^*(\mathbf{h})]^{\mathbf{T}}\right\}$$

for $m_0=0,1,\ldots,n^*$, where $\delta=\min\{\alpha,\beta\}$.

REMARK 6. Examples of difference methods of order higher than 1 are considered in [7] for differential equations and in [12] for differential equations with retarded argument. It is easy to reformulate them for differential – functional equations.

REFERENCES

1. Bal S., *Convergence of a difference method for a system of first order partial differential equations of hyperbolic type*, Ann. Polon. Math. **30** (1974), 19–36.

2. Brandi P., *Sul problema di Cauchy per una equazione differenziale non lineare alle derivate parziali del primo ordine con argomento funzionale*, Atti Sem. Mat. Fis. Univ. Modena **28** (1979), 86–101.
3. Brandi P., Cepitelli R., *Existence, uniqueness and continuous dependence for a hereditary nonlinear functional partial differential equations of the first order*, Atti Sem. Mat. Fis. Univ. Modena **47** (1986), 121–136.
4. Brandi P., Cepitelli R., *On the existence of the solution of nonlinear functional partial differential equations of the first order*, Atti Sem. Mat. Fis. Univ. Modena **29** (1980), 166–186.
5. Kamont Z., *A difference method for the non–linear partial differential equation of the first order with a retarded argument*, Math. Nachr. **107** (1982), 87–93.
6. Kamont Z., *On the Cauchy problem for non–linear partial differential – functional equations of the first order*, Math. Nachr. **88** (1979), 13–29.
7. Kamont Z., Prządka K., *Difference methods for non–linear partial differential equations of the first order*, Ann. Polon. Math. **48** (1988), 227–246.
8. Kowalski Z., *A difference method for the non–linear partial differential equation of the first order*, Ann. Polon. Math. **18** (1966), 235–242.
9. Meis T., Marcowitz U., *Numerical solution of partial differential equations*, New York, 1981.
10. Pliś A., *On difference inequalities corresponding to partial differential inequalities of first order*, Ann. Polon. Math. **20** (1968), 179–181.
11. Prządka K., *Convergence of one–step difference methods for first order partial differential–functional equations*, Atti Sem. Mat. Fis. Univ. Modena **35** (1987), 263–288.
12. Prządka K., *Difference methods for non–linear partial differential–functional equations of the first order*, Math. Nachr. **138** (1988), 105–123.
13. Prządka K., *On the convergence of multistep difference methods for first order partial differential–functional equations*. to appear.
14. Salvadori A., *Sul problema di Cauchy per una struttura ereditaria di tipo iperbolico. Esistenza, unicità e dipendenza continua*, Atti Sem. Mat. Fis. Univ. Modena **32** (1983), 329–356.
15. Szarski J., *Generalized Cauchy problem for differential – functional equations with first order partial derivatives*, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astr. Phys. **24** (1976), 575–580.

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