UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA, FASCICULUS XXX
1993

# A SIMPLE PROOF OF THE EXISTENCE OF THE ALGEBRAIC CLOSURE OF A FIELD

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The aim of this note is to give a simple proof of the existence of the algebraic closure of a field. Several proofs of this theorem are known (see e.g. [1], [2]) but our proof is very short and its idea is very simple. Let us recall the theorem the proof of which we will give.

THEOREM. *Let $K$ be a field. Then there exists a field $L$ which is an algebraic extension of the field $K$ and every non-constant polynomial from $L[x]$ has a zero in $L$.*

PROOF. We need only one simple algebraic fact:

LEMMA. *Let $K$ be a field and let $f \in K[x]$ be a non-constant polynomial. Then there exists an algebraic extension $L$ of the field $K$ in which the polynomial $f$ has a root.*

PROOF. To prove the lemma we may assume that $f$ is irreducible and now it is enough to take $L = K[x]/(f)$. Then is $x + (f)$ a root of $f$ in $L$. □

Now let $S$ be a set such that 1) $K \subset S$, 2) $\operatorname{card} S > \max(\aleph_0, \operatorname{card} K) := \mathcal{N}$. Let $\mathcal{R} = \{L \subset S : L \text{ is an algebraic extension of } K\}$. Let us introduce an order in $\mathcal{R}$ by putting $L_2 > L_1$ iff $L_1 \subset L_2$ is an algebraic extension. By the Kuratowski–Zorn lemma there exists a maximal element $L_0 \in \mathcal{R}$. We will show that $L_0$ is an algebraic closure of the field $K$. Indeed, let us assume the converse, i. e. that there exists a non-constant polynomial $f \in L_0[x]$ which has no zero in $L_0$. By our lemma there exists an algebraic extension $W$ of the field $L_0$ in which $f$ has a root. Since $W$ is an algebraic extension of the field $K$, we have $\operatorname{card} L_0 \le \operatorname{card} W \le \mathcal{N}$, hence $\operatorname{card}(S \setminus L_0) = \operatorname{card} S > \operatorname{card}(W \setminus L_0)$. Thus there exists an injection $i: W \longrightarrow S$ such that $i(x) = x$ for $x \in L_0$. If we implant by $i$ an algebraic structure from $W$ onto $i(W)$ we obtain a new maximal element in $\mathcal{R}$ which is grater than $L_0$. This contradiction finishes the proof. □

## References

1. Lang S., *Algebra*, PWN, Warszawa, 1973.
2. Browkin J., *Teoria ciał*, PWN, Warszawa, 1977.

*Received November 7, 1991*