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Abstract. We extend a result of Blatt – Saff – Simkani on the distribution of the zeros of best polynomial approximants to continuous functions on compacta of $\mathbf{C}$ and point out that the problem is related to a classical result of Leja proved in 1934 concerning Green's function.

0. Introduction

The aim of this note is to complement a series of papers by Blatt, Saff, Simkani and others that have appeared in recent years, concerning the distribution of the zeros of best approximants to a continuous function f on a compact subset E of the complex plane $\mathbf{C}$. They have shown that if $\mathbf{C} \setminus E$ is connected and regular, f is analytic on $\operatorname{int} E$, $f \not\equiv 0$ at any component of $\operatorname{int} E$ and f does not extend to a holomorphic function in an open neighbourhood of E then the limit distribution (with respect to the weak*-topology) of these zeros coincides with the equilibrium measure μ_E for E. In particular, every point of the carrier of μ_E is an accumulation point of the zeros of the best approximants to f.

We point out that the above problem of distribution of the zeros is closely related to the distribution of limit points of systems of Fekete–Leja extremal points of E. The latter problem arose in connection with a fundamental result of Leja proved in 1934, according to which the Green's function for the unbounded component of $\overline{\mathbf{C}} \setminus E$, with pole at ∞, is the limit of sequences involving Lagrange interpolation polynomials with nodes at Fekete extremal points. (See also Myberg [25]). The distribution problem for the extremal points was intensely studied by Leja and his students in the fifties of this century (we refer the reader to [12], [18] and comments on Chapter II in [10]).

Thus, we complete the list of the references in [3], [4] and [8] by the classical results of Leja's school.

We also give a short proof of a theorem that extends the key result of Blatt – Saff – Simkani [4] on the distribution of the zeros of polynomials to the case of non-monic polynomials. This improves, in particular, the main result of [8].

1. Leja's extremal function

Let E be a compact subset of the complex plane $\mathbf{C}$. For each $n \geq 1$, let $X_n = \{x_{n0}, \ldots, x_{nn}\}$ be an arbitrary system of $n+1$ points of E. Consider the *Vandermonde determinant*

$$V(X_n) = \prod_{0 \leq j < k \leq n} |x_{nj} - x_{nk}|$$

and choose a system $Y_n = \{y_{n0}, \ldots, y_{nn}\} \subset E$ such that

$$V_n(E) := V(Y_n) = \max\{V(X_n) : X_n \subset E\}$$

and

$$\Delta^{(0)}(Y_n) := \prod_{k=1}^{n} |y_{n0} - y_{nk}| \leq \Delta^{(j)}(Y_n) := \prod_{\substack{k=0 \\ k \neq j}}^{n} |y_{nj} - y_{nk}|.$$

The system Y_n is called a *Fekete system of extremal points* of E of order n. Leja [13] proved that there exists the limit

$$\lim_{n \to \infty} \left[\Delta^{(0)}(Y_n)\right]^{1/n}$$

which is equal to the *transfinite diameter* of E, $d(E)$, introduced by Fekete [6].

Given an extremal system Y_n, let us denote

$$L_{nj}(z) = L_{nj}(z, Y_n) := \prod_{\substack{k=0 \\ k \neq j}}^{n} \frac{z - y_{nk}}{y_{nj} - y_{nk}}$$

for $j = 0, \ldots, n$. The polynomials L_{nj} are called *fundamental Langrange interpolation polynomials* corresponding to the nodes Y_n. Denote by $D_\infty(E)$ the unbounded component of the complement $\mathbf{C} \setminus E$ of E. Let $\widehat{E}$ denote the set $\overline{\mathbf{C}} \setminus D_\infty(E)$ and let F be the boundary of $D_\infty(E)$. In 1934 Leja [11] proved the following

THEOREM 1.1. *Suppose that* $d(E) > 0$.

1° *For each* $z \in \mathbf{C} \setminus F$, *there exists the limit*

$$L_E(z) = \lim_{n\to\infty} |L_{n0}(z)|^{1/n},$$

the convergence being uniform on every compact subset of $\mathbf{C} \setminus F$.

2°

$$\log L_E(z) = \begin{cases} G_E(z), & \text{if } z \in D_\infty(E), \\ 0, & \text{if } z \in \operatorname{int} \hat{E}, \end{cases}$$

where G_E *denotes the Green's function of* $D_\infty(E)$ *with pole at* ∞ *(see Section 2).*

3° *The set* E^* *of all limit points of the system* $\{Y_1, Y_2, \dots\}$ *of extremal points of* E *coincides with* F.

4° *For each Borel subset* A *of* E *there exists the limit*

$$\mu_E(A) = \lim_{n\to\infty} \mu_n(A)$$

where

$$\mu_n(A) := \frac{1}{n}(\text{number of points of } Y_n \cap A).$$

This remarkable result became the starting point for the so called method of the extremal function L_E, which is known nowadays as *Leja's extremal function* associated with E. The method was created by Leja and then developed by Leja's students at the Jagiellonian University of Cracow. This method admits important applications in complex analysis in one variable as well as in several variables and in approximation theory. By Bedford and Taylor's theory of the *complex Monge - Ampère operator* [1], it appears that a multivariate counterpart of L_E, which is *Siciak's extremal function* (see [19] and [21]), plays a similar role in pluripotential theory (based on plurisubharmonic functions) as that of L_E explained by 2° of Theorem 1.1. There exist at present many papers in the literature on this subject, and we shall restrict ourselves to refering the reader to [13] and [20] in the one-dimensional case (see also Comments on Chapter II in [10]), and to [15], [17], [19], [21], and [24] in the multivariate case.

In the sequel, we extend the function L_E to the whole plane $\mathbf{C}$ by setting $L_E = 1$ on F. By Langrange's interpolation formula, it is not difficult to show that

$$L_E(z) = \lim_{n\to\infty} \left(\sum_{j=0}^{n} |L_{nj}(z)| \right)^{1/n} \qquad \text{for all } z \text{ in } \mathbf{C}$$

(cf. [13], Chapter XI, section 5). Hence, in particular, one gets the following version of the *Bernstein – Walsh inequality*:

INEQUALITY 1.2. *For each polynomial p of degree at most n, we have*

$$|p(z)| \leq \|p\|_E [L_E(z)]^n \quad \text{for all } z \text{ in } \mathbf{C},$$

which will be a crucial point in our futher investigations.

We recall that E is said to be *regular at a point* $\zeta \in F$, if the Green's function G_E has the limit

$$\lim_{\substack{z \to \zeta \\ z \in D_\infty(E)}} G_E(z) = 0.$$

E is said to be *regular* if it is regular at every point $\zeta \in F$. By 2° of Theorem 1.1 this is equivalent to saying that the extremal function L_E is continuous on the whole plane $\mathbf{C}$.

2. Some basic facts from potential theory

If E is a compact subset of $\mathbf{C}$ with $d(E) > 0$, let μ_E be the unique positive measure with $\operatorname{supp} \mu_E \subset E$ and $\mu_E(E) = 1$ that minimizes the *energy integral*

$$I(\mu) := \iint \log(1/|z - w|) d\mu(z) d\mu(w)$$

over all unit measures supported on E (see [22]). Then μ_E is called the *equilibrium measure* (or else *equilibrium distribution*) for E and

$$U^{\mu_E}(z) := \int \log(1/|z - w|) d\mu_E(w)$$

is the *logarithmic potential* of E. The number

$$\operatorname{cap}(E) := \exp(-I(\mu_E))$$

is called the *logarithmic capacity* of E and is known to be equal to $d(E)$ (see [22]). The *Green's function* G_E for $D_\infty(E)$ with pole at ∞ is the (unique) positive harmonic function in $D_\infty(E) \setminus \{\infty\}$ such that

$$G_E(z) = -U^{\mu_E}(z) - \log \operatorname{cap}(E). \tag{2.1}$$

We notice that the function $G_E(z) - \log|z|$ is harmonic at ∞ and that

$$\lim_{z\to\infty}(G_E(z) - \log|z|) = -\log \operatorname{cap}(E). \tag{2.2}$$

3. Distribution of the zeros of the polynomials of the best approximation

In 1929 Bernstein(cf. [2], p. 450) announced the following result: if almost all zeros of the polynomials of best approximation to a positive continuous function on $[-1,1]$ are outside an open ellipse $\mathcal{E}$ with foci -1 and 1 then f extends to a holomorphic function in $\mathcal{E}$. However, he gave a proof of this theorem only for the case of L^2 – weighted approximation. This result was extended by Pleśniak [16] to the case of $L^2(E,\mu)$ – approximation where E is a compact subset of $\mathbf{C}$ and μ is a positive finite measure on E such that the pair (E,μ) satisfies *Leja's polynomial condition.* The theorem announced by Bernstein was proved by Borwein [5] and then extended by Wójcik [23] to the case of uniform approximation on regular compact subsets of $\mathbf{C}$ (cf. also [3]).

In recent years a series of papers were published by Blatt, Saff, Simkani and others concerning the limit distribution of the zeros of the polynomials of best approximation to functions from the algebra $A(E)$ of all continuous functions on E that are holomorphic on $\operatorname{int} E$ (cf. [3], [1] and papers quoted there). Their key theorem can be reformulated as follows (see [4], theorem 2.1).

THEOREM 3.1. *Let E be a compact subset of the complex plane $\mathbf{C}$ with $\operatorname{cap}(E) > 0$. Let N be an infinite subset of the positive integers and $\{p_n\}_{n\in N}$ be a sequence of monic polynomials with $\deg p_n = n$. For each $n \in N$, we denote by $\nu_n = \nu(p_n)$ the discret unit measure defined on Borel sets in $\mathbf{C}$ having mass $1/n$ at each zero of p_n (counting its multiplicity). Suppose that*

$$\limsup_{\substack{n\to\infty \\ (n\in N)}} \|p_n\|_E^{1/n} = \operatorname{cap}(E)$$

and for every compact subset K of $\operatorname{int}\widehat{E}$,

$$\lim_{\substack{n\to\infty \\ (n\in N)}} \nu_n(K) = 0.$$

Then $\nu_n \to \mu_E$ weakly for $n \to \infty$ $(n \in N)$. In other words, for any continuous function φ with compact support in $\mathbf{C}$, we have

$$\int \varphi d\nu_n \longrightarrow \int \varphi d\mu_E \quad \text{as } n \to \infty \ (n \in N).$$

Theorem 3.1 may be adapted to the situation where the polynomials p_n are not necessarily monic (see [8]) which is, in general, more convenient for applications. We also propose here such a version of Theorem 3.1 and give its simple proof.

THEOREM 3.2. *With the assumptions of Theorem 3.1 on E, let $\{p_n\}_{n\in N}$ be a sequence of polynomials with $\deg p_n = n$. Suppose now that*

(i) $$\limsup_{\substack{n\to\infty \\ (n\in N)}} (\|p_n\|_E)^{1/n} = 1;$$

(ii) *there exists $z_0 \in D_\infty(E)$ such that*

$$\limsup_{\substack{n\to\infty \\ (n\in N)}} \left[\frac{1}{n}\log|p_n(z_0)| - G_E(z_0)\right] = 0;$$

(iii) *for every compact subset K of $\operatorname{int}\widehat{E}$*

$$\lim_{\substack{n\to\infty \\ (n\in N)}} \nu_n(K) = 0.$$

Then $\nu_n \to \mu_E$ weakly for $n \to \infty$ $(n \in N)$.

We precede the proof of Theorem 3.2 by the following

LEMMA 3.3. *Let E and $\{p_n\}_{n\in N}$ be as in Theorem 3.2 and suppose that conditions (i) and (ii) hold. Then for every compact subset K of $D_\infty(E)$ $\nu_n(K) \to 0$ as $n \to \infty$ $(n \in N)$.*

PROOF. We set

$$u_n(z) = \frac{1}{n}\log|p_n(z)| - G_E(z) + \frac{1}{n}\sum_k G(z, z_{nk}),$$

where $\{z_{nk}\}$ are the zeros of p_n that lie in K and $G(\cdot, w)$ is the Green's function for $D_\infty(E)$ with pole at w (see e. g. [22]). We notice that for each $n \in N$ the function u_n is subharmonic in $D_\infty(E)$. Moreover, if c is a regular point of $\partial\widehat{E}$, then

$$u_n(c) \le \frac{1}{n}\log\|p_n\|_{E^*},$$

where E^* denotes the support of the measure μ_E. Hence by the generalized maximum principle for subharmonic functions (see e.g. [9], theorem 5.16), we have

$$u_n(z) \le \frac{1}{n}\log\|p_n\|_{E^*} \le \frac{1}{n}\log\|p_n\|_E \tag{3.1}$$

for all z in $D_\infty(E)$. Since for each $z \in D_\infty(E)$ the function $G(z, \cdot)$ is lower semicontinuous, we have

$$G(z, z_{nk}) \ge \inf_{w\in K} G(z, w) =: \delta(z) > 0.$$

Hence, for $n \in N$ and z in $D_\infty(E)$,

$$\delta(z)\nu_n(K) \leq \frac{1}{n}\log\|p_n\|_E + G_E(z) - \frac{1}{n}\log|p_n(z)|.$$

Thus, by assumptions (i) and (ii),

$$\limsup_{\substack{n\to\infty \\ (n\in N)}} \nu_n(k) = 0.$$

PROOF OF THEOREM 3.2. Take $R_0 > 1$ so large that $E \subset \{|z| < R_0\}$. For each $R > R_0$, let $\{z_{n1}, \ldots, z_{nl_n}\}$ be the zeros of p_n that lie in $\{|z| \geq R\}$. By (2.2) and symmetry of Green's functions, from (3.1) we derive

$$u_n(\infty) = \frac{1}{n}\log|a_n| + \log\operatorname{cap}(E) + \frac{1}{n}\sum_{k=1}^{l_n} G_E(z_{nk}) \leq \frac{1}{n}\log\|p_n\|_{E^*}, \tag{3.2}$$

where a_n denotes the leading coefficient of p_n. For each $\varepsilon > 0$, we can choose $R > 1$ such that

$$|\log|z| - G_E(z) - \log\operatorname{cap}(E)| < \varepsilon$$

as $|z| \geq R$. Set

$$q_n^R(z) = a_n(z - z_{n1})\ldots(z - z_{nl_n}).$$

By Lemma 3.3, $l_n/n \to 0$ for $n \to \infty$ $(n \in N)$. Hence, if $R_0 < |z| < R$, we get

$$\begin{aligned}\frac{1}{n}\log|q_n^R(z)| &= \frac{1}{n}\log|a_n| + \frac{1}{n}\sum_k \log|z_{nk}| + o(1)\\ &= \frac{1}{n}\log|a_n| + \frac{1}{n}\sum_k G_E(z_{nk}) + o(1),\end{aligned}$$

and by (3.2)

$$\frac{1}{n}\log|q_n^R(z)| \leq \frac{1}{n}\log\|p_n\|_{E^*} - \log\operatorname{cap}(E) + o(1). \tag{3.3}$$

(Notice that if $l_n = 0$ for all $n \in N$ then (3.3) directly follows from (2.2) and Inequality 1.2).

Now, by Frostman's selection theorem (see [22], theorem II.4) one can choose a subsequence $\{\nu_n\}_{n\in N_1}$ of the sequence $\{\nu_n\}_{n\in N}$ weakly convergent to

a measure ν. By Lemma 3.3 and assumption (iii), the measure ν is supported on $F = \partial D_\infty(E)$ and has total mass 1. For $n \in N$, set

$$h_n(z) = \frac{1}{n}\log|p_n(z)| - \frac{1}{n}\log\|p_n\|_E - G_E(z). \tag{3.4}$$

Then every function h_n is subharmonic in $D_\infty(E)$ and by Inequality 1.2, $h_n \leq 0$ in $D_\infty(E)$, for $n \in N$. Hence by (ii),

$$h(z) := \limsup_{\substack{n\to\infty \\ (n\in N_1)}} h_n(z) = 0 \tag{3.5}$$

quasi–everywhere on $D_\infty(E)$, i. e.

$$\operatorname{cap}(\{z \in D_\infty(E); h(z) < 0\}) = 0.$$

Therefore, if $a \in F$, for every $k \in \mathbf{N}$ one may find $z_k \in D_\infty(E)$ such that $|a - z_k| < 1/k$ and

$$\limsup h_n(z_k) = 0 \quad \text{as} \quad n \to \infty \quad (n \in N_1). \tag{3.6}$$

Let ν_n^R be the restriction of the measure ν_n to the set $\{|z| \leq R\}$. Then $\nu_n^R \longrightarrow \nu$ weakly for $n \to \infty$ $(n \in N_1)$. By (i), (3.3), (3.4) and (2.1) we have

$$\begin{aligned} -\log\operatorname{cap}(E) &\geq \limsup \frac{1}{n}\log|q_n^R(z_k)| \\ &\geq \limsup \frac{1}{n}\log|p_n(z_k)| - G_E(z_k) \\ &\quad + \liminf U^{\nu_n^R}(z_k) - U^{\mu_E}(z_k) - \log\operatorname{cap}(E), \end{aligned}$$

whence by the descent principle (see [9], lemma 5.1) we get

$$U^\nu(z_k) \leq \liminf U^{\nu_n^R}(z_k) \leq U^{\mu_E}(z_k)$$

for $k = 1, 2, \ldots$. Since $U^{\mu_E}(z) \leq I(\mu_E)$ for all z in $\mathbf{C}$ (cf. [22], theorem III.12), and since U^ν is lower semicontinuous, it follows that

$$U^\nu(z_k) \leq \liminf_{k\to\infty} U^\nu(z_k) \leq I(\mu_E)$$

for all a in F. By an argument of the proof of theorem 2.1 of [4], we now have

$$I(\nu) = \int U^\nu(a)d\nu(a) \leq \int I(\mu_E)d\nu(a) = I(\mu_E)\nu(F) = I(\mu_E)$$

and by the uniqueness of the measure minimizing the energy integral $I(\mu)$, we get $\nu = \mu_E$ and, consequently, the whole sequence $\{\nu_n\}_{n\in N}$ converges weakly to μ_E. The proof is completed.

REMARK. In the case where E is regular, (a version of) Theorem 3.2 has been proved by Grothmann [8].

Theorem 3.2 can be viewed as a generalization of Leja's Theorem 1.1. (If $p_n = L_{n0}$ then assertions 1° and 2° of Theorem 1.1 can be easily derived from the proof of Theorem 3.2). As in the case of monic polynomials (see [4]), Theorem 3.2 admits a converse. This complements Leja's result. We first prove

PROPOSITION 3.4. *Let E be a compact subset of $\mathbf{C}$ regular at every point of the support E^* of the equilibrium measure μ_E. Let $\{p_n\}_{n\in N}$ be a sequence of polynomials with $\deg p_n = n$ and leading coefficient a_n such that the sequence $\nu_n = \nu(p_n)$ tends to μ_E weakly for $n \to \infty$ $(n \in N)$. For each $R > 1$ such that $E \subset \{|z| < R\}$, let $\{z_{n1}, \ldots, z_{nl_n}\}$ be the zeros of p_n that lie in $\{|z| \geq R\}$. Then, for $n \to \infty$ $(n \in N)$ and R large enough,*

$$\limsup \left[|a_n| L_E(z_{n1}) \ldots L_E(z_{nl_n})\right]^{1/n} = \limsup \left[\|p_n\|_{E^*}^{1/n} / \operatorname{cap}(E)\right].$$

(Here we put $L_E(z_{n1}) \ldots L_E(z_{nl_n}) = 1$ if $l_n = 0$.)

PROOF. By lemma 1.4 of [16], for each $n \in N$,

$$|a_n| [\operatorname{cap}(E)]^{l_n} L_E(z_{n1}) \ldots L_E(z_{nl_n}) = M_n$$

where

$$M_n = \lim_{m\to\infty} \left|q_n^R(z_0) q_n^R(z_1) \ldots q_n^R(z_{m-1})\right|^{1/m}$$

with

$$q_n^R(z) = a_n(z - z_{n1}) \ldots (z - z_{nl_n}),$$

$\{z_0, z_1, \ldots\} \subset E$ being a *Leja extremal sequence* of E, i.e. if $z_0 \in E$ is arbitrarily chosen, then for $m \geq 1$ z_m is taken to satisfy

$$|(z_m - z_0)(z_m - z_1) \ldots (z_m - z_{m-1})| = \max_{z\in E} |(z - z_0)(z - z_1) \ldots (z - z_{m-1})|.$$

Since $\nu_n \to \mu_E$ weakly, we have $l_n/n \to 0$ for $n \to \infty$ $(n \in N)$ and, consequently, for $R > 1$ large enough,

$$\limsup M_n^{1/n} \geq \limsup \left[\inf_{z\in E} |q_n^R(z)|\right]^{1/n} = \limsup \|q_n^R\|_E^{1/n}.$$

For each $n \in N$, choose $b_n \in E^*$ such that $\|p_n\|_{E^*} = |p_n(b_n)|$. Without loss of generality we may assume that $b_n \to b \in E^*$ as $n \to \infty$ $(n \in N)$. Then by the generalized descent principle (cf. [10], theorem 1.3),

$$\begin{aligned}\limsup \frac{1}{n} \log \|q_n^R\|_{E^*} &\geq \limsup \frac{1}{n} \log |q_n^R(b_n)| \\ &\geq \limsup \frac{1}{n} \log |p_n(b_n)| + \liminf U^{\nu_n^R}(b_n) \\ &\geq \limsup \frac{1}{n} \log \|p_n\|_{E^*} + U^{\mu_E}(b) \\ &= \limsup \frac{1}{n} \log \|p_n\|_{E^*} - \log \operatorname{cap}(E).\end{aligned}$$

This, together with (3.3), completes the proof of the proposition.

It follows from (3.3) that with the assumptions of Theorem 3.1, for R large enough, we have

$$\lim_{\substack{n \to \infty \\ (n \in N)}} |z_{n1} \dots z_{nl_n}|^{1/n} = 1, \tag{3.7}$$

where we put $z_{n1} \dots z_{nl_n} = 1$ if $l_n = 0$.

Conversely, we have

THEOREM 3.5. *Let E and $\{p_n\}_{n \in N}$ satisfy the assumptions of Proposition 3.4. If, for $R > 1$ such that $E \subset \{|z| < R\}$, we have (3.7), then*

$$\lim_{\substack{n \to \infty \\ (n \in N)}} \|p_n / a_n\|_{E^*}^{1/n} = \operatorname{cap}(E).$$

Theorem 3.5 immediately follows from Proposition 3.4 and from the definition of the Chebyshev constant of E (see e.g. [22]). Under the assumption that all zeros of the p_n are uniformly bounded, it has been proved in [4].

Assume now that E is a regular compact subset of $\mathbf{C}$ with connected complement $\mathbf{C} \setminus E$. By Mergelyan's theorem, every function f continuous on E and holomorphic in $\operatorname{int} E$ is the uniform limit of polynomials. If f does not extend to a holomorphic in a neighbourhood of E one may choose a sequence N of positive integers such that the leading coefficients a_n of the polynomials p_n with $\deg p_n = n$ $(n \in N)$ of the best approximation to f must satisfy

PROPERTY 3.6.

$$\lim_{n \to \infty} |a_n|^{1/n} = 1/\operatorname{cap}(E)$$

(see [16], case of L^2 – approximation and [3] or [23], case of uniform approximation). Then, since

$$\lim_{z\to\infty} \frac{1}{n}\log|p_n(z)| - G_E(z) = \log \operatorname{cap}(E) + \frac{1}{n}\log|a_n|,$$

Theorem 3.2 applied to the polynomials p_n $(n \in N)$ and $z_0 = \infty$ gives the result of Blatt and Saff [3] (cf. also [4]). A similar corollary can be obtained in the case of L^p – approximation. As an example we shall consider the L^p – approximation with respect to the Lebesque plane measure λ on *uniformly polynomially cuspidal* (briefly, *UPC*) compact subsets of $\mathbf{C}$ (see [14]).

DEFINITION 3.7. A compact subset $E \subset \mathbf{C} \cong \mathbf{R}^2$ is said to be *UPC* if there exist positive constants M and m, and a positive integer d such that for each point $z \in E$ one may choose a polynomial map $h_z : \mathbf{R} \longrightarrow \mathbf{R}^2$ of degree at most d satisfying the following conditions.

$$h_z([0,1]) \subset E \qquad \text{and} \quad h_z(0) = z$$
$$\operatorname{dist}(h_z(t), \mathbf{C} \setminus E) \geq Mt^m \qquad \text{for all } z \text{ in } E \quad \text{and} \quad t \in [0,1].$$

Every convex compact set E in $\mathbf{C}$ with $\operatorname{int} E \neq \emptyset$ and every $E = \overline{D}$ where D is a bounded domain with Lipschitz boundary are UPC. If E is semianalytic (as a subset of $\mathbf{R}^2$) and $\overline{\operatorname{int} E} = E$ then E is UPC. (For more detailed information on UPC sets we refer the reader to [14].) We shall need

LEMMA 3.8 (cf. [7], theorem 2). *If E is UPC and $p \geq 1$ then there exists constants $M' > 0$ and $m' > 0$ such that for each polynomial q of degree at most n*

$$\|q\|_E \leq M' n^{m'} \left(\int_E |q|^p d\lambda \right)^{1/p}.$$

We note that by the above Lemma one can easily extend Property 3.6 to the case of L^p– approximation (cf. the proof of Property 3.3 in [23]).

Assume now that E is a UPC compact subset of $\mathbf{C}$ such that $\mathbf{C} \setminus E$ is connected. Then, in particular, E is regular (see [14]). Let $L^p_P(E,\lambda)$ $(p \geq 1)$ be the closure of the algebra of all polynomials in $L^p(E,\lambda)$. Then it is clear that the restriction of any $f \in L^p_P(E,\lambda)$ to $\operatorname{int} E$ is equal almost everywhere to a holomorphic function g in $\operatorname{int} E$. Suppose that $g(z) \not\equiv 0$ at any connected component of $\operatorname{int} E$. Then, by Property 3.6 and Lemma 3.8, we derive from Theorem 3.1 the following:

COROLLARY 3.9. *Under the above assumptions, suppose that there is no open neighbourhood U of E and no holomorphic function $\widetilde{f}$ in U such that $\widetilde{f}|_E = f$ almost everywhere in E. Then every point of the boundary of E is a limit point of the set of the zeros of the polynomials of best approximation to f in $L^p(E, \lambda)$.*

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